Research Article

Ahmad N. Alkenani*, Mohammad Ashraf, and Aisha Jabeen

Nonlinear generalized Jordan $(\sigma, \tau)$-derivations on triangular algebras

https://doi.org/10.1515/spma-2017-0008
Received December 22, 2016; accepted April 22, 2017

Abstract: Let $\mathcal{R}$ be a commutative ring with identity element, $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over $\mathcal{R}$ and let $\mathcal{M}$ be $(\mathcal{A}, \mathcal{B})$-bimodule which is faithful as a left $\mathcal{A}$-module and also faithful as a right $\mathcal{B}$-module. Suppose that $\mathfrak{A} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is a triangular algebra which is 2-torsion free and $\sigma, \tau$ be automorphisms of $\mathfrak{A}$. A map $\delta : \mathfrak{A} \to \mathfrak{A}$ (not necessarily linear) is called a multiplicative generalized $(\sigma, \tau)$-derivation (resp. multiplicative generalized Jordan $(\sigma, \tau)$-derivation) on $\mathfrak{A}$ associated with a $(\sigma, \tau)$-derivation (resp. Jordan $(\sigma, \tau)$-derivation) $d$ on $\mathfrak{A}$ if $\delta(xy) = \delta(x)\tau(y) + \sigma(x)d(y)$ (resp. $\delta(x^2) = \delta(x)\tau(x) + \sigma(x)d(x)$) holds for all $x, y \in \mathfrak{A}$. In the present paper it is shown that if $\delta : \mathfrak{A} \to \mathfrak{A}$ is a multiplicative generalized Jordan $(\sigma, \tau)$-derivation on $\mathfrak{A}$, then $\delta$ is an additive generalized $(\sigma, \tau)$-derivation on $\mathfrak{A}$.

Keywords: Triangular algebra; generalized Jordan $(\sigma, \tau)$-derivation; generalized $(\sigma, \tau)$-derivation

MSC: 16W25, 15A78

1 Introduction

Throughout, $\mathcal{R}$ will denote a commutative ring with identity. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are unital algebras over $\mathcal{R}$ and $\mathcal{M}$ is a nonzero $(\mathcal{A}, \mathcal{B})$-bimodule. An $\mathcal{R}$-algebra

$$\mathfrak{A} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \mid a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B} \right\}$$

under the usual matrix operations is called a triangular algebra consisting of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{M}$. In 1960, Chase [2] introduced the concept of triangular ring and obtained several interesting results. In fact, by applying triangular ring Chase [2] constructed a classical example of a left semi-hereditary ring which is not a right semi-hereditary. Motivated by the existence of triangular ring, Cheung [3] initiated the study of various mappings on triangular algebras [4, 5].

An $\mathcal{R}$-linear map $d : \mathcal{A} \to \mathcal{A}$ is called a derivation (resp. Jordan derivation) on $\mathcal{A}$ if $d(ab) = d(a)b + ad(b)$ (resp. $d(a^2) = d(a)a + ad(a)$) holds for all $a, b \in \mathcal{A}$. An $\mathcal{R}$-linear map $\delta : \mathcal{A} \to \mathcal{A}$ is called a generalized derivation (resp. generalized Jordan derivation) on $\mathcal{A}$ associated with a derivation (resp. Jordan derivation) $d$ on $\mathcal{A}$ if $\delta(ab) = \delta(a)b + ad(b)$ (resp. $\delta(a^2) = \delta(a)a + ad(a)$) for all $a, b \in \mathcal{A}$. If the condition of linearity is dropped from the above definitions, then the corresponding maps are called a multiplicative derivation (resp. multiplicative Jordan derivation) and a multiplicative generalized derivation (resp. multiplicative generalized Jordan derivation) respectively.

The study of $(\sigma, \tau)$-derivation is an active research area in rings and algebras. Let $\sigma, \tau$ be two endomorphisms on an algebra $\mathcal{A}$. An $\mathcal{R}$-linear map $d : \mathcal{A} \to \mathcal{A}$ is called a $(\sigma, \tau)$-derivation (resp. Jordan $(\sigma, \tau)$-derivation) on $\mathcal{A}$ if $d(ab) = d(a)b + ad(b)$ (resp. $d(a^2) = d(a)a + ad(a)$) holds for all $a, b \in \mathcal{A}$.
derivation) on \( \mathcal{A} \) if
\[
d(xy) = d(x)\tau(y) + \sigma(x)d(y) \quad \text{(resp. } d(x^2) = d(x)\tau(x) + \sigma(x)d(x) \text{)}
\]
holds for all \( x, y \in \mathcal{A} \). An \( \mathbb{R} \)-linear map \( \delta : \mathcal{A} \to \mathcal{A} \) is called a generalized \((\sigma, \tau)\)-derivation (resp. generalized Jordan \((\sigma, \tau)\)-derivation) on \( \mathcal{A} \) associated with a \((\sigma, \tau)\)-derivation (resp. Jordan \((\sigma, \tau)\)-derivation) \( d \) on \( \mathcal{A} \) if
\[
d(xy) = \delta(x)y + \sigma(x)d(y) \quad \text{(resp. } d(x^2) = \delta(x)x + \sigma(x)d(x) \text{)}
\]
holds for all \( x, y \in \mathcal{A} \). Obviously, an \((I_{\mathcal{A}}, I_{\mathcal{A}})\)-derivation (resp. Jordan \((I_{\mathcal{A}}, I_{\mathcal{A}})\)-derivation) is a derivation (resp. Jordan derivation) on \( \mathcal{A} \), where \( I_{\mathcal{A}} \) is the identity map on \( \mathcal{A} \). If the condition of linearity is dropped from the above definitions, then the corresponding maps are called multiplicative \((\sigma, \tau)\)-derivation (resp. multiplicative generalized Jordan \((\sigma, \tau)\)-derivation) and multiplicative generalized \((\sigma, \tau)\)-derivation (resp. multiplicative generalized Jordan \((\sigma, \tau)\)-derivation) respectively.

It can be easily seen that every derivation on \( \mathcal{A} \) is a Jordan derivation on \( \mathcal{A} \) but the converse need not be true in general. Herstein [8] investigated the converse part and proved that every Jordan derivation on a prime ring of characteristic not 2 into itself is a derivation. This result was extended in different directions to various rings and algebras. In last few decades, the study of multiplicative mappings has attracted the attention of many authors. Firstly, Martindale [11] initiated the study of additivity of multiplicative bijective maps from a prime ring containing a nontrivial idempotent onto an arbitrary ring and showed that every multiplicative bijective map from a prime ring containing a nontrivial idempotent onto an arbitrary ring is necessarily additive. Daif [6] studied the additivity of multiplicative derivation on a 2-torsion free prime ring containing a nontrivial idempotent. Furthermore, Yu and Zhang [12] proved that every nonlinear Lie derivation of triangular algebras is the sum of an additive derivation and a map into its center sending commutators to zero. Recently, Han and Wei [7] studied the Jordan \((\sigma, \tau)\)-derivation on triangular algebras and proved that \( d \) is a Jordan \((\sigma, \tau)\)-derivation on \( \mathfrak{A} \) if and only if \( d \) is a \((\sigma, \tau)\)-derivation on \( \mathfrak{A} \). Further, they extended this result for generalized derivation and established that \( \delta \) is a generalized Jordan \((\sigma, \tau)\)-derivation on \( \mathfrak{A} \) if and only if \( \delta \) is a generalized \((\sigma, \tau)\)-derivation on \( \mathfrak{A} \). Motivated by all these studies, here we characterize the multiplicative generalized Jordan \((\sigma, \tau)\)-derivation and prove that every multiplicative generalized Jordan \((\sigma, \tau)\)-derivation is an additive \((\sigma, \tau)\)-derivation under certain assumptions. In fact, our results generalize, extend and complement several results obtained earlier (see for example Proposition 3.2 and Theorems 4.3 & 4.4 of [7], Theorem 2.1 of [9] and Theorem 2.1 of [13]).

## 2 Preliminaries

Throughout, this paper we shall use the following notions: Let \( \mathcal{A} \) and \( \mathcal{B} \) be unital algebras over \( \mathbb{R} \) and let \( \mathcal{M} \) be an \((\mathcal{A}, \mathcal{B})\)-bimodule which is faithful as a left \( \mathcal{A} \)-module, that is, for \( A \in \mathcal{A} \), \( A\mathcal{M} = 0 \) implies \( A = 0 \) and also as a right \( \mathcal{B} \)-module, that is, for \( B \in \mathcal{B} \), \( MB = 0 \) implies \( B = 0 \). The triangular algebra \( \mathfrak{A} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}) \) is 2-torsion free. In case \( \mathcal{M} \) is faithful the center of \( \mathfrak{A} \) is given by
\[
Z(\mathfrak{A}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : am = mb \text{ for all } m \in \mathcal{M}, a \in \mathcal{A}, b \in \mathcal{B} \right\}.
\]
Define two natural projections \( \pi_\mathcal{A} : \mathfrak{A} \to \mathcal{A} \) and \( \pi_\mathcal{B} : \mathfrak{A} \to \mathcal{B} \) by
\[
\pi_\mathcal{A} \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = a \quad \text{and} \quad \pi_\mathcal{B} \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = b.
\]
Moreover, \( \pi_\mathcal{A}(Z(\mathfrak{A})) \subseteq Z(\mathcal{A}) \) and \( \pi_\mathcal{B}(Z(\mathfrak{A})) \subseteq Z(\mathcal{B}) \) and there exists a unique algebraic isomorphism \( \xi : \pi_\mathcal{A}(Z(\mathfrak{A})) \to \pi_\mathcal{B}(Z(\mathfrak{A})) \) such that \( am = m\xi(a) \) for all \( a \in \pi_\mathcal{A}(Z(\mathfrak{A})), m \in \mathcal{M} \).

Let \( 1_{\mathcal{A}} \) (resp. \( 1_{\mathcal{B}} \)) be the identity of the algebra \( \mathcal{A} \) (resp. \( \mathcal{B} \)) and let \( I \) be the identity of triangular algebra \( \mathfrak{A} \).
\[
p = \begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix}, \quad q = I - p = \begin{pmatrix} 0 & 0 \\ 0 & 1_{\mathcal{B}} \end{pmatrix}
\]
and \( \mathfrak{A}_{11} = p\mathfrak{A}p, \mathfrak{A}_{12} = p\mathfrak{A}q, \mathfrak{A}_{21} = q\mathfrak{A}p, \mathfrak{A}_{22} = q\mathfrak{A}q. \) Thus, \( \mathfrak{A} = p\mathfrak{A}p + p\mathfrak{A}q + p\mathfrak{A}q + q\mathfrak{A}p + q\mathfrak{A}q + q\mathfrak{A}p \).
Similarly, we can obtain that

In the year 2011, Han and Wei [7] studied Jordan

for all

Now,

Since

Theorem 3.1. Let \( \mathfrak{A} = \text{Tri}(A, M, B) \) be a triangular algebra consisting of \( A, B \) with only trivial idempotents and a bimodule \( M \). Let \( \sigma \) and \( \tau \) be two automorphisms of \( \mathfrak{A} \) and \( d : \mathfrak{A} \rightarrow \mathfrak{A} \) be a multiplicative Jordan (\( \sigma, \tau \))-derivation on \( \mathfrak{A} \). Then \( d \) is an additive (\( \sigma, \tau \))-derivation on \( \mathfrak{A} \).

We facilitate our discussion with the following lemma which plays a key role in developing the proof of our theorem:

Lemma 3.2. [10, Theorem 1] Let \( \mathfrak{A} = \text{Tri}(A, M, B) \) be a triangular algebra, where \( A \) and \( B \) have only trivial idempotents. Then an \( \mathbb{R} \)-linear map \( \sigma : \mathfrak{A} \rightarrow \mathfrak{A} \) is said be automorphism of \( \mathfrak{A} \) if and only if it has the form

\[
\sigma \left( \begin{array}{cc} a & m \\ 0 & b \end{array} \right) = \left( \begin{array}{cc} \theta(a) & \theta(a)m - \eta(b) + \nu(m) \\ 0 & \eta(b) \end{array} \right),
\]

where \( \theta : A \rightarrow A \) and \( \eta : B \rightarrow B \) are automorphisms, \( m' \) is a fixed element in \( M \) and \( \nu : M \rightarrow M \) is an \( \mathbb{R} \)-linear bijective mapping such that \( \nu(am) = \theta(a)\nu(m), \nu(mb) = \nu(m)\eta(b) \) for all \( a \in A, b \in B \) and \( m \in M \).

Proof of Theorem 3.1. In order to prove the theorem, using Lemma 3.2 we assume that

\[
\begin{align*}
\sigma(a) & \in A + M, & \tau(a) & \in A + M, & d(a) & = a_1 + a_{12} + a_{22}, \\
\sigma(b) & \in M + B, & \tau(b) & \in M + B, & d(b) & = b_1 + b_{12} + b_{22}, \\
\sigma(m) & \in M, & \tau(m) & \in M, & d(m) & = m_1 + m_{12} + m_{22}, \\
\sigma(p) & = p_m = \left( \begin{array}{cc} 1 & m \\ 0 & 0 \end{array} \right), & \tau(p) & = p_{m'}, & d(p) & = p_1 + p_{12} + p_{22}, \\
\sigma(q) & = q_{-m} = \left( \begin{array}{cc} 0 & -m \\ 0 & 1 \end{array} \right), & \tau(q) & = q_{-m'}, & d(q) & = q_1 + q_{12} + q_{22},
\end{align*}
\]

(3.1)

where \( a, a_{11}, b_{11}, m_{11}, p_{11}, q_{11} ) \in A; m, a_{12}, b_{12}, m_{12}, p_{12}, q_{12} \in M \) and \( b, a_{22}, b_{22}, m_{22}, p_{22}, q_{22} \in B \). Now we separate the proof in various steps:

Step 1: \( d(0) = 0, d(p), d(q) \in M \) and \( d(p) + d(q) = 0 \).

Since

\[
d(0) = d(0)\tau(0) + \sigma(0)d(0) = 0.
\]

Now, \( d(p) = d(p^2) = d(p)\tau(p) + \sigma(p)d(p) \) and using (3.1), we have \( p_{11} = 0 = p_{22} \). This implies that \( d(p) \in M \). Similarly, we can obtain that \( d(q) \in M \).

Again using (3.1), we have

\[
\begin{align*}
0 &= d(pq + qp) \\
&= d(p)\tau(q) + \sigma(p)d(q) + d(q)\tau(p) + \sigma(q)d(p) \\
&= d(p) + d(q).
\end{align*}
\]

(3.2)
Step 2: $d(M) \subseteq M$. 
Since, we have 
\[ d(m) = d(mp + pm) \]
\[ = d(m \tau(p) + \sigma(m)d(p) + d(p)\tau(m) + \sigma(p)d(m)) \]
for all $m \in M$, applying (3.1) and $d(p) \in M$, we get $m_{11} = 0 = m_{22}$ which gives that $d(M) \subseteq M$.

Step 3: $d(A) \subseteq A + M$ and $d(B) \subseteq M + B$. 
For all $a \in A$, we have 
\[ 0 = d(aq + qa) \]
\[ = d(a)\tau(q) + \sigma(a)d(q) + d(q)\tau(a) + \sigma(q)d(a). \]
Now using (3.1) and $d(q) \in M$, we arrive at $a_{22} = 0$ which gives that $d(A) \subseteq A + M$. Similarly, we have $d(B) \subseteq M + B$.

Step 4: $d(A + M) \subseteq A + M$ and $d(M + B) \subseteq M + B$. 
For all $a \in A$ and $m \in M$, it is clear that 
\[ d(m) = d((a + m)q + q(a + m)) \]
\[ = d(a)\tau(q) + \sigma(a + m)d(q) + d(q)\tau(a + m) + \sigma(q)d(a + m). \]
Since $d(q) \in M$ and from (3.1), we obtain that $d(A + M) \subseteq A + M$. Similarly, we can calculate that $d(M + B) \subseteq M + B$.

Step 5: For any $a \in A$, $m \in M$ and $b \in B$
(i) $d(am) = \sigma(a)d(m) + d(a)\tau(m)$,
(ii) $d(mb) = \sigma(m)d(b) + d(m)\tau(b)$.

Using $d(M) \subseteq M$ and $d(A) \subseteq A + M$, we get 
\[ d(am) = d(am + ma) \]
\[ = d(a)\tau(m) + \sigma(a)d(m) + d(m)\tau(a) + \sigma(m)d(a) \]
\[ = d(a)\tau(m) + \sigma(a)d(m) \]
for all $m \in M$ and $a \in A$. In similar way, we can obtain $d(mb) = \sigma(m)d(b) + d(m)\tau(b)$ for all $m \in M$ and $b \in B$.

Step 6: For any $a \in A$, $m, n \in M$ and $b \in B$, we have
(i) $d(a + m) = d(a) + d(m)$,
(ii) $d(n + b) = d(n) + d(b)$.

Applying $d(A + M) \subseteq A + M$ and $d(M) \subseteq M$, we have 
\[ d(an) = d((a + m)n + n(a + m)) \]
\[ = d(a)\tau(n) + \sigma(a + m)d(n) + d(n)\tau(a + m) + \sigma(n)d(a + m) \]
\[ = d(a)\tau(n) + \sigma(a + m)d(n). \]  \hspace{1cm} (3.3)

On the other hand, 
\[ d(an) = d(an + na) \]
\[ = d(a)\tau(n) + \sigma(a)d(n) + d(n)\tau(a) + \sigma(n)d(a) \]
\[ = d(a)\tau(n) + \sigma(a)d(n). \]  \hspace{1cm} (3.4)

Combining (3.3),(3.4) and assuming $\sigma(n) = h_{12}$, we have 
\[ \{d(a + m) - d(a)\}\tau(n) = 0. \]
Using Steps 3 and 4, if we put \( d(a+m) - d(a) = r_{11} + r_{12} \), then we get \( r_{11} = 0 \) which implies that \( d(a+m) - d(a) \in \mathcal{M} \). Also, we have
\[
d(a + m) - d(a) = \sigma(q)(d(a + m) - d(a)) + (d(a + m) - d(a)) \tau(q)
= d((a + m)q + q(a + m)) - d(aq + qa)
= d(m)
\tag{3.5}
\]
for all \( m \in \mathcal{M} \) and \( a \in A \). Similarly, we can show that \( d(n + b) = d(n) + d(b) \) for all \( n \in \mathcal{M} \) and \( b \in B \).

**Step 7:** For any \( a_1, a_2 \in A, m_1, m_2 \in \mathcal{M} \) and \( b_1, b_2 \in B \), we have
\begin{enumerate}[(i)]
\item \( d(m_1 + m_2) = d(m_1) + d(m_2) \),
\item \( d(a_1 + a_2) = d(a_1) + d(a_2) \),
\item \( d(b_1 + b_2) = d(b_1) + d(b_2) \).
\end{enumerate}
For all \( m_1, m_2 \in \mathcal{M} \), we have
\[
d(m_1 + m_2) = d((p + m_1)(q + m_2) + (q + m_2)(p + m_1))
= d(p + m_1) \tau(q + m_2) + \sigma(p + m_1)d(q + m_2) + d(q + m_2) \tau(p + m_1) + \sigma(q + m_2)d(p + m_1)
= d(m_1) \tau(q) + \sigma(p)d(m_2)
= d(m_1) + d(m_2)
\]
which proves the part (i).

Also, using Steps 2, 3 and 5 for all \( a_1, a_2 \in A \) and \( m \in \mathcal{M} \), we obtain
\[
d((a_1 + a_2)m) = d(a_1m) + d(a_2m)
= d(a_1) \tau(m) + \sigma(a_1)d(m) + d(a_2) \tau(m) + \sigma(a_2)d(m).
\tag{3.6}
\]
On the other hand,
\[
d((a_1 + a_2)m) = d(a_1 + a_2) \tau(m) + \sigma(a_1 + a_2)d(m)
= d(a_1 + a_2) \tau(m) + \sigma(a_1)d(m) + \sigma(a_2)d(m).
\tag{3.7}
\]
On combining (3.6) and (3.7), we have
\[
\{ d(a_1 + a_2) - d(a_1) - d(a_2) \} \tau(m) = 0.
\tag{3.8}
\]
Again, using Steps 1 and 3 for all \( a_1, a_2 \in A \) and \( m \in \mathcal{M} \), we obtain that
\[
0 = d(a_1q + qa_1)
= d(a_1) \tau(q) + \sigma(a_1)d(q) + d(q) \tau(a_1) + \sigma(q)d(a_1)
= d(a_1) \tau(q) + \sigma(a_1)d(q).
\tag{3.9}
\]
Now replace \( a_1 \) in (3.9) by \( a_2 \) and \( a_1 + a_2 \) respectively, we get
\[
0 = d(a_2) \tau(q) + \sigma(a_2)d(q)
\tag{3.10}
\]
and
\[
0 = d(a_1 + a_2) \tau(q) + \sigma(a_1 + a_2)d(q).
\tag{3.11}
\]
From (3.9),(3.10) and (3.11), we get
\[
\{ d(a_1 + a_2) - d(a_1) - d(a_2) \} \tau(q) = 0.
\tag{3.12}
\]
Choosing \( d(a_1 + a_2) - d(a_1) - d(a_2) = s_{11} + s_{12} \). Then form (3.8) and (3.12), we have \( s_{11} = 0 = s_{12} \) which gives that \( d(a_1 + a_2) = d(a_1) + d(a_2) \) for all \( a_1, a_2 \in A \). In the similar manner, we get \( d(b_1 + b_2) = d(b_1) + d(b_2) \)
for all $b_1, b_2 \in B$.

**Step 8:** For any $a \in A$, $m \in M$ and $b \in B$, we have

$$d(a + m + b) = d(a) + d(m) + d(b).$$

Since

$$d((a + m)p + p(a + m)) = d((a + m + b)p + p(a + m + b))$$
$$= d(a + m + b)\tau(p) + \sigma(a + m + b)d(p) + d(p)\tau(a + m + b) + \sigma(p)d(a + m + b)$$
$$= d(a + m + b)\tau(p) + \sigma(a)d(p) + d(p)\tau(b) + \sigma(p)d(a + m + b)$$

(3.13)

Again, we have

$$d((a + m)p + p(a + m)) = d(a + m)\tau(p) + \sigma(a + m)d(p) + d(0)$$
$$+ d(p)\tau(a + m) + \sigma(p)d(a + m) + d(bp + pb)$$
$$= d(a + m)\tau(p) + \sigma(a)d(p) + d(p)\tau(b) + \sigma(p)d(a + m)$$
$$+ d(b)\tau(p) + \sigma(b)d(p) + d(p)\tau(b) + \sigma(p)d(b)$$
$$= d(a)\tau(p) + d(m)\tau(p) + \sigma(a)d(p) + d(p)\tau(b)$$
$$+ \sigma(p)d(a) + \sigma(p)d(m).$$

(3.14)

Combining (3.13) and (3.14), we get

$$\sigma(p)(d(a + m + b) - d(a) - d(m) - d(b)) + \{d(a + m + b) - d(a) - d(m) - d(b)\}\tau(p) = 0.$$  

(3.15)

Similarly, we have

$$\sigma(q)(d(a + m + b) - d(a) - d(m) - d(b)) + \{d(a + m + b) - d(a) - d(m) - d(b)\}\tau(q) = 0.$$  

(3.16)

Let us take $d(a + m + b) - d(a) - d(m) - d(b) = t_{11} + t_{12} + t_{22}$. Then form (3.15) and (3.16), we have $t_{11} = t_{12} = t_{22} = 0$ which implies that

$$d(a + m + b) = d(a) + d(m) + d(b)$$

for all $a \in A$, $m \in M$ and $b \in B$.

**Step 9:** For any $a_1, a_2 \in A$ and $b_1, b_2 \in B$, we have

(i) $d(a_1a_2) = \sigma(a_1)d(a_2) + d(a_1)\tau(a_2),$

(ii) $d(b_1b_2) = \sigma(b_1)d(b_2) + d(b_1)\tau(b_2).$

From Step 5 for all $a_1, a_2 \in A$ and $m \in M$, it follows that

$$d((a_1a_2)m) = d(a_1(a_2)m)$$
$$d(a_1a_2)\tau(m) + \sigma(a_1a_2)d(m) = d(a_1)\tau(a_2)m + \sigma(a_1)d(a_2)m$$
$$= d(a_1)\tau(a_2)m + \sigma(a_1)d(a_2)m + \sigma(a_1)\sigma(a_2)d(m)$$

which implies that

$$\{d(a_1a_2) - d(a_1)\tau(a_2) - \sigma(a_1)d(a_2)\}\tau(m) = 0.$$  

(3.17)

Now replace $a_1$ in (3.9) by $a_2$ and $a_1a_2$ respectively, we get

$$0 = d(a_2)\tau(q) + \sigma(a_2)d(q)$$  

(3.18)

and

$$0 = d(a_1a_2)\tau(q) + \sigma(a_1a_2)d(q).$$  

(3.19)
Corollary 3.4. [13, Theorem 2.1] Let $A$ be a triangular algebra consisting of algebras $A$, $B$ and bimodule $M$. Then every Jordan derivation on $A$ is a derivation on $A$. 

Proof. [13, Theorem 2.1] Let $A = A \oplus M \oplus B$ be a triangular algebra consisting of algebras $A$, $B$ and bimodule $M$. Then every Jordan derivation on $A$ is a derivation on $A$. 

Step 10: For all $x, y \in A$, we have $d(x + y) = d(x) + d(y)$. Let us take $x = a_1 + m_1 + b_1$ and $y = a_2 + m_2 + b_2$ where $a_1, a_2 \in A$, $m_1, m_2 \in M$ and $b_1, b_2 \in B$. On using Steps 7 and 8, we arrive

\[
\begin{align*}
(d(a_1, a_2) - d(a_1)\tau(a_2) - \sigma(a_1)d(a_2))\tau(b_2) &= 0. 
\end{align*}
\]

Now assume $d(a_1, a_2) - d(a_1)\tau(a_2) - \sigma(a_1)d(a_2) = u_{11} + u_{12} + u_{22}$. Then form (3.17) and (3.20), we have $u_{11} = u_{12} = u_{22} = 0$ which gives that $d(a_1, a_2) - \sigma(a_1)d(a_2) = \sigma(a_1)d(a_2)$ for all $a_1, a_2 \in A$. In the similar manner, we get $d(b_1, b_2) = \sigma(b_1)d(b_2) + d(b_1)\tau(b_2)$ for all $b_1, b_2 \in B$.

Step II: For all $x, y \in A$, we have $d(xy) = d(x)\tau(y) + \sigma(x)d(y)$. Since $x = a_1 + m_1 + b_1$ and $y = a_2 + m_2 + b_2$. On using Steps 5, 7 and 8, we obtain

\[
\begin{align*}
d(xy) &= d((a_1 + m_1 + b_1)(a_2 + m_2 + b_2)) \\
&= d(a_1 a_2 + a_1 m_2 + m_1 b_2 + b_1 b_2) \\
&= d(a_1)\tau(a_2) + \sigma(a_1)d(a_2) + d(a_1)\tau(m_2) + \sigma(a_1)d(m_2) \\
&+ d(m_1)\tau(b_2) + \sigma(m_1)d(b_2) + d(b_1)\tau(b_2) + \sigma(b_1)d(b_2). 
\end{align*}
\]

On the other hand, we have

\[
\begin{align*}
d(x)\tau(y) + \sigma(x)d(y) &= d(a_1 + m_1 + b_1)\tau(a_2 + m_2 + b_2) + \sigma(a_1 + m_1 + b_1)d(a_2 + m_2 + b_2) \\
&= \sigma(a_1 + m_1 + b_1)(d(a_2) + d(m_2) + d(b_2)) \\
&+ d(m_1)d(b_2) + d(m_1)\tau(b_2) + \sigma(b_1)d(b_2). 
\end{align*}
\]

From (3.22) and (3.23), we get $\delta(xy) = \delta(x)\tau(y) + \sigma(x)d(y)$. This proves the theorem, i.e., a multiplicative Jordan $(\sigma, \tau)$-derivation is an additive $(\sigma, \tau)$-derivation. \hfill $\Box$

Corollary 3.3. [7, Proposition 3.2] Let $A = A \oplus M \oplus B$ be a triangular algebra consisting of algebras $A$, $B$ with only trivial idempotents and a bimodule $M$ and $\sigma$ and $\tau$ be two automorphisms of $A$. Then any Jordan $(\sigma, \tau)$-derivation on $A$ is a $(\sigma, \tau)$-derivation on $A$.

Particularly, if $\sigma$ and $\tau$ are both the identity mapping of the triangular algebra $A$, a similar result still holds even if the condition that $A$ and $B$ have only trivial idempotents is dropped.

Corollary 3.4. [13, Theorem 2.1] Let $A = A \oplus M \oplus B$ be a triangular algebra consisting of algebras $A$, $B$ and bimodule $M$. Then every Jordan derivation on $A$ is a derivation on $A$. 

Proof. [13, Theorem 2.1] Let $A = A \oplus M \oplus B$ be a triangular algebra consisting of algebras $A$, $B$ and bimodule $M$. Then every Jordan derivation on $A$ is a derivation on $A$. 

Download Date | 6/8/19 2:42 PM
4 Nonlinear generalized Jordan \((\sigma, \tau)\)-derivation

The second author together with Ali and Ali [1] studied generalized Jordan \((\sigma, \tau)\)-derivations in prime rings and proved that every generalized Jordan \((\sigma, \tau)\)-derivation on a prime ring of characteristic different from two is a generalized \((\sigma, \tau)\)-derivation. This result was further extended by many authors in various settings. Very recently, Han and Wei [7] proved that under certain restrictions every generalized Jordan \((\sigma, \tau)\)-derivation is a generalized \((\sigma, \tau)\)-derivation on a triangular algebra and vice versa. Motivated by these results, we study multiplicative generalized Jordan \((\sigma, \tau)\)-derivation on triangular algebra and prove the following:

**Theorem 4.1.** Suppose that \(\mathcal{A} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})\) be a triangular algebra having algebras \(\mathcal{A}, \mathcal{B}\) with only trivial idempotents and a bimodule \(\mathcal{M}\). Let \(\sigma\) and \(\tau\) be two automorphisms of \(\mathcal{A}\) and \(\delta : \mathcal{A} \to \mathcal{A}\) be a multiplicative generalized Jordan \((\sigma, \tau)\)-derivation on \(\mathcal{A}\) with associated multiplicative Jordan \((\sigma, \tau)\)-derivation \(d\) on \(\mathcal{A}\). Then \(\delta\) is an additive generalized \((\sigma, \tau)\)-derivation on \(\mathcal{A}\).

**Proof.** Using Theorem 3.1, let us take

\[
\begin{align*}
\delta(a) &= a'_{11} + a'_{12} + a'_{22}, & d(a) &= a_{11} + a_{12}, \\
\delta(b) &= b'_{11} + b'_{12} + b'_{22}, & d(b) &= b_{12} + b_{22}, \\
\delta(m) &= m_{11} + m_{12} + m_{22}, & d(m) &= m_{12}, \\
\delta(p) &= p_{11} + p_{12} + p_{22}, & d(p) &= p_{12}, \\
\delta(q) &= q_{11} + q_{12} + q_{22}, & d(q) &= q_{12}.
\end{align*}
\]

(4.1)

where \(a, a'_{11}, b'_{11}, m'_{11}, p'_{11}, q'_{11}, a_{11} \in \mathcal{A}; m, a'_{12}, b'_{12}, m'_{12}, p'_{12}, q'_{12}, a_{12}, b_{12}, m_{12}, p_{12}, q_{12} \in \mathcal{M}\) and \(b, a'_{22}, b'_{22}, m'_{22}, p'_{22}, q'_{22}, b_{22} \in \mathcal{B}\). In order to prove the theorem, we follow the following steps:

**Step 1:** \(\delta(0) = 0\).

Since

\[
\delta(0) = \delta(0)\tau(0) + \sigma(0)d(0) = 0.
\]

**Step 2:** \(\delta(\mathcal{M}) \subseteq \mathcal{M}\).

Now, \(\delta(p) = \delta(p^2) = \delta(p)\tau(p) + \sigma(p)d(p)\) and using (3.1) and (4.1), we have \(p'_{22} = 0\). On the other hand, \(\delta(q) = \delta(q^2) = \delta(q)\tau(q) + \sigma(q)d(q)\) and from (3.1) and (4.1), we have \(q_{11} = 0\).

Since, we have

\[
\begin{align*}
\delta(m) &= \delta(mp + pm) \\
&= \delta(m)\tau(p) + \sigma(m)d(p) + \delta(p)\tau(m) + \sigma(p)d(m)
\end{align*}
\]

for all \(m \in \mathcal{M}\), applying (3.1) and (4.1), we get \(m'_{22} = 0\). Similarly, on using \(m = mq + qm\), we get \(m'_{11} = 0\).

This implies that \(\delta(\mathcal{M}) \subseteq \mathcal{M}\).

**Step 3:** \(\delta(\mathcal{A}) \subseteq \mathcal{A} + \mathcal{M}\) and \(\delta(\mathcal{B}) \subseteq \mathcal{M} + \mathcal{B}\).

For all \(a \in \mathcal{A}\), we have

\[
0 = \delta(aq + qa) = \delta(a)\tau(q) + \sigma(a)d(q) + \delta(q)\tau(a) + \sigma(q)d(a).
\]

Now from (3.1), (4.1) and \(\delta(q) = q'_{12} + q'_{22}\) we get that \(a'_{22} = 0\) which implies that \(\delta(\mathcal{A}) \subseteq \mathcal{A} + \mathcal{M}\). Similarly, we have \(\delta(\mathcal{B}) \subseteq \mathcal{M} + \mathcal{B}\).

**Step 4:** \(\delta(\mathcal{A} + \mathcal{M}) \subseteq \mathcal{A} + \mathcal{M}\) and \(\delta(\mathcal{M} + \mathcal{B}) \subseteq \mathcal{M} + \mathcal{B}\).

For all \(a \in \mathcal{A}\) and \(m \in \mathcal{M}\), it is easy to see that

\[
\begin{align*}
\delta(m) &= \delta((a + m)q + q(a + m)) \\
&= \delta(a + m)\tau(q) + \sigma(a + m)d(q) + \delta(q)\tau(a + m) + \sigma(q)d(a + m).
\end{align*}
\]
As \( \delta(q) = q_{12} + q'_{12} \) and from (3.1) and (4.1), we have \( \delta(A + M) \subseteq A + M \). Similarly, we can calculate that \( \delta(M + B) \subseteq M + B \).

**Step 5:** For any \( a \in A, m \in M \) and \( b \in B \)

(i) \( \delta(am) = \sigma(a)d(m) + \delta(a)\tau(m) \),
(ii) \( \delta(mb) = \sigma(m)d(b) + \delta(m)\tau(b) \).

Using \( \delta(M) \subseteq M \) and \( \delta(A) \subseteq A + M \), we get

\[
\delta(am) = \delta(am + ma) = \delta(a)\tau(m) + \sigma(a)d(m) + \delta(m)\tau(a) + \sigma(m)d(a)
\]

for all \( m \in M \) and \( a \in A \). Similarly, we can obtain \( \delta(mb) = \sigma(m)d(b) + \delta(m)\tau(b) \) for all \( m \in M \) and \( b \in B \).

**Step 6:** For any \( a \in A, m, n \in M \) and \( a \in B \), we have

(i) \( \delta(a + m) = \delta(a) + \delta(m) \),
(ii) \( \delta(n + b) = \delta(n) + \delta(b) \).

Applying \( \delta(A + M) \subseteq A + M \) and \( \delta(M) \subseteq M \), we have

\[
\delta(an) = \delta \left( (a + m)n + n(a + m) \right) = \delta(a + m)\tau(n) + \sigma(a + m)d(n) + \delta(n)\tau(a + m) + \sigma(n)d(a + m)
\]

(4.2)

On the other way,

\[
\delta(an) = \delta(an + na) = \delta(a)\tau(n) + \sigma(a)d(n) + \delta(n)\tau(a) + \sigma(n)d(a)
\]

(4.3)

From (4.2),(4.3) and using \( \sigma(n) = h_{12} \), we have

\[
\{ \delta(a + m) - \delta(a) \} \tau(n) = 0.
\]

Using Steps 3 and 4 if we put \( \delta(a + m) - \delta(a) = r'_{11} + r'_{12} \), then we get \( r'_{11} = 0 \) which implies that \( \delta(a + m) - \delta(a) \in M \). Also, we have

\[
\delta(a + m) - \delta(a) = \sigma(q)(d(a + m) - d(a)) + \{ \delta(a + m) - \delta(a) \} \tau(q) = \delta((a + m)q + q(a + m) - \delta(a + m - \delta(a)) \tau(q)
\]

(4.4)

for all \( m \in M \) and \( a \in A \). In a similar way, \( \delta(n + b) = \delta(n) + \delta(b) \) for all \( n \in M \) and \( b \in B \).

**Step 7:** For any \( a_1, a_2 \in A, m_1, m_2 \in M \) and \( b_1, b_2 \in B \), we have

(i) \( \delta(m_1 + m_2) = \delta(m_1) + \delta(m_2) \),
(ii) \( \delta(a_1 + a_2) = \delta(a_1) + \delta(a_2) \),
(iii) \( \delta(b_1 + b_2) = \delta(b_1) + \delta(b_2) \).
On using Step 6 for all \(m_1, m_2 \in \mathcal{M}\), we have
\[
\delta(m_1 + m_2) = \delta((p + m_1)(q + m_2) + (q + m_2)(p + m_1)) \\
= \delta(p + m_1)\tau(q + m_2) + \sigma(p + m_1)d(q + m_2) \\
+ \delta(q + m_2)\tau(p + m_1) + \sigma(q + m_2)d(p + m_1) \\
= \{\delta(p) + \delta(m_1)\}\{\tau(q) + \tau(m_2)\} + \{\sigma(p) + \sigma(m_1)\}\{d(q) + d(m_2)\} \\
+ \{\delta(q) + \delta(m_2)\}\{\tau(p) + \tau(m_1)\} + \{\sigma(q) + \sigma(m_2)\}\{d(p) + d(m_1)\} \\
= \delta(pq + qp) + \delta(pm_2 + m_2p) + \delta(qm_1 + m_1q) + \delta(m_1m_2 + m_2m_1) \\
= \delta(m_1) + \delta(m_2)
\]
which proves the part (i).
Also, using Steps 2, 3 and 5 for all \(a_1, a_2 \in \mathcal{A}\) and \(m \in \mathcal{M}\), we obtain
\[
\delta((a_1 + a_2)m) = \delta(a_1m) + \delta(a_2m) \\
= \delta(a_1)\tau(m) + \sigma(a_1)d(m) + \delta(a_2)\tau(m) + \sigma(a_2)d(m).
\] (4.5)
On the other hand,
\[
\delta((a_1 + a_2)m) = \delta(a_1 + a_2)\tau(m) + \sigma(a_1 + a_2)d(m) \\
= \delta(a_1 + a_2)\tau(m) + \sigma(a_1)d(m) + \sigma(a_2)d(m).
\] (4.6)
From (4.5) and (4.6), we have
\[
\{\delta(a_1 + a_2) - \delta(a_1) - \delta(a_2)\}\tau(m) = 0.
\] (4.7)
Again, using Steps 1 and 3 for all \(a_1, a_2 \in \mathcal{A}\) and \(m \in \mathcal{M}\), we obtain that
\[
0 = \delta(a_1q + qa_1) \\
= \delta(a_1)\tau(q) + \sigma(a_1)d(q) + \delta(a_1)\tau(q) + \sigma(a_1)d(q) \\
= \delta(a_1)\tau(q) + \sigma(a_1)d(q).
\] (4.8)
Now, replace \(a_1\) in (4.8) by \(a_2\) and \(a_1 + a_2\) respectively, we get
\[
0 = \delta(a_2)\tau(q) + \sigma(a_2)d(q)
\] (4.9) and
\[
0 = \delta(a_1 + a_2)\tau(q) + \sigma(a_1 + a_2)d(q).
\] (4.10)
From (4.8),(4.9) and (4.10), it follows that
\[
\{\delta(a_1 + a_2) - \delta(a_1) - \delta(a_2)\}\tau(q) = 0.
\] (4.11)
Let us take \(\delta(a_1 + a_2) - \delta(a_1) - \delta(a_2) = s_{11}' + s_{12}'\). Then form (4.7) and (4.11), we have \(s_{11}' = 0 = s_{12}'\) which gives that \(\delta(a_1 + a_2) = \delta(a_1) + \delta(a_2)\) for all \(a_1, a_2 \in \mathcal{A}\). In the similar manner, we get \(\delta(b_1 + b_2) = \delta(b_1) + \delta(b_2)\) for all \(b_1, b_2 \in \mathcal{B}\).

**Step 8:** For any \(a \in \mathcal{A}, m \in \mathcal{M}\) and \(b \in \mathcal{B}\), we have
\[
\delta(a + m + b) = \delta(a) + \delta(m) + \delta(b).
\]
Since
\[
\delta((a + m)p + p(a + m)) = \delta((a + m + b)p + p(a + m + b)) \\
= \delta(a + m + b)\tau(p) + \sigma(a + m + b)d(p) \\
+ \delta(p)\tau(a + m + b) + \sigma(p)d(a + m + b) \\
= \delta(a + m + b)\tau(p) + \sigma(a)d(p) + \delta(p)\tau(b) + \sigma(p)d(a + m + b).
\]
Again, we have

\[ \delta((a + m)p + p(a + m)) = \delta(a + m)\tau(p) + \delta(a + m)d(p) + \delta(0) \]
\[ + \delta(p)\tau(a + m) + \sigma(p)d(a + m) + \delta(bp + pb) \]
\[ = \delta(a + m)\tau(p) + \sigma(a)d(p) + \delta(p)\tau(b) + \delta(p)d(a + m) \]
\[ + \delta(b)\tau(p) + \sigma(b)d(p) + \delta(p)\tau(b) + \sigma(p)d(b) \]
\[ = \delta(a)\tau(p) + \delta(m)\tau(p) + \sigma(a)d(p) + \delta(p)\tau(b) \]
\[ + \sigma(p)d(a) + \sigma(p)d(m). \] (4.13)

Combining (4.12) and (4.13), we obtain

\[ \sigma(p)(d(a + m + b) - d(a) - d(m) - d(b)) + \{ \delta(a + m + b) - \delta(a) - \delta(m) - \delta(b) \}\tau(p) = 0. \] (4.14)

Similarly, we have

\[ \sigma(q)(d(a + m + b) - d(a) - d(m) - d(b)) + \{ \delta(a + m + b) - \delta(a) - \delta(m) - \delta(b) \}\tau(q) = 0. \] (4.15)

Now adding (4.14), (4.15) and using the fact \(d(a + m + b) = d(a) + d(m) + d(b)\), we obtain

\[ \delta(a + m + b) = \delta(a) + \delta(m) + \delta(b) \]

for all \(a \in A, m \in M\) and \(b \in B\).

Step 9: For any \(a_1, a_2 \in A\) and \(b_1, b_2 \in B\), we have

(i) \(\delta(a_1a_2) = \sigma(a_1)d(a_2) + \delta(a_1)\tau(a_2)\),
(ii) \(\delta(b_1b_2) = \sigma(b_1)d(b_2) + \delta(b_1)\tau(b_2)\).

From Step 5 for all \(a_1, a_2 \in A\) and \(m \in M\), it follows that

\[
\begin{align*}
\delta((a_1a_2)m) &= \delta(a_1(a_2m)) \\
\delta(a_1a_2)\tau(m) + \sigma(a_1a_2)d(m) &= \delta(a_1)\tau(a_2m) + \sigma(a_1)d(a_2m) \\
&= \delta(a_1)\tau(a_2m) + \sigma(a_1)d(a_2)\tau(m) + \sigma(a_1)d(a_2)m \\
&= \delta(a_1)\tau(a_2m) + \sigma(a_1)d(a_2)m + \delta(a_1)d(a_2)\tau(m) + \sigma(a_1)d(a_2)m \\
&= \delta((a_1a_2)m) = \delta(a_1(a_2m)) \\
\delta(a_1a_2)\tau(m) + \sigma(a_1a_2)d(m) &= \delta(a_1)\tau(a_2m) + \sigma(a_1)d(a_2m) \\
&= \delta(a_1)\tau(a_2m) + \sigma(a_1)d(a_2)\tau(m) + \sigma(a_1)d(a_2)m \\
&= \delta(a_1)\tau(a_2m) + \sigma(a_1)d(a_2)m + \delta(a_1)d(a_2)\tau(m) + \sigma(a_1)d(a_2)m \\
&= \delta((a_1a_2)m) = \delta(a_1(a_2m)) \\
\end{align*}
\]

which implies that

\[ \{ \delta(a_1a_2) - \delta(a_1)\tau(a_2) - \sigma(a_1)d(a_2) \}\tau(m) = 0. \] (4.16)

Now replace \(a_1\) in (4.8) by \(a_2\) and \(a_1a_2\) respectively, we get

\[ 0 = \delta(a_2)\tau(q) + \sigma(a_2)d(q) \] (4.17)

and

\[ 0 = \delta(a_1a_2)\tau(q) + \sigma(a_1a_2)d(q). \] (4.18)

Multiplying from left (4.17) by \(\sigma(a_1)\) and combining with (4.18), we get

\[ \{ \delta(a_1a_2) - \delta(a_1)\tau(a_2) - \sigma(a_1)d(a_2) \}\tau(q) = 0. \] (4.19)

Let us assume \(\delta(a_1a_2) - \delta(a_1)\tau(a_2) - \sigma(a_1)d(a_2) = u_{11} + u_{12} + u_{22}\). Then form (4.16) and (4.19), we have \(u_{11} = u_{12} = u_{22} = 0\) which gives that \(\delta(a_1a_2) = \sigma(a_1)d(a_2) + \delta(a_1)\tau(a_2)\) for all \(a_1, a_2 \in A\) and \(b_1, b_2 \in B\). In the similar manner, we get \(\delta(b_1b_2) = \sigma(b_1)d(b_2) + \delta(b_1)\tau(b_2)\) for all \(b_1, b_2 \in B\).

Step 10: For all \(x, y \in A\), we have \(\delta(x + y) = \delta(x) + \delta(y)\).
Let us take \( x = a_1 + m_1 + b_1 \) and \( y = a_2 + m_2 + b_2 \) where \( a_1, a_2 \in \mathcal{A}, \ m_1, m_2 \in \mathcal{M} \) and \( b_1, b_2 \in \mathcal{B} \). On using Steps 7 and 8, we arrive

\[
\delta(x + y) = \delta(a_1 + m_1 + b_1 + a_2 + m_2 + b_2) \\
= \delta((a_1 + a_2) + (m_1 + m_2) + (b_1 + b_2)) \\
= \delta(a_1 + a_2) + \delta(m_1 + m_2) + \delta(b_1 + b_2) \\
= \delta(a_1) + \delta(m_1) + \delta(b_1) + \delta(a_2) + \delta(m_2) + \delta(b_2) \\
= \delta(a_1 + m_1 + b_1) + \delta(a_2 + m_2 + b_2) \\
= \delta(x) + \delta(y). \tag{4.20}
\]

**Step 11:** For all \( x, y \in \mathfrak{A} \), we have \( \delta(xy) = \delta(x)\tau(y) + \sigma(x)d(y) \). Since \( x = a_1 + m_1 + b_1 \) and \( y = a_2 + m_2 + b_2 \). Then using Steps 5, 7 and 8, we obtain

\[
\delta(xy) = \delta((a_1 + m_1 + b_1)(a_2 + m_2 + b_2)) \\
= \delta(a_1 a_2 + a_1 m_2 + m_1 b_2 + b_1 b_2) \\
= \delta(a_1)\tau(a_2) + \sigma(a_1)d(a_2) + \delta(a_1)\tau(m_2) + \sigma(a_1)d(m_2) \\
+ \delta(m_1)\tau(b_2) + \sigma(m_1)d(b_2) + \delta(b_1)\tau(b_2) + \sigma(b_1)d(b_2). \tag{4.21}
\]

On the other hand, we have

\[
\delta(x)\tau(y) + \sigma(x)d(y) = \delta(a_1 + m_1 + b_1)\tau(a_2 + m_2 + b_2) + \sigma(a_1 + m_1 + b_1)d(a_2 + m_2 + b_2) \\
= (\delta(a_1) + \delta(m_1) + \delta(b_1))(\tau(a_2) + \tau(m_2) + \tau(b_2)) \\
+ (\sigma(a_1) + \sigma(m_1) + \sigma(b_1))(d(a_2) + d(m_2) + d(b_2)) \\
= \delta(a_1)\tau(a_2) + \sigma(a_1)d(a_2) + \delta(a_1)\tau(m_2) + \sigma(a_1)d(m_2) \\
+ \delta(m_1)\tau(b_2) + \sigma(m_1)d(b_2) + \delta(b_1)\tau(b_2) + \sigma(b_1)d(b_2). \tag{4.22}
\]

From (4.21) and (4.22), we get \( \delta(xy) = \delta(x)\tau(y) + \sigma(x)d(y) \). This proves the theorem, i.e., a multiplicative generalized Jordan \((\sigma, \tau)\)-derivation is an additive generalized \((\sigma, \tau)\)-derivation.

**Corollary 4.2.** [7, Theorem 4.3] Let \( \mathfrak{A} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}) \) be a triangular algebra having algebras \( \mathcal{A}, \mathcal{B} \) with only trivial idempotents and a bimodule \( \mathcal{M} \) and \( \sigma \) and \( \tau \) be two automorphisms of \( \mathfrak{A} \). Then every generalized Jordan \((\sigma, \tau)\)-derivation on \( \mathfrak{A} \) is a generalized \((\sigma, \tau)\)-derivation on \( \mathfrak{A} \).

Particularly, if \( \sigma \) and \( \tau \) are both the identity mapping of the triangular algebra \( \mathfrak{A} \), a similar result still holds even if the condition that \( \mathcal{A} \) and \( \mathcal{B} \) have only trivial idempotents is dropped.

**Corollary 4.3.** [7, Theorem 4.4] Let \( \mathfrak{A} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}) \) be a triangular algebra consisting of algebras \( \mathcal{A}, \mathcal{B} \) and a bimodule \( \mathcal{M} \). Then every generalized Jordan derivation on \( \mathfrak{A} \) is a generalized derivation on \( \mathfrak{A} \).

**Corollary 4.4.** [9, Theorem 2.1] Let \( \mathcal{N} \) be a nest on a Banach space \( X \) and \( \delta \) be an additive generalized Jordan derivation from \( \text{Alg}\mathcal{N} \) into itself. If there exists a non-trivial element in \( \mathcal{N} \) which is complemented in \( X \), then \( \delta \) is an additive generalized derivation.

**Corollary 4.5.** [9, Corollary 2.2] Let \( \mathcal{N} \) be a nest on a Hilbert space \( H \) and \( \delta \) be an additive generalized Jordan derivation from \( \text{Alg}\mathcal{N} \) into itself. Then \( \delta \) is an additive generalized derivation.

**References**

Ahmad N. Alkenani, Mohammad Ashraf, and Aisha Jabeen


