Research Article

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Determinants and inverses of circulant matrices with complex Fibonacci numbers

Abstract: Let $\mathcal{F}_n = circ(F_1^*, F_2^*, \ldots, F_n^*)$ be the $n \times n$ circulant matrix associated with complex Fibonacci numbers $F_1^*, F_2^*, \ldots, F_n^*$. In the present paper we calculate the determinant of $\mathcal{F}_n$ in terms of complex Fibonacci numbers. Furthermore, we show that $\mathcal{F}_n$ is invertible and obtain the entries of the inverse of $\mathcal{F}_n$ in terms of complex Fibonacci numbers.

Keywords: Circulant matrix, determinant, complex Fibonacci sequence, Fibonacci sequence

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1 Introduction and Preliminaries

The circulant matrix $V_n = circ(v_0, v_1, \ldots, v_{n-1})$ associated with complex numbers $v_0, v_1, \ldots, v_{n-1}$ is the $n \times n$ matrix

$$V_n = \begin{pmatrix}
    v_0 & v_1 & \cdots & v_{n-2} & v_{n-1} \\
    v_{n-1} & v_0 & \cdots & v_{n-3} & v_{n-2} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    v_2 & v_3 & \cdots & v_0 & v_1 \\
    v_1 & v_2 & \cdots & v_{n-1} & v_0
\end{pmatrix}.$$  

Circulant matrices have elegant algebraic and matrix theoretic properties. For example, the set of all $n \times n$ circulant matrices Circ($n$) is an $n$-dimensional commutative subalgebra of the algebra of $n \times n$ matrices with the usual matrix operations and

$$\det V_n = \prod_{i=0}^{n-1} \left( \sum_{j=0}^{n-1} e^{i\pi} v_j \right),$$

where $e$ is a primitive $n-th$ root of unity. Moreover, circulant matrices have various applications in mathematics such as coding theory, signal processing, numerical computation, etc. For further detailed properties and applications of circulant matrices the reader can consult the texts of [4]-[6].

In the literature, many scholars dealt with circulant matrices and examined their properties such as their determinants and inverses associated with some integer sequences. In 1970 Lind [8] gave a determinant formula for $F = circ(F_r, F_{r+1}, \ldots, F_{r+n-1})$ ($r \geq 1$). Then Solak [10] examined the matrix norms of $F = circ(F_1, F_2, \ldots, F_n)$ and $L = circ(L_1, L_2, \ldots, L_n)$, where $F_s$ and $L_s$ are the Fibonacci and Lucas numbers,

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respectively. In 2011 Shen and et. al. [9] gave the following determinant formulae for circulant matrices $F$ and $L$

$$\det (F) = (1 - F_{n+1})^{n-1} + F_{n}^{n-2} \sum_{k=1}^{n-1} F_{k} \left( \frac{1 - F_{n+1}}{F_{n}} \right)^{k-1},$$

$$\det (L) = (1 - L_{n+1})^{n-1} + (L_{n} - 2)^{n-2} \sum_{k=1}^{n-1} \left( L_{k+2} - 3L_{k+1} \right) \left( \frac{1 - L_{n+1}}{L_{n} - 2} \right)^{k-1}.$$

They also obtained the entries of the inverses of $F$ and $L$ in terms of Fibonacci and Lucas numbers, respectively. On this topic, Bozkurt and Tam [2] obtained analogues of the results of the paper [9] for circulant matrices associated with Jacobsthal and Jacobsthal-Lucas numbers. Furthermore, Bozkurt and Tam [3] gave a generalization of aforementioned determinant formulae and results on invertibility of these particular circulant matrices. Namely, they calculated the determinant of $W = \text{circ}(W_1, W_2, \ldots, W_n)$, where the sequence $\{W_n\}$ is defined by the recurrence relation $W_n = pW_{n-1} + qW_{n-2}$ $(n \geq 3)$ with $W_1 = a$ and $W_2 = b$ $(a, b, p, q \in \mathbb{Z})$, and they obtained the entries of the inverse of $W$. Then Yazlık and Taşkara [12] generalized the results of Bozkurt and Tam [3] for a circulant matrix whose entries are generalized $k$–Horadam numbers. Further, generalizing above determinantal results for a sequence $\{a_k\}$ of real numbers defined by an $m$–th order linear homogenous recurrence relation $(m \geq 1)$ in [1] the first and third authors of the present paper obtained a formula for the determinant of the circulant matrix $F = \text{circ} (a_1, a_2, \ldots, a_n)$.

On the other hand, recent studies show that there has been an increasing interest on Fibonacci sequence and their generalizations. One of them is the concept of complex Fibonacci numbers $F_n^*$, which was defined by Horadam [7]. The $n$–th complex Fibonacci number is given by the equality $F_n^* = F_n + iF_{n+1}$, where $i$ is the imaginary unit which satisfies $i^2 = -1$. Complex Fibonacci numbers satisfy the same recurrence relation $F_n^* = F_{n-1}^* + F_{n-2}^*$ $(n \geq 2)$ of the classical Fibonacci numbers with different initial conditions, i.e., $F_0^* = i$, $F_1^* = 1 + i$.

Recently, Jiang, Xin and Lu [5] have studied some types of circulant matrices whose entries are Gaussian Fibonacci numbers. The $n$–th Gaussian Fibonacci number $G_n$ is defined by $G_0 = i$, $G_1 = 1$ and $G_n = F_n + iF_{n-1}$ for $n \geq 2$.

In this paper we deal with circulant matrices associated with complex Fibonacci numbers. Let $\mathcal{F}_n = \text{circ}(F_1^*, F_2^*, \ldots, F_n^*)$. Using the same method of Shen et. al. [9] we calculate the determinant of $\mathcal{F}_n$. Then we give Cassini’s identity and d’Ocagne’s identity for the complex Fibonacci numbers. Using these new identities, we obtain the entries of the inverse of $\mathcal{F}_n$ in terms of complex Fibonacci numbers.

We conclude this section with a fundamental tool for the invertibility of circulant matrices.

**Lemma 1.1.** [4] Let $V_n = \text{circ}(v_0, v_1, \ldots, v_{n-1})$ be a circulant matrix. Then we have the following:

(i) $V_n$ is invertible if and only if $f \left( \omega^k \right) \neq 0$ for all $k = 0, 1, 2, \ldots, n - 1$, where $f(x) = \sum_{j=0}^{n-1} v_j x^j$ and $\omega = \exp \left( \frac{2\pi i}{n} \right)$.

(ii) If $V_n$ is invertible then its inverse is also a circulant matrix.

## 2 Main Results

In this section we consider the $n \times n$ circulant matrix $\mathcal{F}_n = \text{circ} (F_1^*, F_2^*, \ldots, F_n^*)$, where $F_i^*$ is the $i$–th complex Fibonacci number.

**Theorem 2.1.** Let $n \geq 3$. Then we have

$$\det \mathcal{F}_n = F_1^* (F_1^* - F_{n+1}^*)^{n-1} + \sum_{k=1}^{n-1} \left( F_k^* - iF_{k+1}^* \right) \left( F_k^* - F_{n+1}^* \right)^{k-1} \left( F_n^* - F_0^* \right)^{n-k-1}.$$
Proof. Let \( n \geq 3 \). Consider the \( n \times n \) matrices as follows:

\[
P = \begin{pmatrix}
1 & 0 & & & 1 \\
-1 & 0 & & & 1 & -1 \\
0 & & & & 1 & -1 & -1 \\
\vdots & & & & 1 & -1 & -1 \\
0 & & & & 1 & \cdots & -1 & 0 \\
0 & & & & 1 & -1 & -1 \\
\end{pmatrix}
\]

and

\[
Q' = \begin{pmatrix}
1 & 0 & & & & & \cdots & 0 & 0 \\
0 & \left( \frac{F_{n-1} - F_0}{F_1 - F_{n+1}} \right)^{n-2} & 0 & \cdots & 0 & 1 \\
0 & \left( \frac{F_{n-1} - F_0}{F_1 - F_{n+1}} \right)^{n-3} & 0 & 1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \frac{F_{n-1} - F_0}{F_1 - F_{n+1}} & 1 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\end{pmatrix}
\]

Then we have

\[
P^r_n Q' = \begin{pmatrix}
F_1 & \theta & F_{n-1} & F_{n-2} & \ldots & F_2 & F_1 \\
F_2 & \mu & F_{n-1} & F_{n-2} & \ldots & F_3 & F_2 \\
0 & 0 & -F_{n+1} & F_{n-1} & \ldots & F_4 & F_3 \\
0 & 0 & F_0 - F_n & F_1 - F_{n+1} & \ldots & F_5 & F_4 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix},
\]

where

\[
\mu = F_1 + \sum_{k=1}^{n-2} F_{k+2} \left( \frac{F_n - F_0}{F_1 - F_{n+1}} \right)^{n-(k+1)}
\]

and

\[
\theta = \sum_{k=1}^{n-1} F_{k+1} \left( \frac{F_n - F_0}{F_1 - F_{n+1}} \right)^{n-(k+1)}
\]

Let \( A_0 \) be the \((n-1) \times (n-1)\) matrix obtained from \( P \) by deleting the first row and the first column of \( P \), i.e.,

\[
A_0 = \begin{pmatrix}
1 & & & & \\
0 & 1 & -1 & & \\
& 1 & -1 & -1 & \\
& & 1 & -1 & -1 \\
\end{pmatrix}
\]

It is clear that \( \det P = \det A_0 \). Let \( A_k \) be the matrix obtained from \( A_{k-1} \) by adding the \( k \)th row of \( A_{k-1} \) to its \((k+1)\)th row and then to its \((k+2)\)th row for each \( k = 1, 2, \ldots, n-3 \), respectively. Also, let \( A_{n-2} \) be the matrix obtained from \( A_{n-3} \) by adding the \((n-2)\)th row to the \((n-1)\)th row. Then, for each \( i = 1, 2, \ldots, n-2 \), we have
Thus $\det A_i = \det A_0$ and hence $\det P = \det A_{n-2}$. Moreover, it is clear that

$$A_{n-2} = \begin{bmatrix}
0 & 1 & & \\
& & \cdots & \\
1 & 0 & & 1
\end{bmatrix}.$$

Thus $\det A_{n-2} = (-1)^{(n-1)(n-2)}$ and hence we have $\det P = (-1)^{(n-1)(n-2)}$. By a similar way, one can easily obtain that $\det Q' = (-1)^{(n-1)(n-2)}$. Therefore, we have

$$\det \mathcal{F}_n = \det P \det \mathcal{F}_n \det Q' = \det (P \mathcal{F}_n Q') = (\mu F_1' - \theta F_2') (F_1' - F_{n+1}')^{n-2}$$

$$= \left( F_1'^2 - F_2' F_n' + \sum_{k=1}^{n-2} (F_1' F_{k+2}' - F_2' F_{k+1}') \left( \frac{F_n' - F_{n-1}'}{F_1' - F_{n+1}'} \right)^{n-k-1} \right) (F_1' - F_{n+1}')^{n-2}$$

$$= F_1' (F_1' - F_{n+1}')^{n-1} + \sum_{k=1}^{n-1} (F_k' - iF_k') \left( \frac{F_n' - F_{n-1}'}{F_1' - F_{n+1}'} \right)^{n-k-1} (F_1' - F_{n+1}')^{k-1}.$$

Thus, the proof is complete. \(\square\)

**Theorem 2.2.** For $n \geq 3$, $\mathcal{F}_n$ is invertible.

**Proof.** Let $n \geq 3$, $\alpha = \frac{1 + \sqrt{5}}{2}$ and $\beta = \frac{1 - \sqrt{5}}{2}$. It is clear that $\alpha + \beta = 1$, $\alpha \beta = -1$ and the Binet formula for $F_n'$ is

$$F_n' = \gamma \alpha^n + \lambda \beta^n,$$

where

$$\gamma = \frac{2 + (5 + \sqrt{5}) i}{10} \text{ and } \lambda = \frac{-2 + (5 - \sqrt{5}) i}{10}.$$

Now, consider the function $f$ for our matrix $\mathcal{F}_n$ in Lemma 1.1. For all $k = 1, 2, \ldots, n - 1$, we have

$$f(\omega^k) = \sum_{j=1}^{n} F_j' (\omega^k)^{j-1}$$

$$= \gamma \alpha \left( \frac{1 - \alpha^n}{1 - \alpha \omega^k} \right) + \lambda \beta \left( \frac{1 - \beta^n}{1 - \beta \omega^k} \right)$$

$$= \gamma \alpha (1 - a^n) (1 - \beta \omega^k) + \lambda \beta (1 - \beta^n) (1 - a \omega^k)$$

$$= \frac{\gamma \alpha + \lambda \beta - (\gamma a^{n+1} + \lambda \beta^{n+1}) + \omega^k [(\gamma + \lambda) - (\gamma a^n + \lambda \beta^n)]}{1 - \omega^k - \omega^{2k}}$$

$$= \frac{F_1' - F_{n+1}' + \omega^k (F_0' - F_n')}{1 - \omega^k - \omega^{2k}}$$

$$= \frac{1 + i - F_{n+1}' + \omega^k (i - F_n')}{1 - \omega^k - \omega^{2k}}.$$
Now we suppose that \(1 - \omega^s - \omega^{2s} = 0\) for some \(s \in \{1, 2, \ldots, n - 1\}\). Then we have \(\omega^n = (1 - \sqrt{5})/2\) but this contradicts \((\omega^s)^n = 1\). Thus, we have \(1 - \omega^k - \omega^{2k} \neq 0\) for all \(k = 1, 2, \ldots, n - 1\). If it is possible, we suppose that for some \(l \in \{1, 2, \ldots, n - 1\}\), \(\sum_{i=1}^{l} F_{n+i} + \omega^l (i - F_n) = 0\) or equivalently \(\omega^l = \frac{F_{n+1} - 1 - i}{1 - F_n}\).

Let \(z_r = \frac{F_{r+1} - 1 - i}{i - F_r}\) for all \(2 \leq r \leq n\). Then we have for the modulus of \(z_r\)

\[
|z_r| = \frac{(F_{r+1} - 1 + i (F_{r+2} - 1)}{F_r + i (1 - F_{r+1})} = \left( \frac{(F_{r+1} - 1)^2 + (F_r + F_{r+1} - 1)^2}{F_r^2 + (1 - F_{r+1})^2} \right)^{\frac{1}{2}} = \left[ \frac{1 + (\frac{F_r - F_{r+1}}{F_r})^2}{(\frac{F_r - F_{r+1}}{F_r})^2 + 1} \right]^{\frac{1}{2}}.
\]

Let \(t(r) = \frac{F_{r+1}}{F_r}\). Then \(t\) is a function of positive integer \(r\), where \(2 \leq r \leq n\). For all \(r\), we obviously have \((-1)^{r-1} \leq F_{r+1} \leq 1\). From Cassini’s identity for the classical Fibonacci numbers, we have

\[
F_{r+1}^2 - F_{r+2}F_r \leq F_{r-1} = F_{r+1} - F_r,
\]

or equivalently \(F_r (1 - F_{r+2}) \leq F_{r+1} (1 - F_{r+1})\). Then

\[
\frac{F_r}{1 - F_{r+1}} \leq \frac{F_{r+1}}{1 - F_{r+1}}.
\]

The last inequality gives that \((r) \leq t (r + 1)\) for all \(2 \leq r \leq n\). In other words, as a function of \(r\), \(t\) is nondecreasing. Moreover, since

\[
\frac{F_r}{1 - F_{r+1}} = \frac{1}{r} - \frac{F_{r+1}}{F_r} \to \frac{1 - \sqrt{5}}{\sqrt{2}},
\]

as \(r \to \infty\) and \(t(2) = -1\), we have \(-1 \leq t (r) \leq \frac{1 - \sqrt{5}}{\sqrt{2}}\).

On the other hand, \(|z_r|\) can be considered as a continuous function of \(t\) and hence we clearly have

\[
|z_r| = \left[ \frac{1 + (t - 1)^2}{t^2 + 1} \right]^{\frac{1}{2}} = \left( \frac{t^2 - 2t + 2}{t^2 + 1} \right)^{\frac{1}{2}}.
\]

Taking the first derivative of \(|z_r|\) with respect to \(t\), we have

\[
\frac{d}{dt} |z_r| = \frac{t^2 - t - 1}{(t^2 + 1)^2 (t^2 - 2t + 2)^{\frac{1}{2}}}.
\]

By the first derivative test, \(|z_r|\) as a function of \(t\) is non-decreasing in the interval \([-1, (1 - \sqrt{5})/2]\). So, for all \(2 \leq r \leq n\), we have \(|z_r| \geq |z_2| = \sqrt{10}/2 > 1\) and in particular, \(|z_n| > 1\). Thus, we obtain \(|\omega^l|^n = |z_n| > 1\) which contradicts \(|\omega^l|^n = 1\). Consequently, we conclude that \(f (\omega^k) \neq 0\) for all \(k = 1, 2, \ldots, n - 1\). Lemma 1.1 completes the proof. □

**Lemma 2.3.** Let \(T = [t_{k,j}]\) be the \((n - 2) \times (n - 2)\) matrix defined by

\[
t_{k,j} = \begin{cases} F_{r+1} - F_{r+2}, & k = j \\ F_r - F_{n+i}, & k = j + 1 \\ 0, & \text{otherwise.} \end{cases}
\]

Then the inverse of \(T\) is \(C = [c_{k,j}]\), where

\[
c_{k,j} = \begin{cases} (F_{r+1} - F_{r+2})^{\frac{1}{2}}, & \text{if } k \geq j, \\ (F_r - F_{n+i})^{\frac{1}{2}}, & \text{if } k < j, \\ 0, & \text{otherwise.} \end{cases}
\]
Proof. Let \( s_{k,i} = \sum_{m=1}^{n-2} t_{k,m} c_{m,i} \) for \( 1 \leq k, j \leq n - 2 \). Consider the \((n-2) \times (n-2)\) matrix \( S = [s_{k,j}] \). For \( k < j \), it is obvious that \( s_{k,j} = 0 \). If we set \( k = j \), then we have

\[
  s_{k,k} = t_{k,k} c_{k,k} = \left( F_{1}^{*} - F_{n+1}^{*} \right) \frac{1}{F_{1}^{*} - F_{n+1}^{*}} = 1.
\]

For \( k \geq j + 1 \), we get

\[
  s_{k,j} = \sum_{m=1}^{n-2} t_{k,m} c_{m,j} = t_{k,k-1} c_{k-1,j} + t_{k,k} c_{k,j}
\]

\[
  = \left( F_{0}^{*} - F_{n}^{*} \right) \frac{(F_{n}^{*} - F_{0}^{*})^{k-j-1}}{(F_{1}^{*} - F_{n+1}^{*})^{k-j}} + \left( F_{1}^{*} - F_{n}^{*} \right) \frac{(F_{n}^{*} - F_{0}^{*})^{k-j}}{(F_{1}^{*} - F_{n+1}^{*})^{k-j+1}} = 0
\]

Thus, we see that \( S = I_{n-2} \), the \((n-2) \times (n-2)\) identity matrix. In similar manner, we can show that \( CT = I_{n-2} \).

This completes the proof. \( \square \)

Now we give a lemma which provides Cassini’s identity and d’Ocagne’s identity for complex Fibonacci numbers. The proof of the following lemma is adapted from the paper of Spivey [11].

**Lemma 2.4.**

\[
  F_{n-1}^{*} F_{n+1}^{*} - \left( F_{n}^{*} \right)^{2} = (-1)^{n} (2 + i),
\]

\[
  F_{m}^{*} F_{n+1}^{*} - F_{n}^{*} F_{m+1}^{*} = (-1)^{n} (2 + i) F_{m-n}.
\]

**Proof.** Consider the matrix

\[
  D_{1} = \begin{pmatrix} F_{0}^{*} & F_{1}^{*} \\ F_{1}^{*} & F_{2}^{*} \end{pmatrix} = \begin{pmatrix} i & 1 + i \\ 1 + i & 1 + 2i \end{pmatrix}.
\]

We construct \( D_{2} \) from \( D_{1} \) by adding the second row of \( D_{1} \) to the first row of \( D_{1} \) and then interchanging the two rows. Continuing this process for \( n \) times, we have

\[
  D_{n} = \begin{pmatrix} F_{n-1}^{*} & F_{n}^{*} \\ F_{n}^{*} & F_{n+1}^{*} \end{pmatrix},
\]

which can be easily proved by induction. The first one of above elementary matrix row operation does not affect the determinant and the second changes only the sign. Therefore, \( \det D_{n} = (-1)^{n-1} \det D_{1} \). Thus, we have

\[
  F_{n-1}^{*} F_{n+1}^{*} - \left( F_{n}^{*} \right)^{2} = (-1)^{n-1} (-2 - i).
\]

We call the last equality Cassini’s identity for complex Fibonacci numbers. Now, to prove d’Ocagne’s identity, consider the matrix

\[
  B_{0} = \begin{pmatrix} F_{n}^{*} & F_{n}^{*} \\ F_{n+1}^{*} & F_{n+1}^{*} \end{pmatrix}.
\]

We can construct \( B_{1} \) by adding the first column of \( D_{n} \) to that of \( B_{0} \). Obviously, we get

\[
  B_{1} = \begin{pmatrix} F_{n+1}^{*} & F_{n}^{*} \\ F_{n+2}^{*} & F_{n+1}^{*} \end{pmatrix}.
\]

Then it is clear that \( \det B_{1} = \det D_{n} = (-1)^{n} (2 + i) \). Then by adding the first column of \( B_{0} \) to that of \( B_{1} \) gives us

\[
  B_{2} = \begin{pmatrix} F_{n+2}^{*} & F_{n}^{*} \\ F_{n+3}^{*} & F_{n+1}^{*} \end{pmatrix}.
\]
By induction, we can prove that
\[ B_s = \begin{pmatrix} F_{n+s} & F_n \\ F_{n+s+1} & F_{n+1} \end{pmatrix}. \]

Using the sum property of determinant, we have \( \det B_s = \det B_{s-1} + \det B_{s-2} \) which shows that \( \{ \det B_s \} \) is a generalized Fibonacci sequence. Therefore, we have
\[ \det B_s = F_{s-1} \det B_0 + F_s \det B_1. \]

Since \( \det B_0 = 0 \) and
\[ \det B_s = F_s \det B_1 = (-1)^{n-1} (-2 - i) F_s \]
we get
\[ F_{n+s} F_{n+1} - F_{n+s+1} F_n = (-1)^n (2 + i) F_s. \]

Setting \( m = n + s \) gives us
\[ F_m F_{n+1} - F_{m+1} F_n = (-1)^n (2 + i) F_{m-n}. \]

Using these new identities for complex Fibonacci numbers, we now determine the entries of the inverse of the circulant matrix \( \mathcal{F}_n \).

**Theorem 2.5.** Let \( n \geq 3 \) and \( \triangle = F_1 \mu - F_2 \theta \). Then the inverse of \( \mathcal{F}_n \) is \( \mathcal{F}_n^{-1} = \text{circ} (a_1, a_2, \ldots, a_n) \), where
\[
\begin{align*}
a_1 &= \frac{1}{\triangle} \left( F_1 + (2 + i) \sum_{k=1}^{n-2} F_{n-k} \frac{(F_n - F_0)(k-1)}{(F_1 - F_{n+1})^k} \right), \\
a_2 &= \frac{1}{\triangle} \left( -F_1^* + (2 + i) \sum_{k=1}^{n-2} F_{n-k} \frac{(F_n^* - F_0^*)(k-1)}{(F_1^* - F_{n+1})^k} \right), \\
a_3 &= -\frac{(2 + i)(F_n - F_0^*)}{\triangle (F_1^* - F_{n+1})}, \\
a_4 &= \frac{(2 + i)(F_n^* - F_0)}{\triangle (F_1^* - F_{n+1})^2},
\end{align*}
\]
and
\[ a_j = \frac{(2 + i)(F_n^* - F_0^*)^{j-3}}{\triangle (F_1^* - F_{n+1})^{j/2}} \]
for \( 5 \leq j \leq n \).

**Proof.** Consider the \( n \times n \) matrix
\[
Q'' = \begin{pmatrix}
1 & -\frac{\theta}{F_1} & \frac{\theta F_1 - \mu F_2}{\triangle} & \frac{\theta F_1 - \mu F_3}{\triangle} & \ldots & \frac{\theta F_1 - \mu F_n}{\triangle} \\
0 & 1 & \frac{F_n - F_0}{\triangle} & \frac{F_n - F_0^*}{\triangle} & \ldots & \frac{F_n - F_0^{n-1}}{\triangle} \\
0 & 0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & 0 & \ldots & 1
\end{pmatrix},
\]
where
\[
\mu = F_1 + \sum_{k=1}^{n-1} F_{k+1} \left( \frac{F_n - F_0}{F_1 - F_{n+1}} \right)^{n-(k+1)},
\]
\[
\theta = \sum_{k=1}^{n-1} F_{k+1} \left( \frac{F_n^* - F_0^*}{F_1^* - F_{n+1}} \right)^{n-(k+1)}.
\]
and
\[ \triangle = F^*_i \mu - F^*_j \theta. \]

Let
\[ U = \begin{bmatrix} F^*_1 & 0 \\ F^*_2 & \mu - F^*_2 \theta \end{bmatrix} \]
and \( T \) be as in Lemma 2.3. Then we have
\[ P\mathcal{F}_n Q' Q'' = U \oplus T, \]
where \( U \oplus T \) denotes the direct sum of \( U \) and \( T \). Further, renaming \( Q = Q' Q'' \), we get
\[ \mathcal{F}_n^{-1} = Q \left( U^{-1} \oplus T^{-1} \right) P. \]

From Lemma 1.1, \( \mathcal{F}_n^{-1} \) is circulant. Thus, we can write \( \mathcal{F}_n^{-1} = \text{circ} (a_1, a_2, \ldots, a_n) \). Since the last row of \( Q \) is
\[ \left( 0, 1, F^*_{n-1} F^*_2 - F^*_{n-1} F^*_1, F^*_2 - F^*_1, \ldots, (F^*_1)^2 - F^*_1 \right), \]
by setting \( K_{n,j} = \left[ Q \left( U^{-1} \oplus T^{-1} \right) \right]_{n,j} \), the \( nj \)-entry of the product \( Q \left( U^{-1} \oplus T^{-1} \right) \), for \( 1 \leq j \leq n \), we obtain
\[
K_{n,1} = -\frac{F^*_2}{\triangle}, \\
K_{n,2} = \frac{F^*_1}{\triangle}, \\
K_{n,j} = \frac{1}{\triangle} \sum_{k=1}^{n-1+j} \frac{(F^*_{n-k-j} F^*_2 - F^*_{n-k-j} F^*_1)}{(F^*_1 - F^*_{n+1})^k} \left( \frac{F^*_n - F^*_0}{F^*_1 - F^*_{n+1}} \right)^{k-1} \\
= \frac{-2 + i}{\triangle} \sum_{k=1}^{n-1+j} \frac{(F^*_{n-k-j} F^*_2 - F^*_{n-k-j} F^*_1)}{(F^*_1 - F^*_{n+1})^k} \left( 3 \leq j \leq n \right)
\]
from the identities in Lemma 2.4. If we multiply the row matrix \( \left[ K_{n,1}, K_{n,2}, \ldots, K_{n,n} \right] \) by the matrix \( P \), we get the last row of \( \mathcal{F}_n^{-1} \). Clearly, by Lemma 1.1, \( \mathcal{F}_n^{-1} = \text{circ} (a_1, a_2, \ldots, a_n) \), where
\[
\begin{align*}
\quad a_1 &= K_{n,2} - K_{n,3} - K_{n,4} = \\
&= \frac{F^*_1}{\triangle} + \frac{2 + i}{\triangle} \sum_{k=1}^{n-2} \frac{(F^*_n - F^*_0)^{k-1}}{(F^*_1 - F^*_{n+1})^k} + \frac{2 + i}{\triangle} \sum_{k=1}^{n-3} \frac{(F^*_n - F^*_0)^{k-1}}{(F^*_1 - F^*_{n+1})^k} = \frac{1}{\triangle} \left( F^*_1 + (2 + i) \sum_{k=1}^{n-2} \frac{(F^*_n - F^*_0)^{k-1}}{(F^*_1 - F^*_{n+1})^k} \right), \\
\end{align*}
\[
\begin{align*}
\quad a_2 &= K_{n,1} - K_{n,3} = -\frac{F^*_2}{\triangle} + \frac{2 + i}{\triangle} \sum_{k=1}^{n-2} \frac{(F^*_n - F^*_0)^{k-1}}{(F^*_1 - F^*_{n+1})^k} = \frac{1}{\triangle} \left( -F^*_2 + (2 + i) \sum_{k=1}^{n-2} \frac{(F^*_n - F^*_0)^{k-1}}{(F^*_1 - F^*_{n+1})^k} \right), \\
\end{align*}
\[
\begin{align*}
\quad a_3 &= K_{n,n} = -\frac{2 + i}{\triangle} (F^*_1 - F^*_{n+1}), \\
\end{align*}
\[
\begin{align*}
\quad a_4 &= K_{n,n-1} - K_{n,n} = -\frac{2 + i}{\triangle} \sum_{k=1}^{2} \frac{(F^*_n - F^*_0)^{k-1}}{(F^*_1 - F^*_{n+1})^k} + \frac{2 + i}{\triangle} F_{2-k} \frac{1}{(F^*_1 - F^*_{n+1})^k} = \frac{(2 + i) (F^*_n - F^*_0)}{\triangle (F^*_1 - F^*_{n+1})^2},
\end{align*}
\]
and

\[ a_j = K_{n,n-j+3} - K_{n,n-j+4} - K_{n,n-j+5} = \]

\[
\frac{2 + i}{\Delta} \sum_{k=1}^{j-2} \frac{(F_n - F_0)^{k-1}}{(F_{j-k} - F_{n+1})^k} + \frac{2 + i}{\Delta} \sum_{k=1}^{j-3} \frac{(F_n - F_0)^{k-1}}{(F_{j-k-1} - F_{n+1})^k} + \frac{2 + i}{\Delta} \sum_{k=1}^{j-4} \frac{(F_n - F_0)^{k-1}}{(F_{j-k-2} - F_{n+1})^k} = \frac{(2 + i) (F_n - F_0)^{j-3}}{\Delta (F_{j-1} - F_{n+1})^{j-2}}
\]

for \(5 \leq j \leq n\).

3 Conclusion

In this paper we deal with circulant matrices whose entries are complex Fibonacci numbers. In a recent study [5] similar results have been presented. We have calculated the determinant of \(\mathcal{F}_n\) by a similar method in [2, 3, 5, 9] but we have used slightly different method to show the invertibility of \(\mathcal{F}_n\). Moreover, our paper shows that above results can be presented for complex Lucas numbers, complex Jacobsthal numbers and their relatives. We have left them for a future work.

References