A formula for all minors of the adjacency matrix and an application

Abstract: We supply a combinatorial description of any minor of the adjacency matrix of a graph. This description is then used to give a formula for the determinant and inverse of the adjacency matrix, $A(G)$, of a graph $G$, whenever $A(G)$ is invertible, where $G$ is formed by replacing the edges of a tree by path bundles.

Keywords: Adjacency Matrix, Linear Subgraphs, Minors, Path Bundle

MSC: 05C50, 05C05, 15A18

1 Introduction

Let $G = (V, E)$ denote a connected simple graph. Unless specified otherwise, we shall take $V = [n] = \{1, \ldots, n\}$ as the vertex set. An edge between two vertices $u$ and $v$ is denoted by $[u, v]$. We write $u \sim v$ to mean that $[u, v]$ is an edge. The adjacency matrix $A(G) = [a_{uv}]$ of the graph $G$, is an $n \times n$ matrix with $a_{uv} = 1$, if $u \sim v$ and $a_{uv} = 0$, otherwise. Note that $a_{uu} = 0$, for all $u \in V$, as the graph is simple.

A subgraph $H$ of $G$ is called an elementary subgraph if each component of $H$ is either an edge ($K_1$) or a cycle of length at least 3. Let $c(H)$ and $c_1(H)$ denote the number of components of $H$ that are cycles and edges, respectively. Then a rephrasing of a result of Harary [5] gives the following statement. For a proof, see either Harary [5] or Bapat [1].

Theorem 1. Let $A$ be the adjacency matrix of a graph $G$ with vertex set $[n]$. Then

$$\det(A) = \sum_H (-1)^{n-c(H) - c_1(H)2}c(H),$$

where the summation runs over all spanning elementary subgraphs $H$ of $G$.

For a tree, the description of the minors of the adjacency matrix has already been supplied by Kim and Shader in [6, Theorem 6]. In this article, we supply the description of the minors of the adjacency matrix for any graph. To start with, let us fix a graph $G$ on $n$ vertices and a positive integer $k$, with $1 \leq k \leq n/2$. Now, let $R = \{r_1, \ldots, r_k\}$ and $C = \{c_1, \ldots, c_k\}$ be two disjoint subsets of $[n]$, where we assume that their elements are written in increasing order, that is, $r_1 < r_2 < \cdots < r_k$ and $c_1 < c_2 < \cdots < c_k$. Also, let $S_k$ denote the symmetric group on $[k]$. Then, for each $\sigma \in S_k$, by an $R$-$C_\sigma$-linear subgraph, we mean a spanning subgraph of $G$ consisting of $k$ vertex disjoint paths $P_i$ from $r_i$ to $c_{\sigma(i)}$, where apart from these paths, the spanning subgraph consists of elementary subgraphs of $G \setminus \{P_1, \ldots, P_k\}$. Let us denote the collection of all $R$-$C_\sigma$ linear subgraphs by $\mathcal{L}_\sigma$. 

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Let \( L \in \mathcal{L}_a \). Let \( P_i \) be the \( r_i \cdot c_{d(i)} \)–Path, for \( i = 1, \ldots, k \). Assume that there are \( r \geq 0 \) components of \( L \) which are edges (not considering the paths) and there are \( t \geq 0 \) components \( C_1, \ldots, C_t \) which are cycles of length at least \( 3 \). Using this information, let us define,

\[
e(L) = (-1)^{\sum ||P_i||} (-1)^{\sum ||G'|| - 2t} = (-1)^{||L||} (-2)^t,
\]

where for any subgraph \( H \) of \( G \), \( ||H|| \) denotes the number of edges in \( H \).

To state and prove our main result, recall that for any two subsets \( R \) and \( S \) of \( [n] \) and any \( n \times n \) matrix \( A \), the notation \( A(R | S) \) represents a submatrix of \( A \), obtained by deleting the rows in \( R \) and the columns in \( S \).

Then, with the definitions and notations as above, we have the following generalization of Theorem 1.

**Theorem 2.** Let \( A \) be the adjacency matrix of a simple graph \( G \) with vertex set \( [n] \) and let \( R = \{ r_1, \ldots, r_k \} \) and \( C = \{ c_1, \ldots, c_k \} \) be disjoint subsets of \( [n] \) written in increasing order. Then

\[
\det A(R | C) = (-1)^{t_1 + \cdots + t_k + c_1 + \cdots + c_k} \sum_{\sigma \in S_k} \left( \text{sgn} \sigma \sum_{L \in \mathcal{L}_a} e(L) \right),
\]

where \( \text{sgn} \sigma \) is the signature of the permutation \( \sigma \).

**Proof.** Let \( \overrightarrow{G} \) be the directed graph obtained from \( G \) by replacing each edge with a pair of oppositely oriented edges. Let \( B \) be the matrix obtained from \( A \) by making the following changes:

a) replace the rows \( r_1, \ldots, r_k \) by the zero vectors,

b) replace the columns \( c_1, \ldots, c_k \) by the zero vectors, and

c) then making the \( (r_i, c_j) \)-entries 1, for \( i = 1, \ldots, k \).

Then \( \det A(R | C) = c(R)c(C) \det B \), where \( c(R) = (-1)^{(r_1 - 1) + \cdots + (r_k - 1)} \) and \( c(C) = (-1)^{c_1 + \cdots + c_k - k} \).

Note that the digraph \( \overrightarrow{H} \) corresponding to the matrix \( B \) is obtained from \( \overrightarrow{G} \) by deleting all the outgoing edges from \( r_i \)'s and all the incoming edges to \( c_j \)'s and then adding a directed edge from \( r_i \) to \( c_j \), for each \( i = 1, \ldots, k \).

Hence, by the coefficient theorem for digraphs [4, Theorem 1.2], we have

\[
\det B = (-1)^n \sum_{\overrightarrow{L} \in \mathcal{L}_a} (-1)^{n(\overrightarrow{L})},
\]

where \( \mathcal{L}_a \) is the set of all linear directed spanning subgraphs \( \overrightarrow{L} \) of \( \overrightarrow{H} \) and \( p(X) \) denotes the number of components of the subgraph \( X \). Observe that components of \( \overrightarrow{L} \) are directed cycles of length at least 2 and hence

\[
\det B = \sum_{\overrightarrow{L} \in \mathcal{L}_a} \prod_{\overrightarrow{C} \text{ is a cycle in } \overrightarrow{L}} (-1)^{||\overrightarrow{C}|| - 1}. \quad (2)
\]

Let us now observe the properties of a directed linear subgraph \( \overrightarrow{L} \) in \( \overrightarrow{H} \) that appears in the above summation. To start with, observe that any cycle in \( \overrightarrow{H} \) that contains the vertex \( r_i \) must also contain the vertex \( c_i \), since there is exactly one directed edge from \( r_i \) to \( c_i \) in \( \overrightarrow{H} \). This observation leads to the following observations as well.

1. First we recall that a component of \( \overrightarrow{L} \) which is a cycle of exactly 2 edges corresponds to an edge in \( G \).

2. Consider a component \( \overrightarrow{C} \) of \( \overrightarrow{L} \) which is a cycle of length \( 3 \) or more. Suppose that \( \overrightarrow{C} \) contains none of the vertices \( r_i \), \( i = 1, \ldots, k \). Let \( C \) be the corresponding cycle in \( G \). Then we would have got the same cycle \( C \) even if we had \( \overrightarrow{C} \) (a cycle that is oppositely oriented to \( \overrightarrow{C} \)) as our component.

3. Consider a component \( \overrightarrow{C} \) of \( \overrightarrow{L} \) which contains \( r_i \). Then this cycle contains the directed edge \([r_i, c_i]\) and it may contain some other edges \([r_i, c_j]\). So, let us assume that \( \overrightarrow{C} \) contains the directed edges \([r_{i_1}, c_{i_1}], \ldots, [r_{i_l}, c_{i_l}]\) (and no more directed edges \([r_i, c_i]\)) in this particular order. Then, in this case, \( \overrightarrow{C} = [r_{i_1}, c_{i_1}, \ldots, r_{i_l}, c_{i_l}, \ldots, r_{i_l}, c_{i_l}, \ldots, r_i, c_i] \) (see Figure 1 for clarity). Notice that \( \overrightarrow{C} \) gives rise to \( s \) vertex disjoint paths, namely,

\[
\overrightarrow{P}_1 = [c_i, \ldots, r_i], \overrightarrow{P}_2 = [c_i, \ldots, r_i], \ldots, \text{ and } \overrightarrow{P}_s = [c_i, \ldots, r_i].
\]
The placements of the directed edges \([r_i, c_i]\) on \(\vec{C}\) naturally defines a permutation of \(i_1, \ldots, i_s\), namely 
\(\mu(\vec{C}) = (i_1, i_2, \ldots, i_s)\), viewed as an element of \(S_k\). Note further that \(\text{sgn} \mu(\vec{C}) = (-1)^{s-1}\). Thus, the contribution of \(\vec{C}\) to \(\det(B)\) is given by 
\[
(-1)^{||\vec{C}||-1} = (-1)^{||P_{i_1}||} \cdots (-1)^{||P_{i_s}||} (-1)^{s-1} = (-1)^{||P_{i_1}|| \cdots ||P_{i_s}||} \text{sgn} \mu(\vec{C}).
\]

Also, note that \(\vec{C}\) corresponds to the vertex disjoint paths \(P_1, P_2, \ldots, P_s\) in \(G\). Since each \(r_i\) is in some directed cycle, we see that the cycles in \(\vec{L}\) give us \(k\) vertex disjoint \(c_i\)-\(r_{\mu(i)}\) paths, where \(\mu = \prod_{C \in \vec{L}} \mu(C)\).

Here we include the possibility of a cycle of length two like \([r_1, c_1, r_1]\), in which case the path under consideration is the directed edge \([c_1, r_1]\).

4. Conversely, given \(k\) vertex disjoint \(c_i\)-\(r_{\mu(i)}\) paths, and some cycles that do not contain any \(r_i\), we put back the directed edges \([r_i, c_i]\) to obtain a class of directed cycles that contain the vertices \(r_i\). Along with the other given cycles we have a directed linear subgraph \(\vec{L}\) of \(H\). Note that this correspondence is unique.

The above observations about the linear directed subgraph \(\vec{L}\) implies the existence of a unique undirected subgraph \(\overrightarrow{L}\) of \(G\) whose components are

1. \(k\) vertex disjoint paths from \(c_i\) to \(r_{\mu(i)}\) (or equivalently from \(r_i\) to \(c_{\sigma(i)}\), taking \(\sigma = \mu^{-1}\)),
2. some \(K_2\)'s not involving any \(r_i\) or \(c_i\), and
3. some cycles of length at least 3 not involving any \(r_i\) or \(c_i\).

Thus, \(L\) is indeed a \(R-C\)-linear subgraph of \(G\).

Conversely, let \(L\) be a \(R-C\)-linear subgraph of \(G\) and suppose that \(L\) consists of the \(k\) vertex disjoint paths \(P_1, \ldots, P_k\), \(r \geq 0\) components that are edges (apart from the paths), and \(t \geq 0\) components \(C_1, \ldots, C_t\) that are cycles of length at least 3. Then, this \(L\) will correspond to exactly \(2^r\) linear directed subgraphs \(\vec{L}_1, \ldots, \vec{L}_{2^r}\) of \(H\) (corresponding to two orientation for each cycle \(C_i\)) in which the common components are

1. the cycles of length 2 corresponding to exactly \(r\) many \(K_2\)'s, and
2. a set of directed cycles that are uniquely determined by the vertex disjoint paths \(P_i\)'s.

Therefore, the contribution to \(\det(B)\) from all these \(\vec{L}_i\)'s can be combined together and rewritten in terms of the linear subgraph \(L\) of \(G\) by

\[
e(L) = (-1)^r(-1)^{|C_1|-1} \cdots (-1)^{|C_t|-1} 2^r(-1)^{|P_i|} \cdots (-1)^{|P_k|} \text{sgn} \sigma
\]

and thus, we obtain

\[
\det(B) = \sum_{\sigma \in S_k} \left( \text{sgn} \sigma \sum_{L \in L_\sigma} e(L) \right).
\]

Remark 3. 

1. It can be observed that the results can be easily generalized for weighted simple graphs by defining \(e(L)\) in Equation (1) appropriately.
2. Let \(R\) and \(C\) be any two subsets of \([n]\) of size \(k\) and let \(R \cap C\) be non-empty. Then the proof of Theorem 2 can be modified as follows:
(a) consider the subgraph $\tilde{G}$ of $G$ that is obtained after deleting the vertices in $R \cap C$ and
(b) use the graph $\tilde{G}$ to find vertex disjoint paths $P_1, \ldots, P_s$, where $s$ is the cardinality of the set $R \setminus (R \cap C)$.

Hence, the condition in Theorem 2 that the sets $R$ and $C$ are disjoint can be relaxed. With this observation, Theorem 2 is similar to ‘all minors matrix tree theorem’ of Chaiken [3].

2 Trees with path bundles as edges

Let $T$ be a tree on vertices $1, \ldots, n$ and $P = [1, 2, \ldots, k]$ be a path. By $T_{k,t}$, we denote the graph obtained by replacing each edge of $T$ by a path bundle of length $k + 1$ which has $t$ paths (see Figure 2). Then, using Theorem 1, we have the following result.

![Fig. 2. Tree and its Path Bundle](image)

**Lemma 4.** Let $T$ be a tree having a perfect matching and let $k$ be an even non-negative integer. Then $\det T_{k,t} = t^n(-1)^{\frac{m}{2}}$.

**Proof.** We prove the result by induction on the length of the tree $T$. So, consider the graph $K_2$, a tree on 2 vertices. Then the graph $T_{k,t}$ corresponds to a path bundle, say $P_{k,t}$, with $k = 2m$, an even positive integer (see Figure 3).

![Fig. 3. $P_{k,3}$](image)

Note that the graph $P_{k,t}$ can be decomposed into elementary linear subgraphs as follows:

1. If the decomposition has a cycle then there are $\frac{t(t-1)}{2}$ ways of choosing the cycle $C_{2k,2}$. In this case, the number of elementary linear subgraphs that are $K_2$’s are $(t - 2)m$.
2. If the decomposition has no cycle then each elementary linear subgraph consist only of $K_2$’s. In this case, there are $t$ possible ways of choosing the complete path $P_{k,2}$ to get a set of $K_2$’s that contains the vertices 1 and 2. The number of $K_2$’s corresponding to $P_{k,2}$ equals $m + 1$. The remaining portion of $P_{k,t}$ consists of $t - 1$ disjoint paths on $k$ vertices and thus the number of $K_2$’s corresponding to the disjoint paths equals $(t - 1)m$. Hence, in this case, the total number of $K_2$’s is $tm + 1$. 
Thus, using Theorem 1 and the condition that \( k \) is even, one has

\[
\det(T_{k,t}) = \det(P_{k,t}) = \frac{t(t-1)}{2} 2(-1)^{kt+1-(r-2)m} + t(-1)^{kt-(m+1)} = t^2(-1)^m + 1.
\]

Hence we see that the statement is valid for \( n = 2 \).

Now, consider a non-singular tree \( T \) on \( n \) vertices with \( n \geq 4 \). Since \( T \) has a perfect matching, let us choose and fix a non-matching edge \( e \). Then the graph \( T - e \) decomposes into two trees, say \( T' \) and \( T'' \) on \( n_1 \) and \( n_2 \) vertices, respectively. Also, note that both the trees have perfect matchings and \( n_1 + n_2 = n \). Furthermore, any linear spanning subgraph of \( T_{k,t} \) consists of a linear spanning subgraph of \( T_{k,t}' \), a linear spanning subgraph of \( T_{k,t}'' \), and \( \frac{k}{2} \) many \( K_2 \)'s. Hence, we can apply the induction hypothesis to get,

\[
\det T_{k,t} = \det T_{k,t}' \det T_{k,t}'' (-1)^\frac{k}{2} = t^n(1(-1)^{\frac{k_1}{2}} t^{n_1}(-1)^{\frac{k_2}{2}} (-1)^\frac{k}{2} = t^n(-1)^{\frac{n-k}{2}}.
\]

Hence, by the principle of mathematical induction, we have the required result. \(\square\)

To proceed further, let us assume that the vertices of the tree \( T \) are already labeled. We use this information to label the vertices of \( T_{k,t} \) in the following way:

For a vertex \( i \) of \( T \), the corresponding vertex of \( T_{k,t} \) will also be labeled \( i \). We list the edges \([i, j]\), for \( i < j \), of \( T \) in the lexicographic order. For example, if the tree has \([1, 4], [1, 5]\) and \([2, 3]\) as edges then the edge \([1, 4]\) will be listed prior to the edges \([1, 5]\) and \([2, 3]\). Then the vertices in the path bundle that corresponds to the \( r \)th edge, say \( e_r = [i, j] \), is labeled as shown in Figure 4.

![Fig. 4. Labeling of a path bundle of \( T_{k,t} \)](image)

We use the above labeling and Lemma 4 to give a necessary and sufficient condition for the graph \( T_{k,t} \) to be non-singular.

**Theorem 5.** The graph \( T_{k,t} \) is non-singular if and only if \( k \) is even and \( T \) has a perfect matching.

**Proof.** If \( k \) is even and \( T \) has a perfect matching then using Lemma 4, we see that \( \det(T_{k,t}) \neq 0 \). Hence, \( T_{k,t} \) is non-singular.

Now, suppose that \( T_{k,t} \) is non-singular. Since each cycle is even, \( T_{k,t} \) is bipartite. Thus \( T_{k,t} \) has a perfect matching, say \( M \). For each vertex \( i \) of \( T \), look at the vertex \( i \) of \( T_{k,t} \). It is matched to a vertex in a path bundle. The edge of \( T \) corresponding to that path bundle gives us an edge of a perfect matching of \( T \). Thus \( T \) is non-singular.

Suppose that \( k \) is odd and without loss of generality, let \( 1 \) be a pendant vertex of \( T \). Then, we inductively define a vector \( x \) as follows:

1. \( x(1) = 1 \).
2. Let \([i, j, k]\) be a path of length 2. Then depending on whether \( x(i) = 1 \) or \(-1\), define \( x(j) = 0 \) and \( x(k) = -x(i) \).

Since \( k \) is odd, it can be verified that \( x \) is well defined and indeed \( x \) is an eigenvector of \( T_{k,t} \) corresponding to the eigenvalue 0. Hence, \( k \) cannot be odd.
Thus, the proof of the theorem is complete. □

**Remark 6.** The above result can be generalized as follows:

Let \( T \) be a tree with vertex set \([n]\) and let \( P = [1, 2, \ldots, k] \) be a path. Also let \( t_1, \ldots, t_{n-1} \) be positive integers and consider the vector \( \mathbf{t}_n = [t_1, t_2, \ldots, t_{n-1}] \). By \( T_{k,t_n} \), we denote the graph obtained by replacing the edge \( e_i \) of \( T \) by a path bundle of length \( k + 1 \) which has \( t_i \) paths. Then the following results hold.

1. \( T_{k,t_n} \) is non-singular if and only if \( k \) is even and \( T \) has a perfect matching.
2. Let \( T_{k,t_n} \) be non-singular with the tree \( T \) having its matching edges as \( e_1, e_3, \ldots, e_{n-1} \) and the remaining as its non-matching edges. Then \( \det T_{k,t_n} = (t_1t_3 \ldots t_{n-1})^2 (-1)^{\frac{1}{2}k!k^{n-1}}. \)

In the remaining part of this manuscript, we will use Theorem 2 to obtain the inverse of the adjacency matrix of the graph \( T_{k,t_n} \), whenever \( T_{k,t_n} \) is non-singular. To do so, we start with the following definition.

A path bundle of \( T_{k,t_n} \) is called a matching (non-matching) path bundle if the corresponding edge in \( T \) is a matching (non-matching) edge.

Let \( T \) be a non-singular tree with a perfect matching \( M \) and let \( k \) be an even non-negative integer. For any two vertices \( r \) and \( s \) of \( T_{k,t_n} \), consider a linear subgraph \( L \subseteq \mathcal{L}_{rs} \), the class of all \( r \)-s-linear subgraphs (recall that these are spanning subgraphs by our definition). Let \( P_{rs} \) be the path from \( r \) to \( s \) in \( L \). Then observe that the condition that all the cycles in \( L \) are even implies that the paths \( P_{rs} \) for different choices of \( L \subseteq \mathcal{L}_{rs} \) are either all even or all odd. With this observation, we state and proof our next result which gives us a way to compute the minors of the adjacency matrix that are non-zero.

**Lemma 7.** Let \( T \) be a non-singular tree with a perfect matching \( M \) and let \( k \) be an even non-negative integer. Let \( r \) and \( s \) be any two vertices of \( T_{k,t_n} \). If \( P \) is an \( r \)-s path containing exactly one vertex of a matching path bundle then there is no \( r \)-s-linear subgraphs of \( T_{k,t_n} \) with \( P_{rs} = P \).

**Proof.** Since \( P \) contains exactly one vertex, say \( u \), of a matching path bundle, it must be a vertex of \( T \). Let \( [u, v] \) be the corresponding matching edge of \( T \). Then the component of \( T_{k,t_n} \) at \( u \) that contains \( v \) has an odd number of vertices. Hence, there is no \( r \)-s-linear subgraphs of \( T_{k,t_n} \).

In view of Lemma 7, we only need to consider those \( r \)-s-linear subgraphs in which the following condition holds: whenever \( P_{rs} \) contains a vertex from a matching path bundle, it must also contain an edge of that path bundle.

Before coming to the main result of this section, we define the notion of a related/ unrelated and interior path bundle and make a few observations. Fix a path \( P \) of \( T_{k,t_n} \). Then

1. a path bundle \( Q \) is said to be related to \( P \) if both \( P \) and \( Q \) have at least one edge in common.
2. a path bundle \( Q \) is said to be an interior path bundle if both the end vertices of \( Q \) are interior vertices of \( P \).

Then, using the ideas involved in the first part of the proof of Lemma 4, one obtains the following observations (see Figure 5 for a better understanding of the idea).

![Fig. 5. Elementary linear subgraphs in Matching and Non-matching path bundles](Image)
Observation 8. Let $T_{k,l}$ be a non-singular path bundle and let $r, s$ be two vertices of $T_{k,l}$. The following statements hold true.

1. In the computation of $\det A(r|s)$ as a product of factors, the contribution from an unrelated matching path bundle equals $t^2(-1)^{\frac{k-1}{2}+1}$.

Proof. Since it is a matching path bundle, the end vertices of the path bundle need to appear in each matching. Thus, the proof is exactly the same as in the proof of Lemma 4 (see the case $n = 2$).

2. In the computation of $\det A(r|s)$ as a product of factors, the contribution from an unrelated non-matching path bundle equals $(-1)^{\frac{k}{2}}$.

Proof. Since it is not a matching path bundle, the end vertices of the path bundle cannot appear in any matching. So, we have exactly $t$ vertex disjoint paths of length $P_k$.

Given two vertices $r$ and $s$ in two distinct path bundles of $T_{k,l}$, by $P(r, s)$, we denote the shortest $r$-$s$-path in $T_{k,l}$. Let $r_1, \ldots, r_m = s_1$ be the points of $T$ that are on this path, where we assume that $r_1$ is closest to $r$ and $r_m = s_1$ is closest to $s$. Thus, such a path has the form $[r \rightarrow r_1 \rightarrow r_2 \rightarrow \cdots \rightarrow r_m = s_1 \rightarrow s]$. In this case, observe that for each $i = 1, \ldots, m - 1$, the path bundles that have both $r_i, r_{i+1}$ as end vertices are the interior path bundles.

With the definitions and notations as above, we have the following theorem. The statement itself consists of 5 different cases and the proof of one of the cases requires 4 sub-cases and hence is a bit lengthy.

Theorem 9. Let $T_{k,l}$ be a non-singular path bundle and let $r, s$ be two vertices of $T_{k,l}$. Then

1. $\det(A(r|s)) = (-1)^{r-s}t^{m-1}(-1)^{\frac{d(r,r_1)d(r_1,s_1)+d(s_1,s)-m-2}{2}}$, whenever $r$ and $s$ belong to two distinct matching path bundles.
2. $\det(A(r|s)) = (-1)^{r-s}t^{m-1}(-1)^{\frac{(r-1)d(r,r_1)d(r_1,s_1)+d(s_1,s)-n}{2}}$, whenever $r$ and $s$ belong to the same matching path bundle.
3. $\det(A(r|s)) = (-1)^{r-s}t^{m-1}(-1)^{\frac{(r-1)d(r,r_1)d(r_1,s_1)+d(s_1,s)-m}{2}}$, whenever $r$ and $s$ belong to two distinct non-matching path bundles.
4. $\det(A(r|s)) = (-1)^{r-s}t^{m-1}(-1)^{\frac{(r-1)d(r,r_1)d(r_1,s_1)+d(s_1,s)}{2}}$, whenever $r$ and $s$ belong to the same non-matching path bundle.
5. $\det(A(r|s)) = (-1)^{r-s}t^{m-1}(-1)^{\frac{d(r,r_1)d(r_1,s_1)+d(s_1,s)-m-2}{2}}$, whenever $r$ belongs to a matching path bundle and $s$ belong a non-matching path bundle.

Proof. Since $T_{k,l}$ is non-singular, by Theorem 5, the integer $k$ is even. Let us now look at the proof of different parts.

Proof of Part 1: Let $r$ and $s$ be in two distinct matching path bundles and let $P(r, s)$ be the corresponding path. Since, we are looking at matching--non-matching--alternating path, $m$ is even. Therefore, in this case, there are $\frac{m+2}{2}$ related matching bundles and $\frac{m}{2}$ related non-matching bundles. Also, there are $t$ choices for an $r$-$s$-path in each such bundle and each of these paths contribute $(-1)^{\frac{k-1}{2}+1}$ to $\det(A(r|s))$. Hence, the total contribution from interior path bundles is $t(-1)^{\frac{k-1}{2}+1}$.

Now, note that both the distances $d(r, r_1)$ and $d(s_1, s)$ in $T_{k,l}$ must be odd, or else, we cannot have a linear subgraph $L$ having the vertices $r_1$ and $s_1$ appearing in the matching bundle. Consider the path bundle that contains $r$. An $r$-$r_1$-path has length $2l_1 + 1$ or $2(k - l_1) + 1$, for some non-negative integer $l_1$. So, its contribution towards $\det A(r|s)$ is $(-1)^{\frac{k}{2}+l_1+1} = (-1)^{\frac{k}{2}+k-l_1+1}$, as $k$ is even. As there are $t$ many $r$-$r_1$-paths in that bundle, the contribution of the bundle to $\det A(r|s)$ is $t(-1)^{\frac{k}{2}+k-l_1+1}$. Or equivalently, the contribution is $t(-1)^{\frac{d(r,r_1)d(r_1,s_1)+d(s_1,s)-m-2}{2}}$. Similarly, the contribution from the path bundle that contains $s$ is $t(-1)^{\frac{d(r,r_1)d(r_1,s_1)+d(s_1,s)-n}{2}}$. 
Thus, the contribution of the related path bundles is $t^{m+1}(-1)^{\frac{(t-1)(t-3)(t-5)}{2}}$ and therefore, using Observation 8, we have

\[
\det A(r|s) = (-1)^{rs}t^{m+1}\left(\frac{(t-1)(t-3)(t-5)}{2}\right)^{\frac{(t-1)(t-3)(t-5)}{2}}t^{n-2}(-1)^{\frac{n-2}{2}}\frac{n-2}{2}.
\]

**Proof of Part 2:** Let $r$ and $s$ be in the same matching path bundle and let $r_1$ and $s_1$ be the end vertices of this path bundle. Also, let $r_1$ be the vertex for which the shortest path from $r$ to $r_1$ does not pass through $s$ (here $r$ could be $r_1$ and $s$ could be $s_1$).

Before proceeding further, note that whenever $d(r, r_1)$ and $d(s, s_1)$ have different parity, then none of the $r\cdot s$-path will appear in a spanning $r\cdot s$-sublinear subgraph. Hence, $\det(A(r|s))$ will be non-zero only if $d(r, r_1)$ and $d(s, s_1)$ have the same parity. We divide the proof into 4 subcases depending on whether $d(r, r_1)$ is even or odd and whether $r$ and $s$ lie on the same path.

**Case I:** Let $r$ and $s$ lie on the same path and let $d(r, r_1)$ be even. The contribution from the linear subgraph $L$ with the shortest $r\cdot s$-path is $(-1)^{\frac{(t+1)(t-1)(t-3)}{2}}$. There are $t-1$ more longer paths. The contributions from $L$ having those paths is $(-1)^{\frac{(t-1)(t-3)(t-5)}{2}}$. Hence, the total contribution from the related path bundle equals $t(-1)^{\frac{(t-1)(t-3)(t-5)}{2}}$. Thus, using Observation 8, we get

\[
\det A(r|s) = (-1)^{rs}t(-1)^{\frac{(t-1)(t-3)(t-5)}{2}}(-1)^{\frac{n-2}{2}}t^{n-2}(-1)^{\frac{n-2}{2}}\frac{n-2}{2}.
\]

**Case II:** Let $r$ and $s$ lie on different paths and let $d(r, r_1)$ be even. Notice that the paths of the form $r \to s_1 \to r_1 \to s$ need not be considered as it creates a component (the component consisting of the shortest path from $r$ to $r_1$) consisting of odd number of vertices. The other $r\cdot s$-paths with their number and contributions are as follows:

(i) $r \to s_1 \to s$ : Exactly one path and contributes $(-1)^{\frac{(t-1)(t-3)(t-5)}{2}}$.
(ii) $r \to r_1 \to s$ : Exactly one path and contributes $(-1)^{\frac{(t-1)(t-3)(t-5)}{2}}$.
(iii) $r \to r_1 \to s$ : Exactly $t-2$ paths and they contribute $t(-1)^{\frac{(t-1)(t-3)(t-5)}{2}}$.

Hence, the total contribution from this path bundle is $t(-1)^{\frac{(t-1)(t-3)(t-5)}{2}}$. Thus, using Observation 8, we get

\[
\det A(r|s) = (-1)^{rs}t(-1)^{\frac{(t-1)(t-3)(t-5)}{2}}(-1)^{\frac{n-2}{2}}t^{n-2}(-1)^{\frac{n-2}{2}}\frac{n-2}{2}.
\]

**Case III:** Let $r$ and $s$ lie on the same path and let $d(r, r_1)$ be odd. In this case, it can be easily verified that the shortest path contributes $(-1)^{\frac{(t-1)(t-3)(t-5)}{2}}(t-1)^2$ and the long paths, that are $(t-1)$ in number, contribute $(-1)^{\frac{(t-1)(t-3)(t-5)}{2}}(t-1)^2$. Hence, the total contribution to $\det A(r|s)$ from this path bundle is a factor of $(t-1)^{\frac{(t-1)(t-3)(t-5)}{2}}$. Thus, using Observation 8, we get

\[
\det A(r|s) = (-1)^{rs}(t-1)t(-1)^{\frac{(t-1)(t-3)(t-5)}{2}}(-1)^{\frac{n-2}{2}}t^{n-2}(-1)^{\frac{n-2}{2}}\frac{n-2}{2}.
\]

**Case IV:** Let $r$ and $s$ lie on different paths and let $d(r, r_1)$ be odd. As in Case II, one cannot consider the paths of the form $r \to r_1 \to s_1 \to s$. The other paths with their number and contribution are as follows:

(i) $r \to s_1 \to s$ : Exactly one path and contributes $(-1)^{\frac{(t-1)(t-3)(t-5)}{2}}$.
(ii) $r \to r_1 \to s$ : Exactly one path and contributes $(-1)^{\frac{(t-1)(t-3)(t-5)}{2}}$.
(iii) $r \to s_1 \to r_1 \to s$ : Exactly $t-2$ paths and they contribute $t(-1)^{\frac{(t-1)(t-3)(t-5)}{2}}$. 


Thus, the total contribution to $\det A(r|s)$ from this path bundle is a factor of $t(-1)^{\frac{d(d(r,s)+d(s,r))}{2}}$.

Hence, using Observation 8, we get

$$\det A(r|s) = (-1)^{r+s}(t(-1)^{\frac{d(r,s)+d(s,r)}{2}})^{\frac{n}{2}} t^{m-2} (-1)^{\frac{d+k}{2}} \frac{n}{2}.$$

**Proof of Part 3:** Let $r$ and $s$ be in two distinct non-matching path bundles and let $P(r, s)$ be the corresponding path. Since, we are looking at non-matching–matching–non-matching–alternating path, $m$ is even. Therefore, in this case, there are $\frac{n}{2}$ related matching bundles and $\frac{m-2}{2}$ related non-matching bundles. As above, it can be verified that the total contribution from interior path bundle is $t(-1)^{\frac{d(d(r,s)+d(s,r))}{2}}$.

Note that both the distances $d(r, r_1)$ and $d(s, s_1)$ in $T_{k,s}$ must be even, or else there does not exist a linear subgraph $L$ having the vertices $r_1$ and $s_1$ in respective non-matching path bundle. Now, consider the path bundle that contains $r$. The only $r$-$r_1$ path has length $2l_1$, for some non-negative integer $l_1$. Hence, its contribution towards $\det A(r|s)$ is $(-1)^{\frac{d(d(r,s)+d(s,r))}{2}}$. In other words, the contribution is $(-1)^{\frac{d(d(r,s)+d(s,r))}{2}}$.

Similarly, the contribution from the path bundle that contains $s$ is $(-1)^{\frac{d(d(r,s)+d(s,r))}{2}}$. Thus, the contribution of the related path bundles is $t^{2l_1}(-1)^{\frac{d(d(r,s)+d(s,r))}{2}}$ and using Observation 8, one has

$$\det A(r|s) = (-1)^{r+s}t^{m-1}(-1)^{\frac{d(r,s)+d(s,r)}{2}} \frac{n+m}{2} t^n(-1)^{\frac{d+k}{2}} \frac{n}{2}.$$

**Proof of Part 4:** Let $r$ and $s$ be in the same non-matching path bundle and let $r_1$ and $s_1$ be the end vertices of this path bundle. Also, let $r_1$ be the vertex for which the shortest path from $r$ to $r_1$ does not pass through $s$ (here again $r$ could be $r_1$ and $s$ could be $s_1$). Note that any $r$-$s$ path that passes through $r_1$ or $s_1$ cannot appear in a spanning $r$-$s$ sublinear subgraph as $r_1$ and $s_1$ are supposed to appear in the matching path bundle. Hence, we must have $r$ and $s$ on the same path of the path bundle and the distances, $d(r_1, r)$ and $d(s_1, s)$, must be odd. Hence, the total contribution of this path bundle towards $\det A(r|s)$ is a factor of $(-1)^{\frac{d(d(r,s)+d(s,r))}{2}}$.

Thus, using Observation 8, we have

$$\det A(r|s) = (-1)^{r+s}(-1)^{\frac{d(r,s)+d(s,r)}{2}} \frac{n+m}{2} t^n(-1)^{\frac{d+k}{2}} \frac{n}{2}.$$

**Proof of Part 5:** Let $r$ be in a matching path bundle and $s$ be in a non-matching path bundle and let $P(r, s)$ be the corresponding path. Notice that it is a matching–non-matching–alternating path and hence $m$ is odd. Also, note that $s_1$ may be same as $r_1$ and in this case $d(r, r_1)$ must be odd and $d(s, s_1)$ must be even. Therefore, there are $\frac{m-1}{2}$ related matching bundles and $\frac{m+1}{2}$ related non-matching bundles. Again, it can be verified that the total contribution from interior path bundle is $t(-1)^{\frac{d(d(r,s)+d(s,r))}{2}}$.

The contribution of the bundle that contains $r$ to $\det A(r|s)$ is a factor of $t(-1)^{\frac{d(d(r,s)+d(s,r))}{2}}$. Similarly, the contribution from the path bundle that contains $s$ is a factor of $(-1)^{\frac{d(d(r,s)+d(s,r))}{2}}$. Hence, the contribution of the related path bundles is $t^{m-1}(-1)^{\frac{d(r,s)+d(s,r)}{2}}$ and using Observation 8, we have

$$\det A(r|s) = (-1)^{r+s}t^{m-1}(-1)^{\frac{d(r,s)+d(s,r)}{2}} \frac{n+m-1}{2} t^n(-1)^{\frac{d+k}{2}} \frac{n}{2}.$$

This completes the proof of Part 5 and hence the proof of the theorem. \[\square\]

**Acknowledgement:** R. B. Bapat gratefully acknowledges the support of the J C Bose Fellowship, Department of Science and Technology, Government of India. The other authors take this opportunity to thank the Indian Statistical Institute Delhi for their hospitality from June 05, 2013 to June 20, 2013.
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