Research Article

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On topological properties of block shift and hierarchical hypercube networks

https://doi.org/10.1515/phys-2018-0101
Received Oct 24, 2018; accepted Nov 14, 2018

Abstract: Networks play an important role in electrical and electronic engineering. It depends on what area of electrical and electronic engineering, for example there is a lot more abstract mathematics in communication theory and signal processing and networking etc. Networks involve nodes communicating with each other. Graph theory has found a considerable use in this area of research. A topological index is a real number associated with chemical constitution purporting for correlation of chemical networks with various physical properties, chemical reactivity. The concept of hyper Zagreb index, first multiple Zagreb index, second multiple Zagreb index and Zagreb polynomials was established in chemical graph theory based on vertex degrees. In this paper, we extend this study to interconnection networks and derive analytical closed results of hyper Zagreb index, first multiple Zagreb index, second multiple Zagreb index, Zagreb polynomials and redefined Zagreb indices for block shift network \((BSN - 1)\) and \((BSN - 2)\), hierarchical hypercube \((HHC - 1)\) and \((HHC - 2)\).

Keywords: hyper Zagreb index, first multiplicative Zagreb index, second multiplicative Zagreb index, Zagreb polynomials, redefined Zagreb indices, block shift networks, hierarchical interconnection networks.

PACS: 02.10.Ox

1 Introduction

Multiprocessor interconnection networks are often required to connect thousands of homogeneously replicated processor-memory pairs, each of which is called a processing node. Instead of using a shared memory, all synchronization and communication between processing nodes for program execution is often done via message passing. Design and use of multi processor interconnection networks have recently drawn considerable attention due to the availability of inexpensive, powerful microprocessors and memory chips. The mesh networks have been recognized as versatile interconnection networks for massively parallel computing. Mesh/torus-like low-dimensional networks have recently received a lot of attention for their better scalability to larger networks, as opposed to more complex networks such as hypercubes. In particular the failure of cooperation on dependent networks has been studied a lot recently in [1].

A number of hierarchical interconnection network (HIN) provide a framework for designing networks with reduced link cost by taking advantage of the locality of communication that exist in parallel applications. HIN employ multiple level. Lower level network provide local communication while higher level networks facilitate remote communication. The multistage networks have long been used as communication networks for parallel computing [2].

The topological properties of certain networks are studied in [3]. Molecules and molecular compounds are often modeled by molecular graphs. A molecular graph is a graph in which vertices are atoms of a given molecule and edges are its chemical bonds. Since the valency of carbon is four, it is natural to consider all graphs with maximum degree \(\leq 4\), as a molecular graph. A graph \(G(V, E)\) with vertex set \(V\) and edge set \(E\) is connected, if there exists a connection between any pair of vertices in \(G\). A network is simply a connected graph having no multiple edges and no loops. Throughout this article, the degree of a vertex \(v \in V(G)\), denoted by \(deg(v)\), is the number of edges incident to \(v\).

A topological index is a numeric quantity associated with a graph which characterizes the topology of a graph and is invariant under graph automorphism. In more pre-
cise way, a topological index \( \text{Top}(G) \) of a graph \( G \), is
a number with the property that for every graph \( H \) isomorphic
to \( G \), \( \text{Top}(H) = \text{Top}(G) \). The concept of topo-
logical index came from work done by Wiener [4], while
he was working on a boiling point of paraffin. He named
this index as the \textit{path number}. Later on, the path number
was renamed as the \textit{Wiener index}. The Wiener index is the
first and the most studied topological index, both from theo-
retical point of view and applications, and defined as the
sum of distances between all pairs of vertices in \( G \), see for
details [5, 6].

One of the oldest topological indices is the first Zagreb
index introduced by I. Gutman and N. Trinajstic based on
degree of vertices of \( G \) in 1972 [7]. The first and the second
Zagreb indices of a graph \( G \) are defined as:

\[
M_1(G) = \sum_{uv \in E(G)} [\deg(u) + \deg(v)]
\]

\[
M_2(G) = \sum_{uv \in E(G)} [\deg(u) \times \deg(v)]
\]

In 2013, G. H. Shirdel, H. R. Pour and A. M. Sayadi [8] in-
roduced a new degree based on Zagreb index named \textit{“ hyper
Zagreb index”} as:

\[
HM(G) = \sum_{uv \in E(G)} [\deg(u) + \deg(v)]^2
\]

M. Ghorbani and N. Azimi defined two new versions ofZa-
greb indices of a graph \( G \) in 2012 [9]. These indices are
the first multiplicative Zagreb index \( PM_1(G) \) and the sec-
ond multiplicative Zagreb index \( PM_2(G) \) and are defined
respectively as:

\[
PM_1(G) = \prod_{uv \in E(G)} [\deg(u) + \deg(v)]
\]

\[
PM_2(G) = \prod_{uv \in E(G)} [\deg(u) \times \deg(v)]
\]

The properties of \( PM_1(G) \) and \( PM_2(G) \) indices for some
chemical structures have been studied in [10].

The first Zagreb polynomial \( M_1(G, x) \) and the second Za-
greb polynomial \( M_2(G, x) \) are defined as:

\[
M_1(G, x) = \sum_{uv \in E(G)} x^{\deg(u) + \deg(v)}
\]

\[
M_2(G, x) = \sum_{uv \in E(G)} x^{\deg(u) \times \deg(v)}
\]

Ranjini et al. [12] reclassified the Zagreb indices, to be spe-
cific, the redefined first, second and third Zagreb indices
for a graph \( G \) as:

\[
ReZG_1(G) = \sum_{uv \in E(G)} \frac{\deg(u) + \deg(v)}{\deg(u) \times \deg(v)}
\]

\[
ReZG_2(G) = \sum_{uv \in E(G)} \frac{\deg(u) \times \deg(v)}{\deg(u) + \deg(v)}
\]

\[
ReZG_3(G) = \sum_{uv \in E(G)} \left[ \deg(u) \times \deg(v) \right]
\]

\[
\times \left[ \deg(u) + \deg(v) \right]
\]

Nowadays, there is an extensive research activity on
HM(\( G \)), \( PM_1(G) \), \( PM_2(G) \) indices and \( M_1(G, x) \), \( M_2(G, x) \)
polynomials and their variants, see also [13–15].

For further study of topological indices of various graph
families, see [16–30].

2 Methods

For the calculation of our outcomes, we used a method-
ology for combinatorial enlisting, a vertex segment pro-
cedure, an edge parcel procedure, diagram theoretical
instruments, logical frameworks, a degree-tallying tech-
nique, and degrees of neighbors system.

3 Results for block shift network

\((BSN - 1)_{n \times n}\)

In this section, we compute certain degree based topo-
logical indices for block shift network (BSN), see Figure
1(a). The Randic type indices for hierarchical interconnec-
tion networks is computed by Haider et. al in [31]. We
compute first and second Zagreb indices, hyper Zagreb in-
dex \( HM(G) \), first multiple Zagreb index \( PM_1(G) \), second
multiple Zagreb index \( PM_2(G) \) and Zagreb polynomials
\( M_1(G, x) \), \( M_2(G, x) \) for \((BSN - 1)_{n \times n}\).

Let \( G \) be a block shift network. The number of ver-
tices and edges in \((BSN - 1)_{n \times n}\) is 16\( n^2 \) and 24\( n^2 - 2 \), re-
spectively. There are two types of edges in \((BSN - 1)_{n \times n}\)
based on degrees of end vertices of each edge. The edge
set of \((BSN - 1)_{n \times n}\) can be divided into two partitions
based on the degree of end vertices. The first edge parti-
tion \( E_1((BSN - 1)_{n \times n}) \) contains 8 edges \( uv \), where \( \deg(u) = 2, \deg(v) = 3 \). The second edge partition \( E_2((BSN - 1)_{n \times n}) \)
contains 24\( n^2 - 10 \) edges \( uv \), where \( \deg(u) = \deg(v) = 3 \).
The first and second Zagreb indices of \((BSN - 1)_{n \times n}\)

Now using Eqs. (1),(2), we have

\[
M_1(G) = \sum_{uv \in E(G)} [\text{deg}(u) \times \text{deg}(v)]
\]
\[
M_2(G) = \sum_{uv \in E_1(G)} [\text{deg}(u) \times \text{deg}(v)]
\]
\[
+ \sum_{uv \in E_2(G)} [\text{deg}(u) \times \text{deg}(v)]
\]
\[
= 5|E_1(G)| + 6|E_2(G)|
\]
\[
= 5(8) + 6(24n^2 - 10)
\]
\[
= 144n^2 - 20.
\]

\[
M_2(G) = \sum_{uv \in E(G)} [\text{deg}(u) \times \text{deg}(v)]
\]

\[
M_2(G) = \sum_{uv \in E_1(G)} [\text{deg}(u) \times \text{deg}(v)]
\]
\[
+ \sum_{uv \in E_2(G)} [\text{deg}(u) \times \text{deg}(v)]
\]
\[
= 6|E_1(G)| + 9|E_2(G)|
\]
\[
= 6(8) + 9(24n^2 - 10)
\]
\[
= 216n^2 - 42.
\]

- **Hyper Zagreb index of \((BSN - 1)_{n \times n}\)**

The hyper Zagreb index using Eq. (3) is computed as:

\[
HM(G) = \sum_{uv \in E(G)} [\text{deg}(u) + \text{deg}(v)]^2
\]

\[
HM(G) = \sum_{uv \in E_1(G)} [\text{deg}(u) + \text{deg}(v)]^2
\]
\[
+ \sum_{uv \in E_2(G)} [\text{deg}(u) + \text{deg}(v)]^2
\]
\[
= 5^2|E_1(G)| + 6^2|E_2(G)|
\]
\[
= 25(8) + 36(24n^2 - 10)
\]
\[
= 864n^2 - 160.
\]

- **Multiplicative Zagreb indices of \((BSN - 1)_{n \times n}\)**

The multiplicative Zagreb indices using Eqs. (4), (5) are computed as:

\[
PM_1(G) = \prod_{uv \in E(G)} [\text{deg}(u) + \text{deg}(v)]
\]

\[
PM_1(G) = \prod_{uv \in E_1(G)} [\text{deg}(u) + \text{deg}(v)]
\]
\[
\times \prod_{uv \in E_2(G)} [\text{deg}(u) + \text{deg}(v)]
\]
\[
= 5^8|E_1(G)| \times 6^8|E_2(G)|
\]
\[
= 5^8 \times 6^{(24n^2-10)}.
\]

\[
PM_2(G) = \prod_{uv \in E(G)} [\text{deg}(u) \times \text{deg}(v)]
\]

\[
PM_2(G) = \prod_{uv \in E_1(G)} [\text{deg}(u) \times \text{deg}(v)]
\]
\[
\times \prod_{uv \in E_2(G)} [\text{deg}(u) \times \text{deg}(v)]
\]
\[
= 6^6|E_1(G)| \times 9^6|E_2(G)|
\]
\[
= 6^6 \times 9^{(24n^2-10)}.
\]

- **The first and second Zagreb polynomials of \((BSN - 1)_{n \times n}\)**
Now using Eqs. (6),(7), we have:

\[ M_1(G, x) = \sum_{uv \in E(G)} x^{\deg(u) + \deg(v)} \]

\[ M_1(G, x) = \sum_{uv \in E_1(G)} x^{\deg(u) + \deg(v)} + \sum_{uv \in E_2(G)} x^{\deg(u) + \deg(v)} = \sum_{uv \in E(G)} x^{\deg(u) + \deg(v)} \]

\[ = \sum_{uv \in E_1(G)} x^2 + \sum_{uv \in E_2(G)} x^6 \]

\[ = |E_1(G)|x^2 + |E_2(G)|x^6 \]

\[ = 8x^2 + (24n^2 - 10)x^6. \]

\[ M_2(G, x) = \sum_{uv \in E(G)} x^{\deg(u) - \deg(v)} \]

\[ M_2(G, x) = \sum_{uv \in E_1(G)} x^{\deg(u) - \deg(v)} + \sum_{uv \in E_2(G)} x^{\deg(u) - \deg(v)} = \sum_{uv \in E(G)} x^{\deg(u) - \deg(v)} \]

\[ = \sum_{uv \in E_1(G)} x^6 + \sum_{uv \in E_2(G)} x^9 \]

\[ = |E_1(G)|x^6 + |E_2(G)|x^9 \]

\[ = 8x^6 + (24n^2 - 10)x^9. \]

The redefined first, second, and third Zagreb index of \((BSN - 1)_{n \times n}\)

Now using the edge partition of the block shift network \(G\), and Eq. (8), we have:

\[ ReZG_1(G) = \sum_{uv \in E(G)} \frac{\deg(u) + \deg(v)}{\deg(u) \times \deg(v)} \]

\[ = \sum_{uv \in E_1(G)} \frac{\deg(u) + \deg(v)}{\deg(u) \times \deg(v)} + \sum_{uv \in E_2(G)} \frac{\deg(u) + \deg(v)}{\deg(u) \times \deg(v)} = 8 \left( \frac{2 + 3}{2 \times 3} \right) + (24n^2 - 10) \left( \frac{3 + 3}{3 \times 3} \right) \]

\[ = \frac{48n^2}{3}. \]

By using Eq. (9), the second redefined Zagreb index is computed as below:

\[ ReZG_2(G) = \sum_{uv \in E(G)} \frac{\deg(u) \times \deg(v)}{\deg(u) + \deg(v)} \]

\[ = \sum_{ab \in E_1(G)} \frac{\deg(u) \times \deg(v)}{\deg(u) + \deg(v)} + \sum_{ab \in E_2(G)} \frac{\deg(u) \times \deg(v)}{\deg(u) + \deg(v)} \]

\[ = 7|E_1(G)| + 8|E_2(G)| \]

\[ = 7(12) + 8(32n^2 - 14) \]

\[ = 512n^2 + 12. \]

4 Results for block shift network \((BSN - 2)_{n \times n}\)

The number of vertices and edges in \((BSN - 2)_{n \times n}\) are \(16n^2\) and \(32n^2 - 2\) respectively see Figure 1(b). There are two types of edges in \((BSN - 2)_{n \times n}\) based on degrees of end vertices of each edge. The edge set of \((BSN - 2)_{n \times n}\) can be divided into two partitions based on the degree of end vertices. The first edge partition \(E_1((BSN - 2)_{n \times n})\) contains 12 edges \(uv\), where \(deg(u) = 3\), \(deg(v) = 4\). The second edge partition \(E_2((BSN - 2)_{n \times n})\) contains \(32n^2 - 14\) edges \(uv\), where \(deg(u) = deg(v) = 4\).

The first and second Zagreb indices of \((BSN - 2)_{n \times n}\)

Now using Eqs. (1),(2), we have

\[ M_1(G) = \sum_{uv \in E(G)} [\deg(u) + \deg(v)] \]

\[ = 7|E_1(G)| + 8|E_2(G)| = 7(12) + 8(32n^2 - 14) = 512n^2 + 12. \]
\[ M_2(G) = \sum_{uv \in E(G)} \left[ \deg(u) \times \deg(v) \right] \]
\[ M_2(G) = \sum_{uv \in E_1(G)} \left[ \deg(u) \times \deg(v) \right] \]
\[ + \sum_{uv \in E_2(G)} \left[ \deg(u) \times \deg(v) \right] \]
\[ = 12|E_1(G)| + 16|E_2(G)| \]
\[ = 12(12) + 16(32n^2 - 14) \]
\[ = 512n^2 - 80. \]

- **Hyper Zagreb index of \((BSN - 2)_{n,n}\)**

The hyper Zagreb index using Eq. (3) is computed as:

\[ HM(G) = \sum_{uv \in E(G)} \left[ \deg(u) + \deg(v) \right]^2 \]
\[ HM(G) = \sum_{uv \in E_1(G)} \left[ \deg(u) + \deg(v) \right]^2 \]
\[ + \sum_{uv \in E_2(G)} \left[ \deg(u) + \deg(v) \right]^2 \]
\[ = 7^2|E_1(G)| + 8^2|E_2(G)| \]
\[ = 49(12) + 64(32n^2 - 14) \]
\[ = 2048n^2 - 308. \]

- **Multiplicative Zagreb indices of \((BSN - 2)_{n,n}\)**

The multiplicative Zagreb indices using Eqs. (4), (5) are computed as:

\[ PM_1(G) = \prod_{uv \in E(G)} \left[ \deg(u) + \deg(v) \right] \]
\[ PM_1(G) = \prod_{uv \in E_1(G)} \left[ \deg(u) + \deg(v) \right] \]
\[ \times \prod_{uv \in E_2(G)} \left[ \deg(u) + \deg(v) \right] \]
\[ = 7^2|E_1(G)| \times 8^2|E_2(G)| \]
\[ = 7^{12} \times 8^{32n^2 - 14}. \]

\[ PM_2(G) = \prod_{uv \in E(G)} \left[ \deg(u) \times \deg(v) \right] \]
\[ PM_2(G) = \prod_{uv \in E_1(G)} \left[ \deg(u) \times \deg(v) \right] \]
\[ \times \prod_{uv \in E_2(G)} \left[ \deg(u) \times \deg(v) \right] \]
\[ = 12|E_1(G)| \times 16|E_2(G)| \]
\[ = 12^{12} \times 16^{32n^2 - 14}. \]

- **The first and second Zagreb polynomials of \((BSN - 2)_{n,n}\)**

Now using Eqs. (6), (7), we have

\[ M_1(G, x) = \sum_{uv \in E(G)} x^{\deg(u) + \deg(v)} \]
\[ M_1(G, x) = \sum_{uv \in E_1(G)} x^{\deg(u) + \deg(v)} \]
\[ + \sum_{uv \in E_2(G)} x^{\deg(u) + \deg(v)} \]
\[ = \sum_{uv \in E_1(G)} x^7 + \sum_{uv \in E_2(G)} x^{16} \]
\[ = |E_1(G)|x^7 + |E_2(G)|x^{16} \]
\[ = 12x^7 + (32n^2 - 14)x^{16}. \]

\[ M_2(G, x) = \sum_{uv \in E(G)} x^{\deg(u) \times \deg(v)} \]
\[ M_2(G, x) = \sum_{uv \in E_1(G)} x^{\deg(u) \times \deg(v)} \]
\[ + \sum_{uv \in E_2(G)} x^{\deg(u) \times \deg(v)} \]
\[ = \sum_{uv \in E_1(G)} x^{12} + \sum_{uv \in E_2(G)} x^{16} \]
\[ = |E_1(G)|x^{12} + |E_2(G)|x^{16} \]
\[ = 12x^{12} + (32n^2 - 14)x^{16}. \]

- **The redefined first, second, and third Zagreb index of \((BSN - 2)_{n,n}\)**

Now using Eq. (8) and the edge partition of the block shift network \((BSN - 2)_{n,n}\), we have:

\[ ReZ_1(G) = \sum_{uv \in E(G)} \frac{\deg(u) + \deg(v)}{\deg(u) \times \deg(v)} \]
\[ ReZ_1(G) = \sum_{ab \in E_1(G)} \frac{\deg(u) + \deg(v)}{\deg(u) \times \deg(v)} \]
\[ + \sum_{ab \in E_2(G)} \frac{\deg(u) + \deg(v)}{\deg(u) \times \deg(v)} \]
\[ = 12(\frac{3 + 4}{3 \times 4}) + (32n^2 - 14)((\frac{4 + 4}{4 \times 4}) \]
\[ = 16n^2. \]

By using Eq. (9), the second redefined Zagreb index is computed as below:

\[ ReZ_2(G) = \sum_{uv \in E(G)} \frac{\deg(u) \times \deg(v)}{\deg(u) + \deg(v)} \]
\[ ReZ_2(G) = \sum_{uv \in E_1(G)} \frac{\deg(u) \times \deg(v)}{\deg(u) + \deg(v)} \]
\[ + \sum_{uv \in E_2(G)} \frac{\deg(u) \times \deg(v)}{\deg(u) + \deg(v)} \]
\[ = 12(\frac{3 \times 4}{3 + 4}) + (32n^2 - 14)((\frac{4 \times 4}{4 + 4}) \]
\[ = \frac{144}{7} + 64n^2 - 58. \]
Now by using Eq. (10), the third redefined Zagreb index is computed as:

\[
ReZG_3(G) = \sum_{uv \in E(G)} \left[ \deg(u) \times \deg(v) \right] \\
\times \left[ \deg(u) + \deg(v) \right]
\]

\[
= \sum_{uv \in E_1(G)} \left[ \deg(u) \times \deg(v) \right] \\
\times \left[ \deg(u) + \deg(v) \right]
+ \sum_{uv \in E_2(G)} \left[ \deg(u) \times \deg(v) \right] \\
\times \left[ \deg(u) + \deg(v) \right]
= 12 \left( (3 \times 4) \times (3 + 4) \right)
+ 32n^2 - 14 \left( (4 \times 4) \times (4 + 4) \right)
= 4096n^2 - 2800.
\]

5 Results for hierarchical hypercube network \((HHC - 1)_{n \times n}\)

In this section, we compute certain degree based topological indices of hierarchical interconnection networks see Figure 2(a). The Randić type indices for hierarchical interconnection networks is computed by Haider et. al in [31]. We compute first and second Zagreb indices, hyper Zagreb index \(HM(G)\), first multiple Zagreb index \(PM_1(G)\), second multiple Zagreb index \(PM_2(G)\) and Zagreb polynomials \(M_1(G, x)\), \(M_2(G, x)\) for hierarchical hypercube network \((HHC - 1)_{n \times n}\) and hierarchical hypercube network \((HHC - 2)_{n \times n}\).

The numbers of vertices and edges in \((HHC - 1)_{n \times n}\) are \(16n + 16\) and \(24n + 20\), respectively. There are two types of edges in \((HHC - 1)_{n \times n}\) based on degrees of end vertices of each edge. The edge set of \((HHC - 1)_{n \times n}\) can be divided into two partitions based on the degree of end vertices. The first edge partition \(E_1((HHC - 1)_{n \times n})\) contains 16 edges \(uv\), where \(\deg(u) = 2, \deg(v) = 3\). The second edge partition \(E_2((HHC - 1)_{n \times n})\) contains 24n + 4 edges \(uv\), where \(\deg(u) = \deg(v) = 3\).

- **The first and second Zagreb indices of \((HHC - 1)_{n \times n}\)**

Now using Eqs. (1), (2), we have

\[
M_1((HHC - 1)_{n \times n}) = \sum_{uv \in E(G)} [\deg(u) + \deg(v)]
\]

\[
M_1(G) = \sum_{uv \in E_1(G)} [\deg(u) + \deg(v)]
+ \sum_{uv \in E_2(G)} [\deg(u) + \deg(v)]
= 5|E_1(G)| + 6|E_2(G)|
= 5(16) + 6(24n + 4)
= 216n + 104.
\]

\[
M_2(G) = \sum_{uv \in E(G)} [\deg(u) \times \deg(v)]
\]

\[
M_2(G) = \sum_{uv \in E_1(G)} [\deg(u) \times \deg(v)]
+ \sum_{uv \in E_2(G)} [\deg(u) \times \deg(v)]
= 6|E_1(G)| + 9|E_2(G)|
= 6(16) + 9(24n + 4)
= 216n + 132.
\]

- **Hyper Zagreb index of \((HHC - 1)_{n \times n}\)**

The hyper Zagreb index using Eq. (3) is computed as:

\[
HM(G) = \sum_{uv \in E(G)} [\deg(u) + \deg(v)]^2
\]

\[
HM(G) = \sum_{uv \in E_1(G)} [\deg(u) + \deg(v)]^2
+ \sum_{uv \in E_2(G)} [\deg(u) + \deg(v)]^2
= 5^2|E_1(G)| + 6^2|E_2(G)|
= 25(16) + 36(24n + 4)
= 864n + 544.
\]

- **Multiple Zagreb indices of \((HHC - 1)_{n \times n}\)**

The multiple-zagreb indices using Eqs. (4), (5) are computed as:

\[
PM_1(G) = \prod_{uv \in E(G)} [\deg(u) + \deg(v)]
\]
The first and second Zagreb polynomials of $HHC - 1_{n \times n}$

Now using Eqs. (6), (7), we have:

$$M_1(G, x) = \sum_{uv \in E(G)} x^{\deg(u) + \deg(v)}$$

$$M_1(G, x) = \sum_{uv \in E_1(G)} x^{\deg(u) + \deg(v)}$$

$$+ \sum_{uv \in E_2(G)} x^{\deg(u) + \deg(v)}$$

$$= \sum_{uv \in E_1(G)} x^{\deg(u) + \deg(v)}$$

$$= |E_1(G)| x^5 + |E_2(G)| x^6$$

$$= 16x^5 + (24n + 4)x^6.$$

$$M_2(G, x) = \sum_{uv \in E(G)} x^{\deg(u) + \deg(v)}$$

$$M_2(G, x) = \sum_{uv \in E_1(G)} x^{\deg(u) + \deg(v)}$$

$$+ \sum_{uv \in E_2(G)} x^{\deg(u) + \deg(v)}$$

$$= \sum_{uv \in E_1(G)} x^{\deg(u) + \deg(v)}$$

$$= |E_1(G)| x^6 + |E_2(G)| x^9$$

$$= 16x^6 + (24n + 4)x^9.$$

The redefined first, second, and third Zagreb index of $HHC - 1_{n \times n}$

Now using Eq. (8) and the edge partition of the block shift network $HHC - 1_{n \times n}$, we have:

$$ReZG_1(G) = \sum_{uv \in E(G)} \frac{\deg(u) + \deg(v)}{\deg(u) \times \deg(v)}$$

$$= \sum_{uv \in E_1(G)} \frac{\deg(u) + \deg(v)}{\deg(u) \times \deg(v)}$$

$$+ \sum_{uv \in E_2(G)} \frac{\deg(u) + \deg(v)}{\deg(u) \times \deg(v)}$$

$$= 16 \left( \frac{3 + 2}{3 \times 2} \right) + (24n + 4) \left( \frac{3 + 3}{3 \times 3} \right)$$

$$= \frac{40}{3} + 48n^2 + 8 \frac{3}{3}.$$
divided into two partitions based on the degree of end vertices. The first edge partition $E_1((HHC - 2)_{n,n})$ contains 24 edges $uv$, where $\deg(u) = 3$, $\deg(v) = 4$. The second edge partition $E_2((HHC - 2)_{n,n})$ contains $32n + 4$ edges $uv$, where $\deg(u) = \deg(v) = 4$.

- **The first and second Zagreb indices of $(HHC - 2)_{n,n}$**

  Now using Eqs. (1),(2), we have

  $M_1(G) = \sum_{uv \in E(G)} [\deg(u) + \deg(v)]$

  $M_1(G) = \sum_{uv \in E_1(G)} [\deg(u) + \deg(v)]$

  $+ \sum_{uv \in E_2(G)} [\deg(u) + \deg(v)]$

  $= 7|E_1(G)| + 8|E_2(G)| = 7(24) + 8(32n + 4)$

  $= 512n + 200$.

  $M_2(G) = \sum_{uv \in E(G)} [\deg(u) \times \deg(v)]$

  $M_2(G) = \sum_{uv \in E_1(G)} [\deg(u) \times \deg(v)]$

  $+ \sum_{uv \in E_2(G)} [\deg(u) \times \deg(v)]$

  $= 12|E_1(G)| + 16|E_2(G)|$

  $= 12(24) + 16(32n + 4)$

  $= 512n + 352$.

- **Hyper Zagreb index of $(HHC - 2)_{n,n}$**

  The hyper Zagreb index using Eq. (3) is computed as:

  $HM(G) = \sum_{uv \in E(G)} [\deg(u) + \deg(v)]^2$

  $HM(G) = \sum_{uv \in E_1(G)} [\deg(u) + \deg(v)]^2$

  $+ \sum_{uv \in E_2(G)} [\deg(u) + \deg(v)]^2$

  $= 7^2|E_1(G)| + 8^2|E_2(G)|$

  $= 49(24) + 64(32n + 4)$

  $= 2048n + 1432$.

- **Multiple Zagreb indices of $(HHC - 2)_{n,n}$**

  The Multiple-Zagreb indices using Eqs. (4), (5) are computed as:

  $\nu PM_1(G) = \prod_{uv \in E(G)} [\deg(u) + \deg(v)]$

  $PM_1(G) = \prod_{uv \in E_1(G)} [\deg(u) + \deg(v)]$

  $\times \prod_{uv \in E_1(G)} [\deg(u) + \deg(v)]$

  $= 7^2|E_1(G)| \times 8^2|E_2(G)|$

  $= 7^{24} \times 8(32n + 4)$.

- **The first and second Zagreb polynomials of $(HHC - 2)_{n,n}$**

  Now using Eqs. (6),(7), we have:

  $M_1(G, x) = \sum_{uv \in E(G)} x^{\deg(u) + \deg(v)}$

  $M_1(G, x) = \sum_{uv \in E_1(G)} x^{\deg(u) + \deg(v)}$

  $+ \sum_{uv \in E_2(G)} x^{\deg(u) + \deg(v)}$

  $= \sum_{uv \in E_1(G)} x^7 + \sum_{uv \in E_2(G)} x^8$

  $= |E_1(G)|x^7 + |E_2(G)|x^8$

  $= 24x^7 + (32n + 4)x^8$.

  $M_2(G, x) = \sum_{uv \in E(G)} x^{\deg(u) \times \deg(v)}$

  $M_2(G, x) = \sum_{uv \in E_1(G)} x^{\deg(u) \times \deg(v)}$

  $+ \sum_{uv \in E_2(G)} x^{\deg(u) \times \deg(v)}$

  $= \sum_{uv \in E_1(G)} x^{12} + \sum_{uv \in E_2(G)} x^{16}$

  $= |E_1(G)|x^{12} + |E_2(G)|x^{16}$

  $= 24x^{12} + (32n + 4)x^{16}$.

- **The redefined first, second, and third Zagreb index of $(HHC - 2)_{n,n}$**

  Now using Eq. (8) and the edge partition of the block shift network $(HHC - 2)_{n,n}$, we have:

  $ReZG_1(G) = \sum_{uv \in E(G)} \frac{\deg(u) + \deg(v)}{\deg(u) \times \deg(v)}$

  $PM_1(G) = \prod_{uv \in E_1(G)} [\deg(u) + \deg(v)]$
In this paper we determined first Zagreb index $M_1(G)$, second Zagreb index $M_2(G)$, hyper Zagreb index $HM(G)$, first multiple Zagreb index $PM_1(G)$, second multiple Zagreb index $PM_2(G)$, Zagreb polynomials $M_1(G, x)$ and $M_2(G, x)$, and redefined Zagreb indices for block shift networks and hierarchical hypercube networks. In future, we are interested in designing some new architectures/networks and then studying their topological indices which will be quite helpful in understanding their underlying topologies.

Acknowledgement: The authors are grateful to the anonymous referees for their valuable comments and suggestions that improved this paper.

References


