Explicit formulas for the constituent matrices. Application to the matrix functions

Abstract: We present a constructive procedure for establishing explicit formulas of the constituents matrices. Our approach is based on the tools and techniques from the theory of generalized Fibonacci sequences. Some connections with other results are supplied. Furthermore, we manage to provide tractable expressions for the matrix functions, and for illustration purposes we establish compact formulas for both the matrix logarithm and the matrix $p$th root. Some examples are also provided.

Keywords: Generalized Fibonacci sequences; Binet formula; Constituents matrices; matrix function; Matrix logarithm; Matrix $p$th root

MSC: Primary 15A16, 11B99; Secondary 65F60

1 Introduction

Matrix functions arise in various fields of mathematics and applied sciences. Since the 19th century several definitions of a matrix function have been proposed in the literature, and it has already given the conditions under which each definition was applicable. In [18] Reinehart focused on the connections between eight definitions of the matrix function.

Throughout these definitions, arise the natural question on the extension of the concept of a (non-polynomial) complex functions to matrix functions. Undoubtedly, the matrix fulfilled this extension must obey to some additional conditions. For more details on this subject we can see [10, 12, 13, 24], and references therein. As a first step, we recall the following definition of the matrix function, which is our main trigger in this study.

Definition 1.1. (Schwerdtfeger, 1938). Let $A$ be a complex square matrix of order $r$, with distinct eigenvalues $\lambda_1, ..., \lambda_s$ with multiplicities $m_1, ..., m_s$ (respectively). Let $f$ be a complex function defined on $\{\lambda_1, ..., \lambda_s\}$ (the spectrum of $A$). The matrix function $f(A)$ is defined by,

$$ f(A) = \sum_{i=1}^{s} A_i \sum_{j=0}^{m_i-1} \frac{f^{(j)}(\lambda_i)}{j!} (A - \lambda_i I)^j = \sum_{i=1}^{s} \sum_{j=0}^{m_i-1} f^{(j)}(\lambda_i) Z_{ij}, $$

where the $A_i$ are called the Frobenius covariants and the $Z_{ij}$ the constituents matrices ($i = 1, ..., s; j = 0, 1, ..., m_i - 1$).

*Corresponding Author: M. Rachidi: Associate researcher with "Equip of DEFA", Département de Mathématiques et Informatique, Faculté des Sciences, Université Moulay Ismail, B.P. 4010, Beni M’hamed, Meknès, Morocco, E-mail: mu.rachidi@hotmail.fr, mu.rachidi@gmail.com

R. Ben Taher: Département de Mathématiques et Informatique, Faculté des Sciences, Université Moulay Ismail, B.P. 4010, Beni M’hamed, Meknès, Morocco, E-mail: bentaher89@hotmail.fr

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The matrices $Z_{ij} \ (i = 1, \ldots, s; \ j = 1, \ldots, m_i - 1)$ are also known as the *components matrices of the matrix $A$. These components matrices depend only on the matrix $A$ and are linearly independent (see [9, 12, 13, 25] for example). Properties of these matrices are largely explored and used in the literature for studying matrix functions (see for instance [9, 10, 12, 13, 19, 21, 22, 25] and references therein). In general, the Frobenius covariants and the constituents matrices are obtained using some constructive processes.

The main goal of this paper is to propose compact formulas for computing the Frobenius covariants and the constituents matrices, also we concentrate ourselves to release some interesting consequences upon some current matrix functions. As a first step, we use tools and techniques from the properties of generalized Fibonacci sequences to determine explicitly the constituents matrices. As a matter of fact, we assemble some further results on the corresponding dynamic solution of the linear homogeneous difference equations, and we show its closed link to the Binet formula of the generalized Fibonacci sequences. More than that, we examine the partial fractions decomposition and we succeed to have an explicit formulation of the dynamic solution. Once that happens, we obtain formulas for the basic Hermite interpolation polynomials as auxiliary result. Thereby, we get a direct expression for the constituent matrices $Z_{ij} \ (i = 1, \ldots, s; \ j = 1, \ldots, m_i - 1)$. Using this material, we arrive to establish tractable formulas of the matrix function under mild additional hypothesis. The constituent matrices were the subject of several studies, where various methods have been developed. That is, similar approaches to ours, have been employed by several authors. More precisely, we explore the relationships between our results with two current methods, one established in [23, 25] by the means of the divided differences approach, and the other based on the techniques involving determination of the partial fraction expansion coefficients, given in [9].

Among the fallout of our investigation, we provide new results on the matrix logarithm and the matrix $\text{pth}$ root of a given matrix $A$, when these two functions are defined on the spectrum of $A$.

The material of this paper is organized as follows. In Section 2, we present some basic results about dynamical solution of an ordinary differential equation emanated from the properties of generalized Fibonacci sequences. We use those results for giving rise to a new proof leading to express under another form the dynamic solution of the scalar homogeneous linear recurrence equations. Over and above, we consider the relationship between the Binet formula for the generalized Fibonacci sequences and the dynamic solution of the scalar homogeneous linear recurrence equations, which leads to release an explicit expression of the Binet formula. To gather and to connect those results to the partial fractions decomposition, we obtain a tractable expression for the constituent matrices of a matrix $A$. In Section 3, the explicit formulas of these matrices allow us to provide compact formulas for some matrix functions under mild additional hypothesis. The connections with the approaches of L. Verde-Star in [23, 25] and F.C. Chang in [9] are broached. Yet, we discuss two applications, the principal matrix logarithm and the principal matrix $\text{pth}$ root, and give new results. Some illustrative examples are also provided.

## 2 Compact formula for constituent matrices via the generalized Fibonacci sequences

### 2.1 Partial fractions decomposition through the generalized Fibonacci sequences

The $r$-generalized Fibonacci sequences are the scalar homogeneous linear recurrence equations of order $r \geq 2$ defined as follows,

$$v_{n+1} = a_0 v_n + \cdots + a_{r-1} v_{n-r+1}, \quad \text{for } n \geq r - 1,$$

where $a_0, \cdots, a_{r-1}, v_0, \cdots, v_{r-1}$ are specified as the coefficients and the initial values (respectively). It was established in [4, 5, 17] that the combinatorial powers of the sequence (1) is given as follows,

$$v_n = \rho(n, r) w_0 + \rho(n - 1, r) w_1 + \cdots + \rho(n - r + 1, r) w_{r-1}, \quad \text{for } n \geq r,$$
where \( w_s = a_{r-1}v_s + \cdots + a_vv_{r-1} \) for \( s = 0, 1, \ldots, r-1 \) and \( \rho(r, r) = 1, \rho(n, r) = 0 \) for \( n < r \) and

\[
\rho(n, r) = \sum_{k_0+2k_1+\cdots+rk_{r-1}=n-r} \frac{(k_0 + \cdots + k_{r-1})!}{k_0! \cdots k_{r-1}!} a_0^{k_0} \cdots a_{r-1}^{k_{r-1}}, \quad \text{for } n \geq r. \tag{2}
\]

On the other hand, it was established in [7] that for every \( r \times r \) square matrix \( A \) of complex entries, with characteristic polynomial \( P_A(z) = z^r - a_0z^{r-1} - \cdots - a_{r-1} \), the exponential \( e^{tA} \) \((t \in \mathbb{R})\) is given by,

\[
e^{tA} = \phi(r-1)(t)A_0 + \phi(r-2)(t)A_1 + \cdots + \phi(t)A_{r-1},
\]

where \( A_0 = I, A_i = A^i - a_0A^{i-1} - \cdots - a_{i-1}I \), for every \( i = 1, \ldots, r-1 \), and \( \phi(k)(t) \) is the \( k \)-derivative of the following function,

\[
\phi(t) = \sum_{n=0}^{\infty} \frac{\rho(n, r, r)}{(n+r-1)!} t^{n+r-1},
\]

such that the \( \rho(n+r, r) \) are given by (2). It was shown that \( \phi(t) \) satisfies the following ordinary differential equation \( y^{(r)}(t) = a_0y^{(r-1)}(t) + a_1y^{(r-2)}(t) + \cdots + a_{r-1}y(t) \). Moreover, using Expression (3), we establish directly that the \( k \)-derivatives of the function \( \phi(t) \) satisfy \( \phi^{(k)}(0) = 0 \) for \( k = 0, 1, \ldots, r-2 \) and \( \phi^{(r-1)}(0) = 1 \). Hence, the function \( \phi(t) \) is solely the so-called dynamical solution of the preceding differential equation (see [3, 6, 7]).

Suppose that \( \lambda_1, \ldots, \lambda_s \) are the distinct roots of the characteristic polynomial \( P_A(z) = z^r - a_0z^{r-1} - \cdots - a_{r-1} \), of multiplicities \( m_1, \ldots, m_s \) (respectively). It is well known that the dynamical solution \( \phi(t) \) can be expressed under the form,

\[
\phi(t) = R_1(t)e^{\lambda_1} + R_2(t)e^{\lambda_2} + \cdots + R_s(t)e^{\lambda_s},
\]

where \( R_j(t) \) \((1 \leq j \leq s)\) are polynomial functions of degree \( m_j - 1 \). More precisely, explicit formulas of the polynomials \( R_j(t) \) \((1 \leq j \leq s)\) are derived from the initial conditions, by solving the following linear system of equations,

\[
\phi(0) = \phi'(0) = \cdots = \phi^{(r-2)}(0) = 0 \quad \text{and} \quad \phi^{(r-1)}(0) = 1.
\]

It was shown in [7] that the polynomials \( R_j(t) \) \((1 \leq j \leq s)\) are given by \( R_j(t) = \sum_{k=0}^{m_j-1} \gamma_{k}^{(j)}(\lambda_1, \ldots, \lambda_s) t^k \) such that

\[
\gamma_{k}^{(j)}(\lambda_1, \ldots, \lambda_s) = (-1)^{r-m_j} \sum_{n_j=0}^{m_j-k-1} \frac{1}{(\lambda_j - \lambda_k)^{n_j}} \left[ \prod_{1 \leq i < j \leq s} \frac{(n_j + m_j - 1)}{n_i} \right], \tag{4}
\]

for \( 0 \leq k \leq m_j - 1 \) and \( 1 \leq j \leq s \).

As it was pointed out in Subsection 4.2 of [7], the coefficients of the polynomials \( R_j(t) \) are merely the coefficients \( a_{i,k} = a_{i,k}(\lambda_1, \ldots, \lambda_s) \) appearing in the following partial fraction decomposition \( \frac{1}{P_A(z)} = \sum_{i=1}^{s} \sum_{k=0}^{m_i-1} \frac{a_{i,k}(\lambda_1, \ldots, \lambda_s)}{(z - \lambda_i)^{k+1}} \) (for more details see Lemma 1 of [14]). This result combined with Formula (4) yields,

**Proposition 2.1.** (Partial fraction decomposition) Under the preceding data, we have \( \frac{1}{P_A(z)} = \sum_{i=1}^{s} a_i(\lambda_1, \ldots, \lambda_s)(z) \frac{1}{(z - \lambda_i)^{m_i}}, \) where

\[
a_i(\lambda_1, \ldots, \lambda_s)(z) = \sum_{k=0}^{m_i-1} \gamma_{k}^{(i)}(\lambda_1, \ldots, \lambda_s)(z - \lambda_i)^{m_i-k-1}, \tag{5}
\]

with the \( \gamma_{k}^{(j)}(\lambda_1, \ldots, \lambda_s) \) given by (4), for \( 1 \leq i \leq s, 0 \leq k \leq m_i - 1 \).

Over and above, we notice that the function \( g(n) = \rho(n + 1, r) \) is nothing else but only the dynamic solution \( g(k) \) of Equation (1), whose initial values are \( g(0) = g(1) = \cdots = g(r - 2) = 0 \) and \( g(r - 1) = 1 \). And a straightforward calculation leads to differ a new expression for \( g(k) \) in terms of the \( \gamma_{k}^{(i)}(\lambda_1, \ldots, \lambda_s) \) \((1 \leq i \leq s, 0 \leq k \leq m_i - 1)\) and we get,
Theorem 2.2. The expression of the dynamic solution \( g(k) \) of Equation (1), submitted to the prescribed initial condition \( g(0) = g(1) = \ldots = g(r - 2) = 0 \) and \( g(r - 1) = 1 \), is given by 

\[
g(n) = \sum_{i=1}^{s} \sum_{k=0}^{m_i-1} \gamma_k^{(i)}(\lambda_1, \ldots, \lambda_s)(\lambda_i)^{n-k}, \quad \text{for every } n \geq r,
\]

where \( \gamma_k^{(i)}(\lambda_1, \ldots, \lambda_s) \) (1 \( \leq i \leq s \), 0 \( \leq k \leq m_i - 1 \)) is labeling by (4).

In point of the fact, we manage to release a new expression of \( \rho(n, r) \) which is given by

\[
\rho(n, r) = \sum_{i=1}^{s} \sum_{k=0}^{m_i-1} \gamma_k^{(i)}(\lambda_1, \ldots, \lambda_s)(\lambda_i)^{n-1-k}, \quad \text{for every } n \geq r.
\]

Thereby, it ensues from the Binet formula for the general solution of the linear recurrence relation (see [11, 16, 20] for example) that

\[
g(n - 1) = \rho(n, r) = \sum_{i=1}^{s} \sum_{j=0}^{m_i-1} C_{i,j} n^j \lambda_i^n, \quad \text{for } n \geq 1,
\]

where the coefficients \( C_{i,j} \) are computed by solving the following linear system of equations

\[
\sum_{i=1}^{s} \sum_{j=0}^{m_i-1} C_{i,j} n^j \lambda_i^n = 0, \quad n = 1, \ldots, r - 1 \quad \text{and} \quad \sum_{i=1}^{s} \sum_{j=0}^{m_i-1} C_{i,j} n^j \lambda_i^n = 1, \quad \text{for } n = r.
\]

A coming together of the two formulations of \( \rho(n, r) \) allows us to derive the explicit expression of the \( C_{i,j} \) in terms of the \( \lambda_i \), \( i = 1, \ldots, s \). Therefore, we get the result,

Proposition 2.3. Let \( g(n - 1) = \rho(n, r) = \sum_{i=1}^{s} \sum_{j=0}^{m_i-1} C_{i,j} n^j \lambda_i^n, \) for \( n \geq 1 \), the dynamic solution of Equation (1), submitted to the prescribed initial condition \( \rho(0, r) = \rho(1, r) = \ldots = \rho(r - 1, r) = 0 \) and \( \rho(r, r) = 1 \). Then, we have

\[
C_{i,j} = \frac{(-1)^{r-m_i} n^j \lambda_i^{m_i-j}}{\prod_{k=0}^{r-m_i} \lambda_i^{m_i-k}},
\]

for every \( 1 \leq i \leq s \) and \( 0 \leq j \leq m_i - 1 \).

Remark 2.4. Let \( \rho(z) = 1 - (a_0 z + a_1 z^2 + \ldots + a_r z^r) \) be the reversed polynomial of \( P_A(z) \). It is known that the \( \rho(n, r) \) appears in the reciprocal \( h(z) = \sum_{k=0}^{m_i} h_k z^k \) of \( \rho(z) \). That is, we show that \( \rho(n, r) = h_{n-r} \), for every \( n \geq r \). Note also that the \( h_k \) are the divided differences of the powers of \( x \) with respect to the roots of \( P_A(z) \), in the Verde-Star’s approach (for more details see Proposition 4.4 and Corollary 4.4 of [23]). Moreover, the dynamic solution \( g(n) \) of equation (1) is given by \( g(n) = h_{n-r+1} \). Similar affirmation has been done in [7] (see also Subsection 2.3 here below).

Remark 2.5. We note also that, the \( \gamma_k^{(i)}_{m_i-k-1} \) are the Taylor coefficients around \( z = \lambda_i \) of the reciprocal of the polynomial \( P_A(z)/(z - \lambda_i)^{m_i} \). Their explicit formula is obtained in several ways, for example, from a product of powers of geometric series, or using divided differences (see Equation (4.15) of [23] and Subsection 2.3).

2.2 Explicit formulas for the constituent matrices

Considering the precedent results, notably those concerning the formulation of partial fractions decomposition in terms of the \( \gamma_k^{(i)}(\lambda_1, \ldots, \lambda_s) \), we aim to emerge the explicit expressions of the constituents matrices \( Z_{ij} \) (\( i = 1, \ldots, s; j = 1, \ldots, m_i - 1 \)) of a matrix \( A \). As a matter of fact, let set

\[
a_{ij}(z) = \sum_{k=0}^{m_i-j-1} \gamma_k^{(i)}_{m_i-k-1}(\lambda_1, \ldots, \lambda_s)(z - \lambda_i)^k,
\]
where the \( \gamma^{[i]}(\lambda_1, \ldots, \lambda_s) \) are given by (4), and consider the polynomials
\[
N_{i0}(z) = a_i(\lambda_1, \ldots, \lambda_s)(z) \prod_{k \neq i} (z - \lambda_k)^{m_k},
\]
\[
N_{ij}(z) = (z - \lambda_i)^j \prod_{k \neq i} (z - \lambda_k)^{m_k} a_{ij}(z), \quad 1 \leq i \leq s, 1 \leq j \leq m_i - 1,
\]
where the \( a_i(\lambda_1, \ldots, \lambda_s)(z) \) are given by (5), and the \( a_{ij} \) \((i = 1, \ldots, s; j = 1, \ldots, m_i - 1)\) are expressed as in (8).

We point out that the \( N_{i0}(A) \) are furthermore the Frobenius covariants of \( A \), as a matter of fact they satisfy the following relations,
\[
I = \sum_{i=1}^s N_{i0}(A) \quad \text{and} \quad N_{i0}(A)N_{j0}(A) = \delta_{ij}N_{i0}(A),
\]
where \( \delta_{ij} \) is the Kronecker symbol. A direct computation yields
\[
N_{ij}(A) = [N_{i0}(A)(A - \lambda_i I)]^j = N_{i0}(A)(A - \lambda_i I)^j.
\]
The matrix \( N_{ij}(A) \) may becomes null matrix \( \Theta_j \) for \( j < m_i \). Let \( n_i \) \((1 \leq n_i \leq m_i)\) be such that the \( N_{in_i}(A) = \Theta_j \), but \( N_{in_i-1}(A) \neq \Theta_j \). The integer \( n_i \) is the so-called index of nilpotency.

We get attention that if \( P_A(z) \) is the minimal polynomial of \( A \), then the multiplicity \( m_i \) of the eigenvalue \( \lambda_i \) represents the index of nilpotency of the \( N_{ij}(A) \), for \( 1 \leq i \leq s, 0 \leq j \leq m_i \). Thereby, the set \( \{N_{i0}(A), N_{i1}(A), \ldots, N_{in_i-1}(A) \} \) is linearly independent.

It is clear that \( \sum_{i=1}^s N_{i0}(z) = 1 \) and this implies that \( N_{i0}(z)_{|z = -\lambda_i} = 1 \), for every \( 1 \leq i \leq s \). Then, since \( N_{ij}(z)_{|z = -\lambda_i} = 0 \) for \( h \neq i \), \( N_{ij}(z)_{|z = -\lambda_i} = 0 \) for \( j \neq k \) and \( N_{ij}(z)_{|z = -\lambda_i} = 1 \), for \( 1 \leq i \leq s \) and \( 0 \leq j \leq n_i - 1 \) and \( N_{ij} \) means the \( k \)-derivative of the polynomial \( N_{ij}(z) \), we obtain the following result.

**Proposition 2.6.** (Interpolation 1) Let \( M_A(z) \) be the minimal polynomial of \( A \), with distinct eigenvalues \( \lambda_1, \ldots, \lambda_s \) of multiplicities \( m_1, \ldots, m_s \). Consider the polynomials \( N_{ij}(z) \) given by the explicit formula (9)-(10). Then, for any function \( f \) defined on the roots of \( P_A(z) \), the polynomial
\[
r(z) = \sum_{i=1}^s \left[ f(\lambda_i)N_{i0}(z) + \frac{f^{(h)}(\lambda_i)}{h!} N_{ih}(z) + \cdots + \frac{f^{(m_i-1)}(\lambda_i)}{(m_i-1)!} N_{in_i-1}(z) \right],
\]
is the unique polynomial in \( z \) of degree less than \( m_1 + m_2 + \ldots + m_s \) satisfying
\[
f^{[i]}(\lambda_i) = f^{[i]}(\lambda_i), \quad \text{for} \quad 1 \leq i \leq s \quad \text{and} \quad 0 \leq j \leq m_i - 1.
\]

Instead of the minimal polynomial \( M_A(z) \), we can take any annihilator polynomial of \( A \), because we will only use the indices of nilpotency of the matrices \( N_{ij}(A) \), for \( 1 \leq i \leq s \) and \( 0 \leq j \leq m_i - 1 \). Thence, we get the following corollary of Proposition 2.6,

**Corollary 2.7.** (Interpolation 2) Let \( R_A(z) \) be an annihilator polynomial of \( A \), with distinct eigenvalues \( \lambda_1, \ldots, \lambda_s \) of multiplicities \( q_1, \ldots, q_s \) (respective). Let \( n_1, \ldots, n_s \) be the indices of nilpotency of the matrices \( N_{ij}(A) \), for \( 1 \leq i \leq s \). For any function \( f \) defined on the roots of \( R_A(z) \), the polynomial
\[
r(z) = \sum_{i=1}^s \left[ f(\lambda_i)N_{i0}(z) + \frac{f^{(h)}(\lambda_i)}{h!} N_{ih}(z) + \cdots + \frac{f^{(n_i-1)}(\lambda_i)}{(n_i-1)!} N_{in_i-1}(z) \right],
\]
is the unique polynomial in \( z \) of degree less than \( n_1 + n_2 + \ldots + n_s \) satisfying
\[
f^{[i]}(\lambda_i) = f^{[i]}(\lambda_i), \quad \text{for} \quad 1 \leq i \leq s \quad \text{and} \quad 0 \leq j \leq n_i - 1.
\]

The standpoint that the matrices \( N_{i0}(A) \) \((0 \leq j \leq n_0 - 1)\) and \( N_{ij}(A) \) \((1 \leq i \leq s, 0 \leq j \leq n_i - 1)\) are given explicitly using Expressions (4)-(5), and Expression (8) allow us to affirm that results of Proposition 2.6 and
Corollary 2.7 on the interpolation are new. That is, in the best of our knowledge these results are not known in the literature under this explicit form.

In the current literature the polynomial \( r(z) \) described explicitly in Proposition 2.6 and Corollary 2.7 is labeled as the Hermite interpolating polynomial, in the case of the minimal polynomial.

To appeal the results established in [12] by Gantmacher, we deduce that \( Z_{ij} = \frac{1}{j!} N_{ij} \) is nothing else but only the constituent matrices of the matrix \( A \). Thence, in light of (9)-(10), we derive the following explicit formula for the constituent matrices of \( A \).

**Theorem 2.8.** Let \( M_A(z) \) be the minimal polynomial of \( A \), with distinct eigenvalues \( \lambda_1, \ldots, \lambda_s \) of multiplicities \( m_1, \ldots, m_s \) (respectively). Then the constituent matrices of \( A \) are given by

\[
Z_{i0} = \left[ \sum_{k=0}^{m_i-1} \gamma_k^i (\lambda_1, \ldots, \lambda_s)(A - \lambda_i)^{m_i-k-1} \right] \left[ \prod_{k<i} (A - \lambda_k)^{m_k} \right], \tag{11}
\]

\[
Z_{ij} = \frac{(A - \lambda_i)^j}{j!} \left[ \prod_{k<i} (A - \lambda_k)^{m_k} \right] \left[ \sum_{k=0}^{m_j-1} \gamma_k^{ij} (\lambda_1, \ldots, \lambda_s)(A - \lambda_i)^{m_j-k-1} \right],
\]

where \( 1 \leq i \leq s, 0 \leq j \leq m_i - 1 \), the \( \gamma_k^{ij}(\lambda_1, \ldots, \lambda_s) \) \((1 \leq i \leq s, 0 \leq k \leq m_i - 1)\) are given by (4).

It is worth noting that the \( Z_{ij} \) \((1 \leq i \leq s, 0 \leq j \leq m_i - 1)\) have been already studied using various approaches by Gantmacher in [12, pp. 104-105], Horn-Johnson in [13, pp. 401-404, 438] and Lancaster and Tismenetsky in [15, Sec. 9.5]. Moreover, one of the referee drew our attention on the closed relationship between our formulas of the constituent matrices and those established in [9, 23, 25]. For this reason, we discuss this aspect in the following subsection.

### 2.3 Discussion and comparisons

In this subsection, we shall bring focus to how our approach for computing the constituent matrices is connected with others existing in the literature, in particular those proposed by F.C. Chang in [9] and L. Verde-Star in [23, 25].

First, we notice that the expression of dynamical solution \( g(n) \) given in Proposition (2.3) allowing also to express \( p(n, r) \) in terms of the characteristics roots \( \lambda_1, \ldots, \lambda_s \), can be regarding in another point of view by utilizing the Verde-Star’s approach. That is, in [7], we had already placed emphasis on some equivalent formulas upon the dynamical solution of the differential equation \( y^{(r)}(t) = a_0 y^{(r-1)}(t) + a_1 y^{(r-2)}(t) + \cdots + a_{r-1} y(t) \) obtained from properties of linearly recurrent sequences, and those established by Verde-Star in [26]. We renew here some essential points which are linked to our paper and we move to give others. We observe that \( R_j(t) = \sum_{k=0}^{m_j-1} L_{j,m_j-1-k} \frac{t^k}{k!} \), where \( L_{k,j} \) \((1 \leq k \leq s, 0 \leq j \leq m_k)\) are the functionals associated with the polynomial \( P(z) = \prod_{i=1}^s (z - \lambda_i)^{m_i} \). Recall that \( L_{k,j} f = E_{h_k} d^i \left( \frac{f}{P_k} \right) \), where \( E_a(f) = f(a), \frac{d^i}{i!} \) \((D \) is ordinary operator of differentiation) and \( P_k(z) = \frac{P(z)}{(z - \lambda_k)^{m_k}} \). Besides, by the means of the sequence of Hörner polynomials \( \{ \omega_k(z) \}_k \) associated with a given polynomial \( \omega(z) = z^{m_1} + b_1 z^{m_2} + b_2 z^{m_3} + \cdots + b_m \) and the function \( \omega^*(t) = (1 - zt) \sum \omega_k(z) t^k \), the expression of \( \frac{1}{\omega^*(t)} \) given in [23] is \( \frac{1}{\omega^*(t)} = \sum_{n=0}^{\infty} h_n t^n \).

We have \( h_n = \sum_{j=0}^{m_n} \left( \begin{array}{c} m_n \\ j \\
\end{array} \right) (-1)^{|j|} b_1^{j_1} b_2^{j_2} \cdots b_m^{j_m} \), where the summation runs over the multi indices \( j = (j_1, j_2, \cdots, j_m) \) with nonnegative coordinates such that \( a(j) = k \), where \( a(j) = j_1 + 2 j_2 + \cdots + mj_m \), and \(|j| = j_1 + j_2 + \cdots + j_m\) (for more details, see proof of Proposition 4.4 in [23]). When we take \( \omega(z) = P_A(z) \), we derive that the sequence for \( h_n \) is nothing else but the Fibonacci sequence \( p(n + r, r) \) defined by Expression (2). Yet, another formulation of \( h_n \) in terms of the roots of \( P_A(z) \) is provided in Corollary 4.4 of [23]. We ob-
serve that employing our techniques we get an equivalent formula in Proposition 2.3. And involving the \(\gamma^{|i|}_k\) given by (4), which are expressed in terms of the roots of \(P_A(z)\), we observe that \(\gamma^{|i|}_k\) are merely the coefficients \(a_{i,k}(1, \ldots, s)\) appearing in the partial fraction decomposition \(\frac{1}{P_A(z)} = \sum_{i=1}^{s} \sum_{k=0}^{m_i-1} a_{i,k}(1, \ldots, s) (z - \lambda_i)^k\). Owing to the fact that \(R_j(t) = \sum_{k=0}^{m_j-1} \gamma_k^{|j|}(1, \ldots, s) t^k = \sum_{k=0}^{m_j-1} L_{j,m_j-k-1} 1^k\), it was mentioned above, we may have the equivalent result given in Equation (4.15) of [23].

Second, Proposition 2.6 and Corollary 2.7, may be very useful for resolving diverse questions in the current literature, concerning the Hermite interpolation polynomials, where various methods have been developed. In [25], the basic Hermite interpolation polynomials is presented with the idea of a constructive process of the spectral decomposition of a square matrix. And as a consequence, properties of the idempotent (and nilpotent) matrices are established.

Furthermore, following the approach used by Chang in [9], the Hermite polynomials are expressed with the aid of the coefficients of the partial fractions decomposition for the reciprocal of the minimal (or characteristic) polynomial. Note that, the equivalent formula of the \(a_{ij}(z)\) given in (8) appears in [9] as the numerators in the partial fraction decomposition of \(1/P_A(z)\). In addition, their computations require some higher derivations of the partial fraction.

In light of the above discussion, it turns out that the formulas (11)-(12) of the constituent matrices, obtained from Expressions (4)-(5), (8) and (9)-(10), are some of the most direct and explicit expressions for computing the constituent matrices. We point out that the material of our approach is simple and requires only some elementary background tools.

### 3 Matrix functions and some applications

In this section, we provide some fallouts of the results established in the preceding section. We start by giving a proposition in the general setting, we then supply some of its applications. Even better, we give a new formula for both the matrix logarithm and the matrix \(p\)th root. Over and above, some examples will be explored for purposes of illustration. As we have already mentioned the definition given by Schwerdtfeger says that for a matrix \(A\), with distinct eigenvalues \(\lambda_1, \ldots, \lambda_s\) with indices \(n_1, \ldots, n_s\) (respectively), we get

\[
\begin{align*}
  f(A) &= \sum_{i=1}^{s} \sum_{j=0}^{m_i-1} f^{(j)}(\lambda_i) A_j (A - \lambda_i)^j = \sum_{i=1}^{s} \sum_{j=0}^{m_i-1} f^{(j)}(\lambda_i) Z_{ij},
\end{align*}
\]

where the \(Z_{ij}\) \((1 \leq i \leq s, 0 \leq j \leq m_i - 1)\) are the constituents matrices, that depend only on \(A\) (but not \(f\)). The novel for our result is to exhibit the explicit expression of \(f(A)\), under the additional hypotheses that \(f\) is defined on the spectrum of the matrix \(A\). That is, our approach reposes primordially on the use of Theorem 2.8. Thence, we get.

**Theorem 3.1.** Let \(M_A(z)\) be the minimal polynomial of \(A\), with distinct eigenvalues \(\lambda_1, \ldots, \lambda_s\) of multiplicities \(m_1, \ldots, m_s\) (respectively). Let \(f\) be a function defined on the roots of \(M_A(z)\) (or equivalently on the spectrum of \(A\)). Then, we have

\[
f(A) = \sum_{i=1}^{s} [f(\lambda_i) Z_{i0}(A) + \ldots + f^{(b)}(\lambda_i) Z_{ih}(A) + \ldots + f^{(m_i-1)}(\lambda_i) Z_{im_i-1}(A)],
\]

where the \(Z_{ij}\) \((1 \leq i \leq s, 0 \leq j \leq m_i - 1)\), the constituent matrices of \(A\), are given by (11)-(12) and also (4).

Let \(P_A(z)\) be the characteristic (or any annihilator) polynomial of \(A\), with distinct eigenvalues \(\lambda_1, \ldots, \lambda_s\) of multiplicities \(m_1, \ldots, m_s\) and indices \(n_1, \ldots, n_s\) (respectively). Let \(f\) be a function defined on the roots of \(P_A(z)\).
Then, in light of the Corollary 2.7, we have

\[ f(A) = \sum_{i=1}^{s} [f(\lambda_i) Z_{i0}(A) + \ldots + f^{(h)}(\lambda_i) Z_{ih}(A) + \ldots + f^{(n_i-1)}(\lambda_i) Z_{in_i-1}(A)], \]

(14)

where the \( Z_{ij} \) (\( 1 \leq i \leq s, \ 0 \leq j \leq n_i - 1 \)), the constituent matrices of \( A \) are given by (11) (12) and also (4).

We point out that Expressions (13)-(14) of the matrix function \( f(A) \), with the aid of (11)-(12) and (4) are not known in the literature under this explicit form. As a consequence and in order to illustrate the importance of the preceding Theorem 3.1, let consider the function \( f(A) = \text{Log}(A) \), when we suppose that \( f(z) = \text{Log}(z) \) is defined on the spectrum of \( A \), which means that the \( k \)-derivative \( \text{Log}^{(k)}(\lambda_i) \) exists for every \( 1 \leq i \leq s \), we obtain a new expression of \( \text{Log}(A) \).

**Proposition 3.2.** Let \( M_A(z) \) be the minimal polynomial of \( A \), with distinct eigenvalues \( \lambda_1, \ldots, \lambda_s \) of multiplicities \( m_1, \ldots, m_s \) (respectively). We suppose that \( \text{Log}(z) \) is defined on the spectrum of \( A \). We have

\[ \text{Log}(A) = \sum_{i=1}^{s} \left[ \text{Log}(\lambda_i) Z_{i0}(A) + \ldots + \frac{(-1)^{h_i}}{\lambda_i^{n_i}} Z_{ih}(A) + \ldots + \frac{(-1)^{n_i-1}}{\lambda_i^{n_i-1}} Z_{in_i-1}(A) \right], \]

(15)

where the \( Z_{ij} \) (\( 1 \leq i \leq s, \ 0 \leq j \leq m_i - 1 \)), the constituent matrices of \( A \), are given by (11)-(12) and also (4).

**Example 3.3.** Let consider the matrix \( A \) of order 3 such that its minimal polynomial of the matrix takes the form \( P(z) = M_A(z) = (z - 1)^2(z - 2) \). Therefore, applying the formula (4) we obtain,

\[ \gamma_0^{(1)}(1, 2) = -1, \ \gamma_1^{(1)}(1, 2) = -1, \ \gamma_0^{(2)}(1, 2) = -1, \]

Thereafter, we have \( N_{10}(A) = -A(A - 2I_3) \), \( N_{11}(A) = -(A - 2I_3)(A - I_3) \) and \( N_{20}(A) = -(A - I_3)^2 \). Thus the formula (15) yields

\[ \text{Log}(A) = -(A - 2I_3)(A - I_3) - (A - 1)^2 \text{Log}(2). \]

Note that, some explicit formulas for the principal matrix logarithm have been established in [1], using the Binet formula of sequences (1).

Another interesting application is provided by applying Theorem 3.1 for \( f(z) = \sqrt[3]{z} = z^{1/3} \) (\( p \in \mathbb{N} \)). That is, we manage to get also a new expression of the matrix \( p \)th root \( \sqrt[p]{A} = A^{1/p} \) of a given matrix \( A \).

**Proposition 3.4.** Let \( P_A(z) \) an annihilator polynomial of \( A \) with distinct eigenvalues \( \lambda_1, \ldots, \lambda_s \) of multiplicities \( m_1, \ldots, m_s \) and indices \( n_1, \ldots, n_s \). We suppose that \( f(z) = z^\frac{1}{p} \) is defined on the spectrum of \( A \). Then, we have

\[ A^{\frac{1}{p}} = \sum_{i=1}^{s} \left[ \sqrt[p]{\lambda_i} Z_{i0}(A) + \frac{1}{p} A^{\frac{1}{p} - 1} Z_{i1}(A) + \ldots + \frac{1}{p} \left( \frac{1}{p} - i \right) A^{\frac{1}{p} - h_i} Z_{ih}(A) + \ldots + \frac{1}{p} \left( \frac{1}{p} - i \right) Z_{in_i-1}(A) \right], \]

(16)

where the \( Z_{ij} \) (\( 1 \leq i \leq s, \ 0 \leq j \leq n_i - 1 \)), the constituent matrices of \( A \), are given by (11)-(12) and also (4).

For purpose of illustration, we examine the following example.

**Example 3.5.** Let consider the same matrix \( A \) taken in the precedent example, we aim to compute \( A^{1/3} \). As a matter of fact, it follows from formula (16) that

\[ \sqrt[3]{A} = (A - 2I_3)(-\frac{4}{3}A + \frac{1}{3}I_3) - \sqrt[3]{2}(A - I_3)^2. \]

Recently, some explicit formulas of the principal matrix \( p \)th root have been obtained in [2, 8]. Note that, the Binet formula of sequence (1) and Expression (2) play a central role in the results of [2].
Concluding remarks and perspective

The interpolating polynomial definition of a matrix function has been extensively studied in the literature. Among the methods used, the theory of constituent matrices plays an important role, and continues to be a subject of research, for the study of functions of matrices. In Section 2 we developed a method to provide an explicit form for the constituent matrices. In Section 3 we illustrate our results with two typical applications of the literature.

Our results on the new Binet formulas, the partial fraction decomposition and the explicit formula of the constituent matrices, have interesting perspectives. Indeed, it is important to mention that our results may be applied to other type of usual functions, for example the resolvent of a matrix. More than that, the partial fraction decomposition of $1/P_A(z)$ is currently used in various papers. However, its explicit formula obtained with the aid of (4)-(5), is not known under this compact form in the literature. So, the result of Proposition 2.1 may be very useful for various subjects related to this topic.

Finally, we emphasis here that the results of Subsection 2.1 bring as well to solve the Vandermonde linear systems and some generalized Vandermonde systems, but in order to not digress from our target, we do not explore this question in this paper. A research paper on this subject, where some important new results are established, is actually submitted.

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References


