Orthogonal diagonalization for complex skew-persymmetric anti-tridiagonal Hankel matrices

DOI 10.1515/spma-2016-0008
Received May 13, 2015; accepted November 12, 2015

Abstract: In this paper, we obtain an eigenvalue decomposition for any complex skew-persymmetric anti-tridiagonal Hankel matrix where the eigenvector matrix is orthogonal.

Keywords: Anti-tridiagonal matrices; Hankel matrices; orthogonal diagonalization; skew-persymmetric matrices.

1 Introduction

An $n \times n$ complex anti-tridiagonal Hankel matrix is a matrix of the form

$$\text{antitridiag}_n(a_1, a_0, a_{-1}) := \begin{pmatrix}
0 & \cdots & \cdots & 0 & a_1 & a_0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_1 & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_0 & a_{-1} & 0 & \cdots & \cdots & 0
\end{pmatrix}, \quad n \in \mathbb{N}, \quad a_1, a_0, -a_1 \in \mathbb{C},$$

where $\mathbb{N}$ is the set of natural numbers (that is, the set of positive integers) and $\mathbb{C}$ denotes the set of (finite) complex numbers.

In [1, Theorem 4] an orthogonal diagonalization for any real persymmetric anti-tridiagonal Hankel matrix was presented. The following result, which was given in [2, Theorem 5 and Lemma 1], is a generalization of [1, Theorem 4] to complex matrices ([3] is another recent reference where the eigenvalues of certain persymmetric anti-tridiagonal Hankel matrices are given).

Theorem 1. Let $n \in \mathbb{N}$ and $a_1, a_0 \in \mathbb{C}$. Then

$$\text{antitridiag}_n(a_1, a_0, a_{-1}) = Y_n \text{diag}(\tau_1, \ldots, \tau_n) Y_n^{-1},$$

where $\text{diag}(\tau_1, \ldots, \tau_n) := (\tau_j \delta_{j,k})_{j,k=1}^n$, with $\delta$ being the Kronecker delta and

$$\tau_j = (-1)^{j+1} \left( a_0 + 2a_1 \cos \frac{j\pi}{n+1} \right), \quad j \in \{1, \ldots, n\},$$

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and $Y_n$ is the $n \times n$ real symmetric orthogonal matrix whose entries are given by
\[
[Y_n]_{j,k} = \sqrt{\frac{2}{n+1}} \sin \frac{j\pi n}{n+1}, \quad j, k \in \{1, \ldots, n\}.
\]

In the present paper we give an orthogonal diagonalization for any complex skew-persymmetric anti-tridiagonal Hankel matrix, i.e., for any matrix of the form $\text{antitridiag}_n(a, 0, -a)$ with $a \in \mathbb{C}$ and $n \in \mathbb{N}$.

## 2 Orthogonal diagonalization

We first give an orthogonal diagonalization for any complex skew-persymmetric anti-tridiagonal Hankel matrix of odd order.

**Theorem 2.** Consider $a \in \mathbb{C}$ and $n = 2p + 1$ with $p \in \mathbb{N} \cup \{0\}$. Let $Y_n$ be as in Theorem 1. Then
\[
\text{antitridiag}_n(a, 0, -a) = U_n \text{diag}(\lambda_1, \ldots, \lambda_n) U_n^{-1},
\]
where
\[
\lambda_j = -2a \cos \frac{j\pi}{n+1}, \quad j \in \{1, \ldots, n\},
\]
and $U_n$ is the $n \times n$ real orthogonal matrix given by
\[
[U_n]_{j,k} = s_{j,k} [Y_n]_{j,k}, \quad j, k \in \{1, \ldots, n\},
\]
with
\[
s_{j,k} = \begin{cases} 
(-1)^{\frac{j+1}{2}} & \text{if } j \text{ is odd}, \\
(-1)^{\frac{j+1+n+2k}{2}} & \text{if } j \text{ is even}, 
\end{cases} \quad j, k \in \{1, \ldots, n\}. \quad (1)
\]

**Proof.** See A. \hfill \Box

We now give an orthogonal diagonalization for any complex skew-persymmetric anti-tridiagonal Hankel matrix of even order.

**Theorem 3.** Consider $a \in \mathbb{C}$ and $n = 2p$ with $p \in \mathbb{N}$. Let $Y_n$ be as in Theorem 1. Then
\[
\text{antitridiag}_n(a, 0, -a) = U_n \text{diag}(\lambda_1, \ldots, \lambda_n) U_n^{-1},
\]
where
\[
\lambda_j = -2a \cos \frac{j\pi}{n+1}, \quad j \in \{1, \ldots, n\},
\]
and $U_n$ is the $n \times n$ real orthogonal matrix given by
\[
[U_n]_{j,k} = \sqrt{2} s_{j,k} [Y_n]_{j,k}, \quad j, k \in \{1, \ldots, n\},
\]
with
\[
s_{j,k} = \begin{cases} 
(-1)^{\frac{j+1}{2}} (-1)^{\frac{n+2k}{4}} & \text{if } j \text{ is odd}, \\
(-1)^{\frac{j+1+n+2k}{2}} (-1)^{\frac{n+2k}{2}} & \text{if } j \text{ is even}, 
\end{cases} \quad j, k \in \{1, \ldots, n\}. \quad (2)
\]

**Proof.** See B. \hfill \Box

In [4, 5] Rimas gave a diagonalization for the matrix $\text{antitridiag}_n(1, 0, -1)$ of the form
\[
\text{antitridiag}_n(1, 0, -1) = W_n \text{diag} \left( -2 \cos \frac{\pi}{n+1}, -2 \cos \frac{2\pi}{n+1}, \ldots, -2 \cos \frac{n\pi}{n+1} \right) W_n^{-1}, \quad (3)
\]
where the eigenvector matrix $W_n$ is real and not orthogonal. Although $W_n$ is not orthogonal it satisfies

$$[W_n^T \ W_n]_{j,k} = 0$$

whenever $j, k \in \{1, \ldots, n\}$ and $j \neq k$, because eigenvectors of a real symmetric matrix corresponding to different eigenvalues are orthogonal (see, e.g., [6, p. 548]).

From (3) we have

$$\text{antitridiag}_n(a, 0, -a) = W_n \text{diag}(\lambda_1, \ldots, \lambda_n) W_n^{-1}$$

for all $a \in \mathbb{C}$. Thus, the difference between the diagonalization in (4) and the one given in Theorems 2 and 3 for the matrix antitridiag$_n(a, 0, -a)$ lies in the eigenvector matrix. Since the algebraic multiplicity of $\lambda_k$ is 1 with $a \neq 0$, its geometric multiplicity is also 1 (see, e.g., [7, Section 4.12]), and consequently, there exists $r_k \in \mathbb{R}$ such that

$$[U_n]_{1:n,k} = r_k [W_n]_{1:n,k}, \quad k \in \{1, \ldots, n\},$$

where $[U_n]_{1:n,k}$ and $[W_n]_{1:n,k}$ denote the $k$th column of the matrix $U_n$ and $W_n$, respectively. Therefore,

$$1 = [I_n]_{k,k} = [U_n^T \ U_n]_{k,k} = [U_n]_{1:n,k}^T [U_n]_{1:n,k} = r_k^2 [W_n]_{1:n,k}^T [W_n]_{1:n,k} = r_k^2 \|[W_n]_{1:n,k}\|^2_2,$$

and hence,

$$|r_k| = \frac{1}{\|[W_n]_{1:n,k}\|^2_2}, \quad k \in \{1, \ldots, n\},$$

where $\| \cdot \|_2$ denotes the Euclidean norm. Thus,

$$[U_n]_{1:n,k} \in \left\{ \frac{[W_n]_{1:n,k}}{\|[W_n]_{1:n,k}\|^2_2}, -\frac{[W_n]_{1:n,k}}{\|[W_n]_{1:n,k}\|^2_2} \right\}, \quad k \in \{1, \ldots, n\}.$$  (5)

Observe that to prove (5) is another way to prove Theorems 2 and 3.

We finish by mentioning that Theorems 2 and 3 are useful to obtain the natural powers of the considered matrix antitridiag$_n(a, 0, -a)$:

$$(\text{antitridiag}_n(a, 0, -a))^q = U_n \text{diag}(\lambda_1^q, \ldots, \lambda_n^q) U_n^T, \quad q \in \mathbb{N}.$$  

The natural powers of complex skew-persymmetric anti-tridiagonal Hankel matrices were also studied in [4, 5, 8, 9]. In [8] the expression for those powers was obtained by using the eigenvalue decomposition of the complex tridiagonal Toeplitz matrices presented in [10].

**Acknowledgement:** The authors thank the two anonymous reviewers for their helpful comments.

This work was supported in part by the Spanish Ministry of Economy and Competitiveness through the RACHEL project (TEC2013-47141-C4-2-R).

## A Proof of Theorem 2

We divide the proof into two steps. In the first step we prove that $U_n$ is an orthogonal matrix, or equivalently, that $U_n^T U_n = I_n$, where $\top$ denotes transpose and $I_n$ is the $n \times n$ identity matrix. For convenience we rewrite (1) as

$$s_{j,k} = (-1)^{j+1} \left((-1)^{\frac{1}{2(n-k)}}\right)^{j+1}, \quad j, k \in \{1, \ldots, n\},$$

where $[x]$ denotes the smallest integer not less than $x$. Applying the basic trigonometric formula $-2 \sin x \sin y = \cos(x + y) - \cos(x - y)$ (see, e.g., [11, p. 97]) yields

$$[U_n^T \ U_n]_{j,k} = \sum_{h=1}^n [U_n^T]_{j,h} [U_n]_{h,k} = \sum_{h=1}^n [U_n]_{h,j} [U_n]_{h,k}.$$
\[ \sum_{h=1}^{n} (-1)^{j+k} \left( (-1)^{\frac{j+k}{2}} \right)^{h+1} [Y_n]_{h,j} (-1)^{j+k} \left( (-1)^{\frac{j+k}{2}} \right)^{h+1} [Y_n]_{h,k} \]

\[ = \frac{2}{n+1} \sum_{h=1}^{n} (-1)^{j+k} (h+1)^{\frac{h+1}{n+1}} \sin \frac{hj\pi}{n+1} \sin \frac{hk\pi}{n+1} \]

\[ = \frac{2}{n+1} \sum_{h=1}^{n} ((-1)^{j+k})^{h+1} \left[ -1 \frac{\cos (j+k)h\pi}{n+1} - \cos (j-k)h\pi \right] \]

\[ = \frac{(-1)^{j+k+1}}{n+1} \sum_{h=1}^{n} ((-1)^{j+k})^{h} \left[ \cos (j+k)h\pi - \cos (j-k)h\pi \right], \quad j, k \in \{1, \ldots, n\}. \]

Consequently, using

\[ \sum_{h=1}^{n} \cos \frac{mh\pi}{n+1} = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ n & \text{if } m \in \Omega, \\ -1 & \text{if } m \in 2N \setminus \Omega, \end{cases} \]

and

\[ \sum_{h=1}^{n} (-1)^{h} \cos \frac{mh\pi}{n+1} = \begin{cases} -\frac{1}{n+1} \sum_{h=1}^{n+1} \cos \frac{jh\pi}{n+1} & \text{if } j = k, \\ \frac{1}{n+1} \sum_{h=1}^{n+1} \cos \frac{jh\pi}{n+1} & \text{if } j \neq k \text{ and } j+k \text{ is even}, \\ 0 & \text{if } j \neq k \text{ and } j+k \text{ is odd}, \end{cases} \]

where \( m \in \mathbb{N} \cup \{0\} \) and \( \Omega = \{2(n+1)k : k \in \mathbb{N} \cup \{0\}\} \) (see [1, Eqs. (5) and (7)]), we obtain

\[ U_n^T U_n \]

In the second step we prove that antitridiag\( \alpha(0, -\alpha) = U_n D_n U_n^{-1}, \) or equivalently, antitridiag\( \alpha(0, -\alpha) = U_n D_n U_n^T, \) where \( D_n = \text{diag}(\lambda_1, \ldots, \lambda_n). \) Applying the basic trigonometric formulas \(-2 \sin x \sin y = \cos(x + y) - \cos(x - y)\) and \(\cos(x + y) = \cos x \cos y - \sin x \sin y\) (see, e.g., [11, pp. 96-97]) yields

\[ \cos \frac{h\pi}{n+1} \sin \frac{j\pi}{n+1} \sin \frac{k\pi}{n+1} = \cos \frac{h\pi}{n+1} \left[ \frac{1}{2} \left( \cos \frac{(j+k)h\pi}{n+1} - \cos \frac{(j-k)h\pi}{n+1} \right) - \frac{1}{2} \left( \cos \frac{(j+k)h\pi}{n+1} \cos \frac{h\pi}{n+1} - \cos \frac{(j-k)h\pi}{n+1} \cos \frac{h\pi}{n+1} \right) \right] \]

\[ = -\frac{1}{2} \left[ \cos \frac{(j+k+1)h\pi}{n+1} + \cos \frac{(j+k-1)h\pi}{n+1} \cos \frac{(j-k+1)h\pi}{n+1} - \cos \frac{(j-k+1)h\pi}{n+1} \right] \]

for all \( h, j, k \in \{1, \ldots, n\}, \) and hence,

\[ U_n D_n U_n^T \]

\[ = \sum_{h=1}^{n} \sum_{l=1}^{n} [U_n]_{h,l} [D_n]_{l,h} \left[ U_n^T \right]_{l,k} = \sum_{h=1}^{n} [U_n]_{h,k} \sum_{l=1}^{n} [D_n]_{l,h} \left[ U_n^T \right]_{l,k} = \sum_{h=1}^{n} [U_n]_{h,k} \sum_{l=1}^{n} [D_n]_{l,h} \left[ U_n^T \right]_{l,k} \]

\[ = \frac{-4a}{n+1} \sum_{h=1}^{n} (-1)^{j+k} \left( (-1)^{\frac{j+k}{2}} \right)^{h+1} \left( (-1)^{\frac{j+k}{2}} \right)^{h+1} \cos \frac{h\pi}{n+1} \sin \frac{j\pi}{n+1} \sin \frac{k\pi}{n+1} \]

\[ = \frac{4a}{n+1} \sum_{h=1}^{n} \left( (-1)^{j+k} \right)^{h} \cos \frac{h\pi}{n+1} \sin \frac{j\pi}{n+1} \sin \frac{k\pi}{n+1} \]

\[ = \frac{4a}{n+1} \sum_{h=1}^{n} \left( (-1)^{j+k} \right)^{h} \left[ \cos \frac{(j+k+1)h\pi}{n+1} + \cos \frac{(j+k-1)h\pi}{n+1} \cos \frac{(j-k+1)h\pi}{n+1} - \cos \frac{(j-k+1)h\pi}{n+1} \cos \frac{(j-k-1)h\pi}{n+1} \right] \]
where \( t_{j,k} = (-1)^{j+k} \left( \frac{n}{2} \right)^{j+k} \) and \( j, k \in \{1, \ldots, n\} \). Therefore, using (6) and (7) we conclude that

\[
\begin{bmatrix} U_n D_n U_n^\top \end{bmatrix}_{j,k} = \begin{cases} \frac{a_{ij}^k}{\pi} \left[ n + (-1)^{j+k} - (-1)^k \right] & \text{if } j + k = n, \\ \frac{a_{ij}^k}{\pi} \left[ (-1)^{j+k} n - (-1)^k \right] & \text{if } j + k = n + 2, \\ \frac{a_{ij}^k}{\pi} \left[ 0 + 0 - 0 \right] = 0 & \text{if } j + k \text{ is even}, \\ \frac{a_{ij}^k}{\pi} \left[ (-1)^{j+k} - (-1)^k \right] = 0 & \text{otherwise}, \end{cases}
\]

And

\[
A(-1)^{j+k} = \begin{cases} a(-1)^{j+k+1} = a & \text{if } j + k = n \text{ and } j \text{ is even}, \\ a(-1)^{j+k+1} = a & \text{if } j + k = n \text{ and } j \text{ is odd}, \\ a(-1)^{j+k+1} = a(-1)^{j+k+2} = -a & \text{if } j + k = n + 2 \text{ and } j \text{ is even}, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
\text{antitri diag}_n (a, 0, -a)_{j,k}, \quad j, k \in \{1, \ldots, n\}.
\]

\( \square \)

**B Proof of Theorem 3**

We divide the proof into two steps. In the first step we prove that \( U_n \) is an orthogonal matrix. For convenience we rewrite (2) as

\[
s_{j,k} = \frac{(-1)^{j+k}(1 + (-1)^{n/2})}{2}, \quad j, k \in \{1, \ldots, n\}.
\]

Applying (6) and (7) yields

\[
\begin{bmatrix} U_n^\top U_n \end{bmatrix}_{j,k} = \sum_{h=1}^{n} [U_n]_{h,j} [U_n]_{h,k}
\]

\[
= \frac{1}{n+1} \sum_{h=1}^{n} \left[ 1 + (-1)^{n/2} \right] \left[ 1 + (-1)^{n/2} \right] \text{sin} \frac{hj\pi}{n+1} \text{sin} \frac{hk\pi}{n+1}
\]

\[
= \frac{1}{n+1} \sum_{h=1}^{n} \left[ (1 + (-1)^{n/2}) \cos \frac{(j+k)h\pi}{n+1} - \cos \frac{(j-k)h\pi}{n+1} \right]
\]

\[
= \frac{1}{n+1} \sum_{h=1}^{n} \left[ (1 + (-1)^{n/2}) \cos \frac{(j+k)h\pi}{n+1} - \cos \frac{(j-k)h\pi}{n+1} \right]
\]

\[
= \frac{-1}{n+1} \sum_{h=1}^{n} \left[ \cos \frac{(j+k)h\pi}{n+1} - \cos \frac{(j-k)h\pi}{n+1} \right]
\]

\[
= \frac{-1}{n+1} \sum_{h=1}^{n} \left[ \cos \frac{(j+k)h\pi}{n+1} - \cos \frac{(j-k)h\pi}{n+1} \right]
\]

\[
= \begin{cases} \frac{-1}{n+1} [1 - n] - (-1)^{j+k} [0 - 0] = 1 & \text{if } j = k, \\ \frac{-1}{n+1} [1 - (-1)] - (-1)^{j+k} [0 - 0] = 0 & \text{if } j \neq k \text{ and } j + k \text{ is even}, \\ 0 - 0 = 0 & \text{if } j \neq k \text{ and } j + k \text{ is odd}, \end{cases}
\]

\[
= [I_n]_{j,k}, \quad j, k \in \{1, \ldots, n\}.
\]
In the second step we prove that antitridiag$_n(a, 0, -a) = U_nD_nU_n^T$, or equivalently, antitridiag$_n(a, 0, -a) = U_nD_n^TU_n^T$, where $D_n = \text{diag}(\lambda_1, \ldots, \lambda_n)$. We have

$$
\begin{align*}
\left[U_nD_nU_n^T\right]_{j,k} &= \sum_{h=1}^{n} [U_n]_{j,h}[D_n]_{h,k}[U_n]_{k,h} = \frac{(-1)^{\frac{j+k+1}{2}+\frac{j-k}{2}}} {2} \left(\frac{n+1}{2}\right) \left(1 + (-1)^{\frac{j+k}{2}+\frac{j-k}{2}}\right) \lambda_h[Y_n]_{j,h}[Y_n]_{k,h} \\
&= \frac{(-1)^{\frac{j+k+1}{2}+\frac{j-k}{2}}(1)} {2} \sum_{h=1}^{n} \left(1 + (-1)^{\frac{j+k}{2}} + (-1)^k\right) \lambda_h[Y_n]_{j,h}[Y_n]_{k,h} \\
&= \frac{(-1)^{\frac{j+k+1}{2}+\frac{j-k}{2}}(1)} {2} \left(\frac{n+1}{2}\right) \left(1 + (-1)^{\frac{j+k}{2}} + (-1)^k\right) \lambda_h[Y_n]_{j,h}[Y_n]_{k,h} \\
&= \frac{-2at_{j,k}} {2\cos(\frac{\pi}{n+1})\sin(\frac{j\pi}{n+1})\sin(\frac{k\pi}{n+1})} \sum_{h=1}^{n} \lambda_h[Y_n]_{j,h}[Y_n]_{k,h} \\
&= \frac{-2at_{j,k}} {2\cos(\frac{\pi}{n+1})} \sin(\frac{j\pi}{n+1})\sin(\frac{k\pi}{n+1})}, \quad j, k \in \{1, \ldots, n\}.
\end{align*}
$$

where $t_{j,k} = (-1)^{\frac{j+k+1}{2}+\frac{j-k}{2}}(1 + (-1)^{\frac{j+k}{2}})$, $r_{j,k} = (-1)^{\frac{j+k+1}{2}+\frac{j-k}{2}}(-1)^{\frac{j-k}{2}} (-1)^k$, and $j, k \in \{1, \ldots, n\}$. Using (6), (7), and (8) we conclude that

$$
\begin{align*}
\left[U_nD_nU_n^T\right]_{j,k} &= \frac{-2at_{j,k}} {2\cos(\frac{\pi}{n+1})} \sin(\frac{j\pi}{n+1})\sin(\frac{k\pi}{n+1})} \\
&= \left\{\begin{array}{ll}
\frac{\frac{(-1)^{\frac{j+k+1}{2}+\frac{j-k}{2}}(1)} {2}} {2\cos(\frac{\pi}{n+1})} & j + k = n, \\
\frac{\frac{(-1)^{\frac{j+k+1}{2}+\frac{j-k}{2}}(1)} {2}} {2\cos(\frac{\pi}{n+1})} & j + k = n + 2, \\
\frac{\frac{(-1)^{\frac{j+k+1}{2}+\frac{j-k}{2}}(1)} {2}} {2\cos(\frac{\pi}{n+1})} & j + k = n + 4, \\
0 & \text{otherwise,}
\end{array}\right.
\end{align*}
$$

$$
\left[\text{antitridiag}_n(a, 0, -a)\right]_{j,k}, \quad j, k \in \{1, \ldots, n\}.
$$

References


