On the determinants of some kinds of circulant-type matrices with generalized number sequences

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Abstract: Recently, determinant computation of circulant type matrices with well-known number sequences has been studied, extensively. This study provides the determinants of the RFMLR, RLMFL, RFPRLrR and RLPFPrL circulant matrices with generalized number sequences of second order.

1 Introduction

Circulant type matrices have applications in many fields of science such as algebraic coding theory, number theory, graph theory, acoustics, numeric analysis and so on. Also, these type matrices are used in signal processing, image processing and linear equation systems.

Recently, a new class of circulant matrix family, called RFMLR, RLMFL, RFPRLrR and RLPFPrL circulant matrices, has been studied, mostly.

Definition 1. [7] A row first-minus-last right (RFMLR) n-square circulant matrix is defined as below

\[ R = RFMLRcircfr(c_1, c_2, \ldots, c_n) = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \\ c_n & c_1 - c_n & \cdots & c_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_3 & c_4 - c_3 & \cdots & c_2 \\ c_2 & c_3 - c_2 & \cdots & c_1 - c_n \end{pmatrix} \]

(1)

Definition 2. [7] A row last-minus-first left (RLMFL) circulant matrix is defined as below:

\[ S = RLMFLcircfr(c_1, c_2, \ldots, c_n) = \begin{pmatrix} c_1 & \cdots & c_{n-1} & c_n \\ c_2 & \cdots & c_n - c_1 & c_1 \\ \vdots & \ddots & \vdots & \vdots \\ c_{n-1} & \cdots & c_{n-3} - c_{n-2} & c_{n-2} \\ c_n - c_1 & \cdots & c_{n-2} - c_{n-1} & c_{n-1} \end{pmatrix} \]

(2)

As it can be seen, RFMLR and RLMFL circulant matrices have a conventional first row and any other row derived from the previous one with the following process, respectively: Get the \((i + 1)\)-st row by subtracting the last (first) element of the \(i\)-th row from the first (last) element of the \(i\)-th row, and then shifting the elements of the \(i\)-th row cyclically one position to the right (left). Clearly, the RFMLR and RLMFL circulant matrices are obtained by not only its first row but also its any row [7].
**Definition 3.** [11] A row first-right r-right (RFP{LR}) circulant matrix with the first row \((a_1, a_2, \ldots, a_n)\) denoted by \(RFP{LR}cir{fr}(a_1, a_2, \ldots, a_n)\), is an n-squared matrix given with the following form

\[
D = \begin{pmatrix}
a_1 & a_2 & \cdots & a_n \\
ra_n & a_1 + ra_n & \cdots & a_{n-1} \\
ra_{n-1} & ra_n + ra_{n-1} & \cdots & a_{n-2} \\
\cdots & \cdots & \cdots & \cdots \\
ra_2 & ra_3 + ra_2 & \cdots & a_1 + ra_n
\end{pmatrix}.
\]

**Definition 4.** [11] A row last-plus-first r-left (RFP{FrL}) circulant matrix with the first row \((a_1, a_2, \ldots, a_n)\), denoted by \(RFP{FrL}cir{fr}(a_1, a_2, \ldots, a_n)\), is an n-squared matrix given with the following form

\[
L = \begin{pmatrix}
a_1 & \cdots & a_{n-1} & a_n \\
a_2 & \cdots & a_n + ra_1 & ra_1 \\
a_3 & \cdots & ra_1 + ra_2 & ra_2 \\
\cdots & \cdots & \cdots & \cdots \\
a_n + ra_1 & \cdots & ra_{n-2} + ra_{n-1} & ra_{n-1}
\end{pmatrix}.
\]

Obviously, the RFP{LR} and RFP{FrL} circulant matrices are obtained by its first row for any \(r\).

In literature, there is huge amount of papers on applications of circulant type matrices with some famous number sequences. For example, Shen et al. gave determinants and inverses of circulant matrices with Fibonacci and Lucas numbers [1]. Bozkurt and Tam considered determinants and inverses of circulant matrices with Jacobsthal and Jacobsthal-Lucas numbers [2]. Moreover, Bozkurt obtained determinants and inverses of circulant matrices with a generalized number sequences [3].

The two-term generalized number sequence \(\{W_n\}\) is defined by

\[
W_n = pW_{n-1} - qW_{n-2}, \quad n \geq 2
\]

with initial conditions \(W_0 = a\) and \(W_1 = b\), here \(a, b, p, q \in \mathbb{Z}\) [3, 4]. Let \(\alpha\) and \(\beta\) be the roots of \(x^2 - px + q = 0\). Then, the Binet formula of generalized number sequence \(\{W_n\}\) is

\[
W_n = \frac{A\alpha^n + B\beta^n}{\alpha - \beta},
\]

where \(A = b - a\beta, B = a\alpha - b, \alpha + \beta = p, a\beta = q\) and \(\alpha - \beta = \sqrt{p^2 - 4q}\).

Jiang et al. obtained simple formulas for determinants of RFMLR circulant matrices with Perrin, Padovan, Tribonacci and Generalized Lucas numbers [6]. Shen et al. acquired formulas for determinants of the RFMLR and RLMFL circulant matrices involving certain famous numbers such as Fibonacci, Lucas, Pell and Pell-Lucas numbers [5]. Moreover, the authors [5] defined basic RFMLR circulant matrix \(\otimes_{(1,-1)}\) as the following form

\[
\otimes_{(1,-1)} = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 \\
1 & -1 & 0 & \cdots & 0
\end{pmatrix}.
\]

Characteristic polynomial of \(\otimes_{(1,-1)}\) is \(h(x) = x^n + x - 1\) which has only simple roots, given by \(x_i (i = 1, 2, \ldots, n)\). Here \(\otimes_{(1,-1)}\) is equal to \(\mathbf{I}_n - \otimes_{(1,-1)}[5]\). So, a matrix \(R\) can be written as below:

\[
R = f(\otimes_{(1,-1)}) = \sum_{i=1}^{n} c_i \otimes_{(1,-1)}^{i-1}
\]

Further, \(\otimes_{(1,-1)}\) is equal to \(\mathbf{I}_n - \otimes_{(1,-1)}\)[5]. So, a matrix \(R\) can be written as below:
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if and only if $R$ is a RFMLR circulant matrix [5]. Here $f(x) = \sum_{i=1}^{n} c_{i} x^{i-1}$ is called the representer of RFMLR circulant matrix $R$ [5]. Let $A$ and $B$ be RFMLR circulant matrices of $n$-order. Then, $AB$ and $A^{-1}$ matrices are RFMLR circulant matrices [5].

Let define $M = RLMFLcircfr(c_1, \ldots, c_n)$ and $N = RFMLRcircfr(c_n, \ldots, c_1)$. Then, Shen et al. [5] found

$$M = NJ_n$$

(9)

here $J_n$ is backward identity matrix form

$$J_n = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}.$$  

(10)

Xu et al. obtained formulas for RFP$_r$L$_r$R and RLP$_r$Fr$L$ circulant matrices with some famous numbers such as Fibonacci, Lucas, Pell and Pell-Lucas numbers [11]. Moreover, the authors [11] defined basic RFP$_r$L$_r$R circulant matrix $\ominus_{(r,r)}$ as the following form

$$\ominus_{(r,r)} = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
r & r & 0 & \cdots & 0
\end{pmatrix}.$$  

(11)

Characteristic polynomial of $\ominus_{(r,r)}$ is $w(x) = x^n - rx - r$, which has only simple roots, given by $\epsilon_k (k = 1, 2, \ldots, n)$. Here $\ominus_{(r,r)}^m$ is equal to $RFP_cL_cRcircfr(0,0,\ldots,0,1,0,\ldots,0)$. Further, $\ominus_{(r,r)}^n$ is equal to $RI_n - r\ominus_{(r,r)}$ [11]. So, a matrix $D$ can be written as below:

$$D = \mathcal{S}(\ominus_{(r,r)}) = \sum_{i=1}^{n} a_i \ominus_{(r,r)}^{i-1}$$

(12)

if and only if $D$ is a RFP$_r$L$_r$R circulant matrix [11]. Here $\mathcal{S}(x) = \sum_{i=1}^{n} c_{i} x^{i-1}$ is called the representer of RFP$_r$L$_r$R circulant matrix $D$ [11]. Let $P$ and $T$ be RFP$_r$L$_r$R circulant matrices of $n$-order. Then, $PT$ and $P^{-1}$ matrices are RFMr$L_r$R circulant matrices [11].

Let define $L = RLP_cF_ccircfr(a_1, \ldots, a_n)$ and $U = RFP_rL_circfr(a_n, \ldots, a_1)$. Then, Xu et al. [11] found

$$L = UJ_n.$$  

(13)

here $J_n$ is defined by (10).

Determinant is one of the basic rules in mathematics and it is useful for solving many problems. Few methods are known for computing determinant but much of them need a plent of procedure. For example, Gauss elimination method needs about $2n^3/3$ arithmetic steps for a matrix of order $n$. Therefore, determinant computation has been considered for special matrices with special entries.
2 On determinants of the RFMLR and RLMFL circulant matrices with Generalized number sequences

In this section, we give formulas for determinants of RFMLR and RLMFL circulant matrices with two-term generalized number sequences.

Lemma 5. [5] The eigenvalues of $R = RFMLR_{cfr}(c_1, c_2, \ldots, c_n)$ are given by

$$\lambda_i = f(x_i) = \sum_{k=1}^{n} c_k x_i^{k-1}, \quad i = 1, 2, \ldots, n.$$  

So, the determinants of $R$ matrix is given by

$$\det R = \prod_{i=1}^{n} \sum_{k=1}^{n} c_k x_i^{k-1}, \quad (14)$$

where $x_i$'s ($i = 1, 2, \ldots, n$) are the roots of the characteristic polynomial of $\ominus(1, -1)$.

Lemma 6. [5] Let $x_i$ ($i = 1, 2, \ldots, n$) be the roots of the characteristic polynomial of $\ominus(1, -1)$. So,

$$\prod_{i=1}^{n} (ax_i^2 + bx_i + c) = c^n + a^{-1}(a + b + c) + c(y_1^{n-1} + y_2^{n-1}) - (y_1^n + y_2^n), \quad (15)$$

where $a, b, c$ are real numbers, and

$$y_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2}; \quad y_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2}. \quad (16)$$

Let us define $E_n = RFMLR_{cfr}(W_1, W_2, \ldots, W_n)$ for $n \geq 3$, $p^2 - 4q \geq 0$ and $a, b \geq 0$. Then

$$\det E_n = \frac{(b - W_{n+1})^n + (-qW_n)^{n-1}(b - qa)}{1 + q^{n-1}(1 + q - p) + (z_1^{n-1} + z_2^{n-1}) - (z_1^n + z_2^n)} + \frac{(b - W_{n+1})(y_1^{n-1} + y_2^{n-1}) - (y_1^n + y_2^n)}{1 + q^{n-1}(1 + q - p) + (z_1^{n-1} + z_2^{n-1}) - (z_1^n + z_2^n)}$$

where $W_n$ given by (5),

$$y_{1,2} = \frac{(qa - qW_n - W_{n+1}) \mp \sqrt{(W_{n+1} + qW_n - qa)^2 + 4(b - W_{n+1})(qW_n)}}{2} \quad (17)$$

$$z_{1,2} = \frac{p \mp \sqrt{p^2 - 4q}}{2}. \quad (18)$$

Proof. If $n = 1$, then $\det E_1 = b$. For $n = 2$,

$$E_2 = \begin{pmatrix} W_1 & W_2 \\ W_2 & W_1 - W_2 \end{pmatrix} = \begin{pmatrix} b & pb - qa \\ pb - qa & b(1 - p) + qa \end{pmatrix}.$$
So, \( \det E_2 = b^2(1 - p) + qab - (pb - qa)^2 \). For \( n \geq 3 \), by using (6) and Lemma 5, we obtain

\[
\det E_n = \prod_{i=1}^{n}(W_1 + W_2x_i + \cdots + W_nx_i^{n-1})
\]

\[
= \prod_{i=1}^{n}\left( \frac{Aa + B\beta}{\alpha - \beta} + \frac{Aa^2 + B\beta^2}{\alpha - \beta}x_i + \cdots + \frac{Aa^n + B\beta^n}{\alpha - \beta}x_i^{n-1} \right)
\]

\[
= \prod_{i=1}^{n}\frac{Aa}{\alpha - \beta} \left( 1 - (ax_i)^n \right) + \frac{B\beta}{\alpha - \beta} \left( 1 - (bx_i)^n \right)
\]

\[
= \prod_{i=1}^{n}\frac{Aa + B\beta - x_i^n(Aa^{n+1} + B\beta^{n+1}) - a\beta x_i(A + B) + a\beta x_i^{n+1}(Aa^n + B\beta^n)}{(\alpha - \beta)(1 - (a + \beta)x_i + a\beta x_i^2)}
\]

\[
= \prod_{i=1}^{n}\frac{W_1 - x_i^nW_{n+1} - x_iqW_0 + x_i^{n+1}qW_n}{1 - px_i + qx_i^2}.
\]

Because of \( x_i^n + x_i - 1 = 0 \), we obtain

\[
\det E_n = \prod_{i=1}^{n}\frac{(-qW_n)x_i^2 + (W_{n+1} + qW_n - qa)x_i + b - W_{n+1}}{1 - px_i + qx_i^2}.
\]

By Lemma 6, we have

\[
\prod_{i=1}^{n}(-qW_n)x_i^2 + (W_{n+1} + qW_n - qa)x_i + b - W_{n+1} = (b - W_{n+1})^n + (-qW_n)^{n-1}(b - qa)
\]

\[
+ (b - W_{n+1})(y_1^{n-1} + y_2^{n-1}) - (y_1^n + y_2^n)
\]

and

\[
\prod_{i=1}^{n}1 - px_i + qx_i^2 = 1 + q^n(1 + q - p) + (z_1^{n-1} + z_2^{n-1}) - (z_1^n + z_2^n)
\]

here \( y_1, y_2, z_1 \) and \( z_2 \) are defined by (17) and (18). So, the proof is completed. \( \square \)

**Theorem 7.** Let \( K_n = \text{RFMLRcircfr}(W_n, W_{n-1}, \ldots, W_1) \) for \( p^2 - 4q \geq 0 \) and \( a, b \geq 0 \). Then

\[
\det K_n = \frac{q^n(W_n - a)^n + (-b)^{n-1}(qW_n - W_{n+1})}{q^n + p - q + 1 + (u_1^{n-1} + u_2^{n-1}) - (u_1^n + u_2^n)} + \frac{q(W_n - a)(t_1^{n-1} + t_2^{n-1}) - (t_1^n + t_2^n)}{q^n + p - q + 1 + (u_1^{n-1} + u_2^{n-1}) - (u_1^n + u_2^n)}
\]

where

\[
t_{1,2} = \frac{(W_{n+1} - qa - b) \mp \sqrt{(qa + b - W_{n+1})^2 + 4bq(W_n - a)}}{2}
\]

and

\[
u_{1,2} = \frac{p \mp \sqrt{p^2 - 4q}}{2}.
\]

**Proof.** The matrix \( K_n \) has the following form

\[
K_n = \begin{pmatrix}
W_n & W_{n-1} & \cdots & W_1 \\
W_1 & W_n - W_1 & \cdots & W_2 \\
\vdots & \vdots & \ddots & \vdots \\
W_{n-2} & W_{n-3} - W_{n-2} & \cdots & W_{n-1} \\
W_{n-1} & W_{n-2} - W_{n-1} & \cdots & W_n - W_1
\end{pmatrix}
\]
Then by using (6) and Lemma 5, we obtain
\[
\det K_n = \prod_{i=1}^{n} (W_n + W_{n-1}x_i + \cdots + W_1x_i^{n-1})
\]
\[
= \prod_{i=1}^{n} \left( \frac{Aa^n + B\beta^n}{\alpha - \beta} + \frac{Aa^{n-1} + B\beta^{n-1}}{\alpha - \beta}x_i + \cdots + \frac{Aa + B\beta}{\alpha - \beta}x_i^{n-1} \right)
\]
\[
= \prod_{i=1}^{n} \frac{-bx_i^2 + (qa + b - W_{n+1})x_i + q(W_n - a)}{x_i^2 - (\alpha + \beta)x_i + a\beta}.
\]

By Lemma 6, we have
\[
\prod_{i=1}^{n} -bx_i^2 + (qa + b - W_{n+1})x_i + q(W_n - a) = q^n(W_n - a)^n + (-b)^{n-1}(qW_n - W_{n+1})
\]
\[
+ q(W_n - a)(t_1^{n-1} + t_2^{n-1}) - (t_1^n + t_2^n)
\]

and
\[
\prod_{i=1}^{n} x_i^2 - (\alpha + \beta)x_i + a\beta = \prod_{i=1}^{n} x_i^2 - px_i + q
\]
\[
= q^n + p - q + 1 + (u_1^{n-1} + u_2^{n-1}) - (u_1^n + u_2^n)
\]

here \(t_1, t_2, u_1\) and \(u_2\) are given by (19) and (20).

**Theorem 8.** Let \(G_n = RLMFL\circ r(W_1, W_2, \ldots, W_n)\) for \(p^2 - 4q \geq 0\) and \(a, b \geq 0\). Then
\[
\det G_n = \begin{cases} 
\det K_n, & n \equiv 0, 1 \pmod{4} \\
-\det K_n, & n \equiv 2, 3 \pmod{4} 
\end{cases}
\]

where the matrix \(K_n\) is defined as (21).

**Proof.** The matrix \(G_n\) has the following form:
\[
G_n = \begin{pmatrix}
W_1 & \cdots & W_{n-1} & W_n \\
W_2 & \cdots & W_{n-1} & W_n \\
\vdots & \ddots & \vdots & \vdots \\
W_{n-1} & \cdots & W_{n-2} & W_{n-2} \\
W_n - W_1 & \cdots & W_{n-2} - W_{n-1} & W_{n-1}
\end{pmatrix}
\]

From (9), we can write
\[
\det G_n = \det(K_nJ_n) = \det K_n \det J_n
\]

here the matrix \(J_n\) is defined by (10) and
\[
\det J_n = (-1)^{\frac{n(n-1)}{2}}.
\]

So,
\[
\det G_n = (-1)^{\frac{n(n-1)}{2}} \det K_n.
\]

Consequently,
\[
\det G_n = \begin{cases} 
\det K_n, & n \equiv 0, 1 \pmod{4} \\
-\det K_n, & n \equiv 2, 3 \pmod{4} 
\end{cases}
\]

which is desired.
3 On determinants of the RFPrLrR and RLPrFrL circulant matrices with Generalized number sequences

In this section, we give formulas for determinants of RFPrLrR and RLPrFrL circulant matrices with two-term generalized number sequences.

Lemma 9. [11] The eigenvalues of \( U = RFrLrcirc_{cr}(a_1, a_2, \ldots, a_n) \) are given by

\[
\lambda_i = f(\varepsilon_k) = \sum_{i=1}^{n-1} a_i \varepsilon_k^{-i}, \quad i = 1, 2, \ldots, n.
\]

So, the determinants of \( U \) matrix is given by

\[
\det U = \prod_{k=1}^{n} \left( a_i \varepsilon_k^{-i} \right).
\]  \hfill (22)

Lemma 10. [11] Let \( \varepsilon_k \) (\( k = 1, 2, \ldots, n \)) be the roots of the characteristic polynomial of \( \ominus_{(r,t)} \), So,

\[
\prod_{k=1}^{n} (a \varepsilon_k^2 + b \varepsilon_k + c) = e^n - re[(as)^n + (at)^{n-1}] - r[(as)^n + (at)^n] + r^2(c - b + a)
\]  \hfill (23)

here \( a, b, c \) are real numbers, and

\[
s = \frac{-b + \sqrt{b^2 - 4ac}}{2a}; \quad t = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.
\]  \hfill (24)

Theorem 11. Let us define \( F_n = RFPrLrRcirc_{cr}(W_1, W_2, \ldots, W_n) \) for \( n \geq 3, r, q, W_n \neq 0, p^2 - 4q \geq 0 \) and \( a, b \geq 0 \). Then

\[
\det F_n = \frac{(b - rW_{n+1})^n - (rqW_n)^n - (br - r^2W_{n+1})(h_1^{n-1} + h_2^{n-1})}{1 + q^{n-1}(1 + p)(z_1^{n-1} + z_2^{n-1}) - (z_1^n + z_2^n)} + \frac{r^2(qrW_n)^{n-1}(qa + b) - r(qrW_n)^n(h_1^n + h_2^n)}{1 - qr^{n-1}[l_1^{n-1} + l_2^{n-1}] - q(l_1^n + l_2^n) + p + q + 1}
\]

where \( W_n \) given by \( (5) \),

\[
h_{1,2} = \frac{(qa - qrW_n + rW_{n+1}) + \sqrt{(qrW_n - qa - rW_{n+1})^2 - 4qrW_n(b - rW_{n+1})}}{2qrW_n}
\]  \hfill (25)

and

\[
l_{1,2} = \frac{p + \sqrt{p^2 - 4q}}{2q}.
\]  \hfill (26)

Proof. For \( n \geq 3 \), by using \( (6) \) and Lemma 10, we obtain

\[
\det F_n = \prod_{k=1}^{n} \left( W_1 + W_2 \varepsilon_k + \cdots + W_n \varepsilon_k^{n-1} \right)
\]

\[
= \prod_{i=1}^{n} \left( \frac{Aa + Bb}{a - \beta} + \frac{Aa^2 + Bb^2}{a - \beta} \varepsilon_k + \cdots + \frac{Aa^n + Bb^n}{a - \beta} \varepsilon_k^{n-1} \right)
\]

\[
= \prod_{i=1}^{n} \left( (rqW_n) \varepsilon_k^2 + (-rW_{n+1} + qrW_n - qa) \varepsilon_k + b - rW_{n+1} \right)
\]

By Lemma 11, we have

\[
\prod_{i=1}^{n} \left( (rqW_n) \varepsilon_k^2 + (-rW_{n+1} + qrW_n - qa) \varepsilon_k + b - rW_{n+1} \right) = (b - rW_{n+1})^n - (rqW_n)^n - (br - r^2W_{n+1})(h_1^{n-1} + h_2^{n-1}) + r^2(qrW_n)^{n-1}(qa + b) - r(qrW_n)^n(h_1^n + h_2^n)
\]

\[
+ r^2(qrW_n)^{n-1}(qa + b) - r(qrW_n)^n(h_1^n + h_2^n)
\]
Then by using (6) and Lemma 10, we obtain
\[
\prod_{i=1}^{n} a\beta e_k^n - (\alpha + \beta)e_k + 1 = \prod_{i=1}^{n} q(\alpha^n + p\epsilon_k + 1)
\]
where
\[
\prod_{i=1}^{n} q(\alpha^n + p\epsilon_k + 1) = 1 - rq^{n-1}[l_1^{n-1} + l_2^{n-1} - q(l_1^n + l_2^n) + p + q + 1]
\]
here \(h_1, h_2, l_1\) and \(l_2\) are defined by (25) and (26). So, this proof is completed. \(\square\)

**Theorem 12.** Let \(Y_n = RFPrLRcirc r(W_n, W_{n-1}, \ldots, W_1)\) for \(r, b \neq 0, p^2 - 4q \geq 0\) and \(a, b \geq 0\). Then
\[
\det Y_n = \frac{g_1(27)}{2rb}
\]
where
\[
g_{1,2} = (W_{n+1} + qra - rb) \pm \sqrt{(rb - qra - W_{n+1})^2 + 4rbq(W_n - ra)}
\]
and
\[
n_{1,2} = p \mp \sqrt{p^2 - 4q}.
\]

**Proof.** The matrix \(Y_n\) has the following form
\[
Y_n = \begin{pmatrix}
W_n & W_{n-1} & \cdots & W_1 \\
rW_1 & W_n + rW_1 & \cdots & W_2 \\
\vdots & \vdots & \ddots & \vdots \\
rW_{n-2} & rW_{n-3} + rW_{n-2} & \cdots & W_{n-1} \\
rW_{n-1} & rW_{n-2} + rW_{n-1} & \cdots & W_n + rW_1
\end{pmatrix}.
\]

Then by using (6) and Lemma 10, we obtain
\[
\det Y_n = \prod_{k=1}^{n} (W_n + W_{n-1}e_k + \cdots + W_1e_k^{n-1})
\]

\[
= \prod_{i=1}^{n} \left( \frac{A\alpha^n + B\beta^n}{\alpha - \beta} + \frac{A\alpha^{n-1} + B\beta^{n-1}}{\alpha - \beta} e_k + \cdots + \frac{A\alpha + B\beta}{\alpha - \beta} e_k^{n-1} \right)
\]

\[
= \prod_{i=1}^{n} \frac{rb\epsilon_k^n + (rb - qra - W_{n+1})e_k + q(W_n - ra)}{e_k^n + (\alpha + \beta)e_k + \alpha\beta}
\]

By Lemma 11, we have
\[
\prod_{i=1}^{n} rb\epsilon_k^n + (rb - qra - W_{n+1})e_k + q(W_n - ra) = q^n(W_n - ra)^n + r^n(\alpha^n + \beta^n) - rq(W_n - ra)(\alpha^{n-1} + \beta^{n-1}) - r(rb)^n(g_1^n + g_2^n)
\]
and
\[
\prod_{i=1}^{n} \frac{rb\epsilon_k^n + (rb - qra - W_{n+1})e_k + q(W_n - ra)}{e_k^n + (\alpha + \beta)e_k + \alpha\beta}
\]
\[
= q^n - rq(\alpha^{n-1} + \beta^{n-1}) - r(\alpha^n + \beta^n) + p + q + 1
\]
here \(g_1, g_2, v_1\) and \(v_2\) are given by (27) and (28). So this proof is completed. \(\square\)
Theorem 13. Let $H_n = RLPrFL_{circ}(W_1, W_2, \ldots, W_n)$ for $p^2 - 4q \geq 0$ and $a, b \geq 0$. Then
\[\det H_n = \begin{cases} 
\det Y_n, & n \equiv 0, 1 \pmod{4} \\
-\det Y_n, & n \equiv 2, 3 \pmod{4}
\end{cases}\]
where the matrix $Y_n$ is defined as (29).

Proof. The matrix $H_n$ has the following form:
\[
H_n = \begin{pmatrix}
W_1 & \cdots & W_{n-1} & W_n \\
W_2 & \cdots & W_n + rW_1 & rW_1 \\
\vdots & \ddots & \vdots & \vdots \\
W_{n-1} & \cdots & rW_{n-3} + rW_{n-2} & rW_{n-2} \\
W_n + rW_1 & \cdots & rW_{n-2} + rW_{n-1} & rW_{n-1}
\end{pmatrix}.
\]
From (13), we can write
\[\det H_n = \det(Y_n J_n) = \det Y_n \det J_n\]
here the matrix $J_n$ is defined by (10) and
\[\det J_n = (-1)^{\frac{n(n-1)}{2}}.\]
So,
\[\det H_n = (-1)^{\frac{n(n-1)}{2}} \det Y_n.\]
Consequently,
\[\det H_n = \begin{cases} 
\det Y_n, & n \equiv 0, 1 \pmod{4} \\
-\det Y_n, & n \equiv 2, 3 \pmod{4}
\end{cases}\]
Therefore, this proof is completed. \qed

References


