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Construction of symmetric Hadamard matrices of order 4ν for $\nu = 47, 73, 113$

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Abstract: We continue our systematic search for symmetric Hadamard matrices based on the so called propus construction. In a previous paper this search covered the orders 4ν with odd $\nu \leq 41$. In this paper we cover the cases $\nu = 43, 45, 47, 49, 51$. The odd integers $\nu < 120$ for which no symmetric Hadamard matrices of order 4ν are known are the following:

$$47, 59, 65, 67, 73, 81, 89, 93, 101, 103, 107, 109, 113, 119.$$

By using the propus construction, we found several symmetric Hadamard matrices of order 4ν for $\nu = 47, 73, 113$.

Keywords: Symmetric Hadamard matrices, Propus array, cyclic difference families, Diophantine equations

1 Introduction

In this paper we continue the systematic investigation, begun in [1], of the propus construction of symmetric Hadamard matrices.

Let us recall that a *Hadamard matrix* is a $\{1, -1\}$ -matrix H of order m whose rows are mutually orthogonal, i.e. $HH^T = mI_m$, where I_m is the identity matrix of order m . We say that H is *skew-Hadamard matrix* if also $H + H^T = 2I_m$. The famous *Hadamard conjecture* asserts that Hadamard matrices exist for all orders m which are multiples of 4. (They also exist for $m = 1, 2$.) Similar conjectures have been proposed for symmetric Hadamard matrices and skew-Hadamard matrices, see e.g. [2, V.1.4]. The smallest orders 4ν for which such matrices have not been constructed are 668 for Hadamard matrices, 276 for skew-Hadamard matrices, and 188 for symmetric Hadamard matrices. Let us also mention that symmetric Hadamard matrices of orders 116, 156, 172 have been constructed only very recently, see [1, 3].

Since the size of a Hadamard matrix or a skew or symmetric Hadamard matrix can always be doubled, while preserving its type, we are interested mostly in the case where these matrices have order 4ν with ν odd.

The propus construction is based on the so called *Propus array*

$$H = \begin{bmatrix} -C_1 & C_2R & C_3R & C_4R \\ C_3R & RC_4 & C_1 & -RC_2 \\ C_2R & C_1 & -RC_4 & RC_3 \\ C_4R & -RC_3 & RC_2 & C_1 \end{bmatrix}. \quad (1)$$

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In this paper, except in section 4, the matrices C_i will be circulants of order ν and the matrix R will be the back-circulant identity matrix of order ν ,

$$R = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & & 1 & 0 \\ \vdots & & & & \\ 0 & 1 & & 0 & 0 \\ 1 & 0 & & 0 & 0 \end{bmatrix}.$$

The matrix H will be a Hadamard matrix if

$$\sum_i C_i C_i^T = 4\nu I_{4\nu}. \quad (2)$$

(Superscript T denotes transposition of matrices.) If also $C_1^T = C_1$ and $C_2 = C_3$ then H will be a symmetric Hadamard matrix.

To construct the circulants C_i satisfying the above conditions we use the cyclic propus difference families (A_1, A_2, A_3, A_4) with parameters $(\nu; k_1, k_2, k_3, k_4; \lambda)$ such that $A_2 = A_3$ and at least one of the base blocks A_1, A_4 is symmetric. The parameters must satisfy the three equations

$$\sum_{i=1}^4 k_i(k_i - 1) = \lambda(\nu - 1), \quad (3)$$

$$\sum_{i=1}^4 k_i = \lambda + \nu, \quad (4)$$

$$k_2 = k_3. \quad (5)$$

We refer to such parameter sets as the *propus parameter sets*.

For the definitions of the terms that we use here and the facts we mention below, we refer the reader to [1]. Without any loss of generality, we impose the following additional restrictions:

$$\nu/2 \geq k_1, k_2; \quad k_1 \geq k_4. \quad (6)$$

For convenience we say that the propus parameter sets satisfying these additional conditions are *normalized*.

For a given odd ν there exist at least one normalized propus parameter set, see [1, Theorem 1]. However, there exist even ν for which this is not true, see [1, Theorem 2].

It is conjectured in [1] that for each odd ν there exists at least one propus difference family in the cyclic group \mathbf{Z}_ν of integers modulo ν . But this may fail if we specify not only ν (odd) but also the parameters $k_1, k_2 = k_3, k_4$. Our computations suggest that these exceptional propus parameter sets must have all k_i equal to each other. For instance, there is no cyclic propus difference family having the parameters $(25; 10, 10, 10, 10; 15)$. (This is also true for the propus difference families over the elementary abelian group $\mathbf{Z}_5 \times \mathbf{Z}_5$.)

One of the authors developed a computer program to search for propus difference families. For the description of the algorithm used in the program we refer the reader to [1]. We used that program on PCs to construct many such families for odd (or even) ν . The first version of the program was used in the range $\nu < 43$. The second, improved version, was capable of finding solutions for $\nu \leq 51$. Some of the timings for these computations are given in section A.

In section 2 we give several examples of symmetric Hadamard matrices of new orders 188, 292, and 452.

In section 3 we list the normalized propus parameter sets for odd $\nu \in \{43, 45, \dots, 59\}$ and for each of them we indicate whether propus families with that parameter set exist and, if they do, which of the blocks A or D can be chosen to be symmetric. This list together with a similar list in [1] shows that there is a rich supply of propus type symmetric Hadamard matrices for orders 4ν with odd $\nu < 50$. Sporadic examples are also known for $\nu = 53, 55, 57$. The first undecided case is $\nu = 59$.

In section 4 we focus on the case where $\nu = s^2$ is an odd square. We compute the number of propus parameter sets $(\nu; x, y, y, z; \lambda)$ with $\nu = s^2$ by dropping the normalization condition $x \geq z$. This number, N_s ,

is also the number of positive odd integer solutions of a simple quadratic Diophantine equation, namely (9). When s is an odd prime then we conjecture that $N_s - s - 1 \in \{+1, -1\}$. We refer to the cases where v is odd and all k_i are equal as exceptional cases. They occur only when $v = s^2$. We also conjecture that every prime $s \equiv 1 \pmod{4}$ can be written uniquely as $s = (a^2 + b^2)/(a - b)$ where a and b are positive integers and $1 < a \leq (s - 1)/2$. Moreover, the denominator $a - b$ is either a square or 2 times a square.

Finally in section A, for each of the normalized propus parameter sets with odd $v = 43, 45, \dots, 51$, but excluding the exceptional parameter set $(49; 21, 21, 21, 21; 35)$, we list one or two examples of propus difference families.

2 Symmetric Hadamard matrices of new orders

The smallest order $4v$ for which no symmetric Hadamard matrix was known previously is $188 = 4 \cdot 47$. There are four propus parameter sets

$(47; 20, 22, 22, 18; 35)$, $(47; 22, 20, 20, 19; 34)$, $(47; 23, 19, 19, 21; 35)$, $(47; 23, 22, 22, 17; 37)$

with $v = 47$. In each case we constructed many such matrices, but here we record just two examples for each parameter set. In all four cases, A is symmetric in the first and D symmetric in the second example. As $B = C$ we omit the block C . The examples are separated by semicolons.

$(47; 20, 22, 22, 18; 35)$

$[1, 2, 6, 7, 12, 14, 15, 18, 22, 23, 24, 25, 29, 32, 33, 35, 40, 41, 45, 46],$

$[0, 1, 2, 3, 4, 7, 9, 10, 13, 14, 19, 26, 28, 30, 32, 34, 35, 36, 37, 39, 42, 46],$

$[0, 1, 2, 10, 12, 15, 20, 23, 26, 27, 28, 30, 33, 34, 39, 42, 43, 45];$

$[0, 1, 3, 4, 6, 7, 10, 11, 13, 15, 18, 19, 24, 29, 31, 33, 35, 37, 38, 45],$

$[0, 1, 2, 5, 8, 9, 10, 12, 13, 18, 19, 23, 24, 25, 27, 29, 31, 32, 38, 39, 41, 44],$

$[9, 10, 11, 12, 14, 16, 20, 21, 23, 24, 26, 27, 31, 33, 35, 36, 37, 38];$

$(47; 22, 20, 20, 19; 34)$

$[1, 4, 5, 7, 8, 9, 11, 12, 16, 18, 21, 26, 29, 31, 35, 36, 38, 39, 40, 42, 43, 46],$

$[0, 1, 2, 3, 7, 15, 16, 19, 21, 23, 26, 27, 28, 29, 30, 32, 34, 37, 38, 44],$

$[0, 1, 2, 3, 8, 9, 11, 12, 13, 18, 20, 21, 26, 27, 32, 34, 36, 41, 44];$

$[0, 1, 2, 3, 7, 9, 10, 12, 14, 16, 17, 18, 20, 23, 26, 27, 28, 35, 37, 42, 43, 45],$

$[0, 1, 2, 3, 9, 11, 13, 14, 19, 23, 26, 27, 29, 30, 32, 33, 34, 35, 38, 43],$

$[0, 4, 6, 11, 15, 16, 18, 19, 22, 23, 24, 25, 28, 29, 31, 32, 36, 41, 43];$

$(47; 23, 19, 19, 21; 35)$

$[0, 1, 4, 5, 7, 9, 10, 11, 12, 15, 19, 22, 25, 28, 32, 35, 36, 37, 38, 40, 42, 43, 46],$

$[0, 1, 2, 3, 5, 8, 9, 10, 12, 13, 17, 19, 22, 24, 28, 30, 34, 36, 37],$

$[0, 1, 2, 3, 13, 14, 17, 18, 19, 21, 25, 26, 27, 30, 32, 34, 35, 40, 41, 43, 44];$

$[0, 1, 2, 3, 5, 6, 7, 8, 11, 13, 15, 16, 18, 22, 23, 24, 26, 27, 29, 33, 38, 40, 45],$

$[0, 1, 2, 3, 5, 11, 12, 17, 22, 25, 29, 30, 31, 33, 34, 35, 37, 38, 41],$

$[0, 2, 4, 7, 8, 10, 11, 16, 17, 21, 23, 24, 26, 30, 31, 36, 37, 39, 40, 43, 45];$

$$\begin{aligned}
& (47; 23, 22, 22, 17; 37) \\
& [0, 1, 4, 11, 12, 13, 15, 16, 18, 19, 21, 22, 25, 26, 28, 29, 31, 32, 34, 35, 36, 43, 46], \\
& [0, 1, 2, 3, 4, 5, 8, 10, 13, 18, 19, 21, 23, 25, 27, 29, 30, 36, 39, 41, 42, 43], \\
& [0, 1, 2, 5, 6, 10, 11, 12, 15, 21, 25, 26, 33, 38, 40, 41, 45]; \\
& [0, 1, 2, 3, 4, 7, 8, 10, 11, 12, 13, 16, 19, 21, 23, 25, 26, 29, 31, 33, 34, 35, 45], \\
& [0, 1, 2, 3, 4, 8, 9, 12, 13, 14, 17, 18, 19, 20, 26, 27, 29, 31, 34, 37, 40, 44], \\
& [0, 2, 6, 13, 15, 18, 20, 21, 22, 25, 26, 27, 29, 32, 34, 41, 45].
\end{aligned}$$

Let us give a concrete example. We choose the first parameter set above, $(47; 20, 22, 22, 18; 35)$, and its first propus difference family, namely:

$$\begin{aligned}
A &= [1, 2, 6, 7, 12, 14, 15, 18, 22, 23, 24, 25, 29, 32, 33, 35, 40, 41, 45, 46], \\
B = C &= [0, 1, 2, 3, 4, 7, 9, 10, 13, 14, 19, 26, 28, 30, 32, 34, 35, 36, 37, 39, 42, 46], \\
D &= [0, 1, 2, 10, 12, 15, 20, 23, 26, 27, 28, 30, 33, 34, 39, 42, 43, 45].
\end{aligned}$$

The binary $\{+1, -1\}$ -sequences $a, b = c, d$ associated with the base blocks $A, B = C, D$ are:

$$\begin{aligned}
a &= [1, -1, -1, 1, 1, 1, -1, -1, 1, 1, 1, 1, -1, 1, -1, -1, 1, 1, -1, 1, 1, 1, -1, \\
&\quad -1, -1, -1, 1, 1, 1, -1, 1, 1, -1, -1, 1, -1, 1, 1, 1, 1, -1, -1, 1, 1, 1, -1, -1], \\
b = c &= [-1, -1, -1, -1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, 1, 1, -1, 1, 1, 1, \\
&\quad 1, 1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, -1, -1, -1, 1, -1, 1, 1, -1, 1, 1, 1, -1], \\
d &= [-1, -1, -1, 1, 1, 1, 1, 1, 1, 1, -1, 1, -1, 1, 1, -1, 1, 1, 1, 1, -1, 1, 1, -1, \\
&\quad 1, 1, -1, -1, -1, 1, -1, 1, 1, -1, -1, 1, 1, 1, 1, -1, 1, 1, -1, -1, 1, -1, 1].
\end{aligned}$$

Let $C_1, C_2 = C_3, C_4$ be the circulant matrices whose first rows are the sequences $a, b = c, d$. Note that C_1 is symmetric. By plugging these circulants into the propus array (1) we obtain a symmetric Hadamard matrix of order 188.

Next we give the symmetric Hadamard matrices of order $4v$ where $v = 73, 113$. No symmetric Hadamard matrices of these orders were known previously.

We build the four base blocks $A, B = C, D$ as a union of orbits of a subgroup H of \mathbf{Z}_v^* acting on the finite field \mathbf{Z}_v . We choose $H = \{1, 8, 64\}$ for $v = 73$ and $H = \{1, 16, 28, 30, 49, 106, 109\}$ for $v = 113$.

For $v = 73$ we use the parameter set $(73; 36, 36, 36, 28; 63)$. The base blocks of the two propus difference families are:

$$\begin{aligned}
A &= \bigcup_{i \in I} iH, \quad I = \{1, 2, 3, 4, 9, 11, 18, 21, 26, 27, 36, 43\} \\
B = C &= \bigcup_{j \in J} jH, \quad J = \{2, 4, 5, 6, 9, 12, 14, 17, 27, 34, 35, 36\} \\
D &= \{0\} \cup \bigcup_{k \in K} kH, \quad K = \{1, 2, 3, 6, 7, 9, 18, 42, 43\}; \\
A &= \bigcup_{i \in I} iH, \quad I = \{1, 2, 3, 4, 6, 9, 12, 18, 25, 27, 35, 36\} \\
B = C &= \bigcup_{j \in J} jH, \quad J = \{2, 5, 7, 9, 13, 17, 25, 26, 33, 35, 36, 42\}
\end{aligned}$$

$$D = \{0\} \cup \bigcup_{k \in K} kH, \quad K = \{4, 6, 13, 18, 27, 34, 35, 36, 42\}.$$

For $v = 113$ we use the parameter set $(113; 56, 49, 49, 56; 97)$. The base blocks of the four propus difference families are:

$$\begin{aligned} A &= \bigcup_{i \in I} iH, \quad I = \{1, 4, 5, 6, 13, 17, 18, 20\} \\ B = C &= \bigcup_{j \in J} jH, \quad J = \{1, 5, 9, 11, 12, 17, 39\} \\ D &= \bigcup_{k \in K} kH, \quad K = \{2, 3, 5, 10, 11, 12, 18, 20\}; \\ A &= \bigcup_{i \in I} iH, \quad I = \{1, 4, 5, 6, 13, 17, 18, 20\} \\ B = C &= \bigcup_{j \in J} jH, \quad J = \{1, 2, 4, 11, 12, 13, 17\} \\ D &= \bigcup_{k \in K} kH, \quad K = \{1, 2, 3, 5, 11, 12, 18, 20\}; \end{aligned}$$

$$\begin{aligned} A &= \bigcup_{i \in I} iH, \quad I = \{1, 4, 5, 6, 13, 17, 18, 20\} \\ B = C &= \bigcup_{j \in J} jH, \quad J = \{1, 2, 4, 11, 12, 13, 17\} \\ D &= \bigcup_{k \in K} kH, \quad K = \{3, 4, 5, 8, 9, 12, 13, 20\}; \\ A &= \bigcup_{i \in I} iH, \quad I = \{1, 3, 4, 10, 12, 13, 18, 39\} \\ B = C &= \bigcup_{j \in J} jH, \quad J = \{2, 5, 9, 10, 17, 20, 39\} \\ D &= \bigcup_{k \in K} kH, \quad K = \{2, 3, 9, 11, 12, 17, 20, 39\}. \end{aligned}$$

The first three families share the same block A , and the second and third family differ only in block D . In spite of that, the four families are pairwise nonequivalent. The equivalence is defined as follows.

We say that two cyclic propus difference families (A_1, A_2, A_3, A_4) and (B_1, B_2, B_3, B_4) having the same parameter set $(v; k_1, k_2, k_3, k_4; \lambda)$ are *equivalent* if there is an automorphism ϕ of the cyclic group \mathbf{Z}_v such that B_i is a translate of A_i for each i .

3 Normalized propus parameter sets

We list here all normalized propus parameter sets $(v; x, y, y, z; \lambda)$ for odd $v = 43, 45, \dots, 59$. The cyclic propus families consisting of four base blocks $A, B, C, D \subseteq \mathbf{Z}_v$ having sizes x, y, y, z , respectively, and such that

$B = C$ and A or D is symmetric give symmetric Hadamard matrices of order $4v$. (If only D is symmetric we have to switch A and D before plugging the blocks into the propus array.) If $x = z \neq y$ then the parameter set $(v; y, x, x, y; \lambda)$ is also normalized and is included in our list. In the former case the two base blocks of size y have to be equal, while in the latter case the base blocks of size x have to be equal.

The four base blocks, subsets of Z_v , are denoted by A, B, C, D . We require all propus difference families to have $B = C$. If we know such a family exists with symmetric block A , we indicate this by writing the symbol A after the parameter set, and similarly for the symbol D . If we know that there exists a propus family with both A and D symmetric, then we write the symbol AD . Finally, the question mark means that the existence of a cyclic propus difference family remains undecided.

The symbol T indicates that the parameter set belongs to the Turyn series of Williamson matrices. Since in that case all four base blocks are symmetric, the symbol T implies AD . Further, the symbol X indicates that the parameter set belongs to another infinite series (see [3, Theorem 5]) which is based on the paper [5] of Xia, Xia, Seberry, and Wu. In our list below the symbol X implies D . More precisely, for a difference family A, B, C, D in the X -series two blocks are equal, say $B = C$, and one of the remaining blocks is skew, block A in our list, and the last one is symmetric, block D .

For odd v in the range $43, 45, \dots, 51$ there is only one propus parameter set, $(49; 21, 21, 21, 21; 36)$, for which we failed to find a cyclic propus difference family. (We believe that such family does not exist.)

Normalized propus parameter sets with v odd, $43 \leq v \leq 59$

$(43; 18, 21, 21, 16; 33)$	A, D	$(43; 19, 18, 18, 18; 30)$	A, D
$(43; 21, 17, 17, 20; 32)$	A, D	$(43; 21, 19, 19, 16; 32)$	A, D
$(43; 21, 21, 21, 15; 35)$	A, D	$(45; 18, 21, 21, 18; 33)$	A, D
$(45; 19, 20, 20, 18; 32)$	AD, T	$(45; 21, 18, 18, 21; 33)$	A, D
$(45; 21, 20, 20, 17; 33)$	A, D	$(45; 21, 22, 22, 16; 36)$	A, D
$(45; 22, 19, 19, 18; 33)$	A, D, X	$(47; 20, 22, 22, 18; 35)$	A, D
$(47; 22, 20, 20, 19; 34)$	A, D	$(47; 23, 19, 19, 21; 35)$	A, D
$(47; 23, 22, 22, 17; 37)$	A, D	$(49; 21, 21, 21, 21; 35)$?
$(49; 22, 22, 22, 19; 36)$	A, D	$(49; 22, 24, 24, 18; 39)$	A, D
$(49; 23, 20, 20, 22; 36)$	AD, T	$(49; 23, 23, 23, 18; 38)$	A, D
$(51; 21, 25, 25, 20; 40)$	AD, T	$(51; 23, 22, 22, 21; 37)$	A, D
$(53; 22, 24, 24, 22; 39)$?	$(53; 24, 22, 22, 24; 39)$?
$(53; 24, 25, 25, 20; 41)$?	$(53; 26, 22, 22, 23; 40)$	D, X
$(55; 23, 26, 26, 22; 42)$	AD, T	$(55; 24, 25, 25, 22; 41)$?
$(55; 24, 27, 27, 21; 44)$?	$(55; 26, 23, 23, 24; 41)$?
$(55; 27, 24, 24, 22; 42)$?	$(55; 27, 25, 25, 21; 43)$?
$(57; 25, 25, 25, 24; 42)$?	$(57; 27, 25, 25, 23; 43)$?
$(57; 27, 26, 26, 22; 44)$?	$(57; 28, 28, 28, 21; 48)$	D, X
$(59; 26, 28, 28, 23; 46)$?	$(59; 27, 25, 25, 26; 44)$?
$(59; 28, 29, 29, 22; 49)$?		

In order to justify the claims made in this list, we give in section A examples of the propus difference families having the required properties. (For $v = 47$ the examples are listed in section 2.)

4 Exceptional series of propus parameter sets

We say that a propus parameter set $(v; k_1, k_2, k_3, k_4 : \lambda)$ is *exceptional* if $k_1 = k_2 = k_3 = k_4$. The exceptional parameter sets are parametrized by just one integer $s > 1$,

$$\Pi_s = (s^2; \binom{s}{2}, \binom{s}{2}, \binom{s}{2}, \binom{s}{2}; s(s-2)). \tag{7}$$

There exists a cyclic propus difference family with parameter set Π_3 . There exists also a propus difference family (A, B, C, D) over the group $\mathbf{Z}_3 \times \mathbf{Z}_3$ with the same parameter set and such that A is symmetric and $B = C = D$. By using the finite field $\mathbf{Z}_3[\alpha]$ where $\alpha^2 = -1$, we can take

$$A = \{0, \alpha, -\alpha\}, \quad B = C = D = \{\alpha, 1 - \alpha, \alpha - 1\}. \tag{8}$$

For $s = 5$, it is reported in [1] that there are no propus difference families in \mathbf{Z}_{25} having Π_5 as its parameter set. We performed another exhaustive search and found no such families in $\mathbf{Z}_5 \times \mathbf{Z}_5$.

For $s = 7$, our non-exhaustive searches found no cyclic propus difference families having the parameter set Π_7 . However, we found a cyclic difference family with parameter set Π_7 and $B = C$ with neither A nor D symmetric:

$$\begin{aligned} A &= [0, 1, 2, 3, 4, 8, 11, 12, 14, 19, 21, 24, 26, 27, 29, 37, 38, 41, 44, 45, 46], \\ B &= C = [0, 1, 2, 3, 5, 7, 11, 14, 15, 17, 24, 27, 28, 29, 32, 35, 38, 43, 44, 45, 47], \\ D &= [0, 1, 2, 5, 6, 8, 10, 11, 12, 14, 16, 18, 21, 22, 23, 30, 31, 32, 36, 37, 41]. \end{aligned}$$

While computing the propus parameter sets $(v; x, y, y, z; \lambda)$ in the case when $v = s^2$ is an odd square, we observed an interesting feature. Namely, if in the definition of normalized propus difference sets we drop only the condition that $x \geq z$ and if s is an odd prime then the number, N_s , of such parameter sets is either s or $s + 2$. It follows from the proof of [1, Theorem 1] that N_s is equal to the number of odd positive integer solutions of the Diophantine equation

$$\xi^2 + 2\eta^2 + \zeta^2 = 4s^2. \tag{9}$$

After making additional computations, we decided to propose the following conjecture.

Conjecture 1. For any odd prime s , $N_s - s - 1 \in \{+1, -1\}$.

We have verified our conjecture for all odd primes less than 10000. There are 1228 such primes. For 606 of them we have $N_s = s$ and for the remaining 622 we have $N_s = s + 2$. Thus the sequence $N_s - s - 1$ is a $\{+1, -1\}$ -sequence when s runs through odd primes < 10000 . We have sketched the partial sums of this sequence on Figure 1. It shows that, for the first 1228 values of s , the partial sums of the sequence $N_s - s - 1$ are mostly positive.

If s is a prime congruent to 1 (mod 4), we observed that apart from Π_s there is another normalized propus parameter set with $v = s^2$ and $k_2 = k_3 = \binom{s}{2}$. Let us denote this new parameter set by

$$\Pi'_s = (s^2; \binom{s}{2} + \alpha, \binom{s}{2}, \binom{s}{2}, \binom{s}{2} - \beta; s(s-2) + \alpha - \beta). \tag{10}$$

The integers α and β are positive and satisfy the quadratic Diophantine equation

$$\alpha^2 + \beta^2 = s(\alpha - \beta). \tag{11}$$

We propose another conjecture.

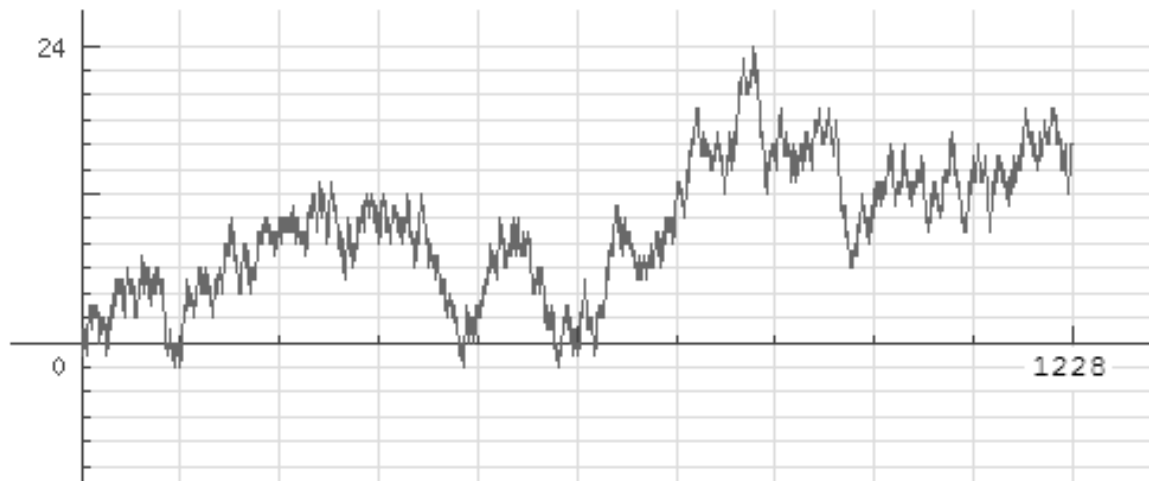


Figure 1: Partial sums of the sequence $N_s - s - 1$, s odd prime

Conjecture 2. For any odd prime $s \equiv 1 \pmod{4}$ the Diophantine equation (11), in the unknowns α and β , has a unique solution (a, b) , where a and b are positive integers and $1 < a \leq (s-1)/2$. Moreover, $a - b$ is either a square or 2 times a square.

We have verified that this conjecture holds for $s < 100000$. If we drop the condition $1 < a \leq (s-1)/2$, then there exists one more solution, namely $(s-a, b)$. Note also that the two solutions share the same b , and so the integer b is uniquely determined by s .

A Appendix

The cyclic propus difference families listed below, except some of the families that belong to one of the two infinite series T and X , have been constructed by using a computer program written by one of the authors. The program was run on two PCs, each with a single 64-bit processor. For $v = 39$ it takes about 5 minutes to obtain a solution, about 20 minutes for $v = 41$, about 1 hour for $v = 43$, about 3 or 4 hours for $v = 45$, about 12 hours for $v = 47$, about 2 days for $v = 49$, and 5 days for $v = 51$. In all families below the base block $B = C$, and to save space we omit the block C . The families are terminated by semicolons.

(43; 18, 21, 21, 16; 33)
 [1, 3, 6, 9, 14, 15, 16, 19, 20, 23, 24, 27, 28, 29, 34, 37, 40, 42],
 [0, 1, 2, 4, 5, 10, 12, 14, 15, 16, 17, 20, 21, 23, 24, 26, 27, 28, 32, 34, 41],
 [0, 1, 2, 3, 9, 10, 13, 15, 18, 21, 29, 34, 36, 37, 38, 39];
 [0, 1, 2, 3, 7, 8, 9, 10, 16, 17, 20, 22, 25, 28, 36, 41],
 [0, 1, 2, 3, 6, 7, 9, 10, 12, 13, 14, 18, 20, 27, 29, 30, 31, 33, 34, 39, 41],
 [1, 3, 6, 9, 14, 15, 16, 19, 20, 23, 24, 27, 28, 29, 34, 37, 40, 42];

(43; 19, 18, 18, 18; 30)
 [0, 4, 9, 10, 11, 15, 16, 18, 19, 21, 22, 24, 25, 27, 28, 32, 33, 34, 39],
 [0, 1, 2, 3, 11, 12, 17, 19, 20, 23, 24, 25, 27, 29, 31, 33, 36, 40],
 [0, 1, 2, 3, 5, 10, 12, 15, 18, 23, 25, 26, 28, 29, 36, 39, 40, 41];

[0, 1, 2, 5, 9, 10, 14, 16, 20, 23, 24, 27, 29, 30, 32, 34, 36, 38, 40],
 [0, 1, 2, 3, 4, 8, 9, 10, 11, 14, 18, 21, 26, 27, 30, 32, 40, 42],
 [2, 7, 8, 9, 10, 13, 15, 18, 21, 22, 25, 28, 30, 33, 34, 35, 36, 41];

(43; 21, 17, 17, 20; 32)

[0, 1, 3, 6, 7, 9, 11, 14, 16, 20, 21, 22, 23, 27, 29, 32, 34, 36, 37, 40, 42],
 [0, 1, 2, 3, 5, 6, 7, 13, 15, 24, 25, 28, 29, 32, 37, 39, 40],
 [0, 1, 2, 3, 10, 12, 14, 15, 18, 19, 20, 25, 28, 29, 31, 32, 34, 35, 37, 39];
 [0, 1, 2, 3, 4, 7, 12, 13, 14, 18, 20, 23, 24, 28, 30, 32, 33, 34, 36, 38, 41],
 [0, 1, 2, 5, 8, 10, 15, 17, 18, 19, 21, 24, 25, 30, 36, 37, 40],
 [1, 3, 4, 5, 6, 7, 8, 13, 18, 21, 22, 25, 30, 35, 36, 37, 38, 39, 40, 42];

(43; 21, 19, 19, 16; 32)

[0, 1, 6, 11, 12, 13, 16, 17, 19, 20, 21, 22, 23, 24, 26, 27, 30, 31, 32, 37, 42],
 [0, 1, 2, 6, 8, 9, 12, 15, 17, 20, 22, 23, 24, 26, 27, 35, 36, 39, 41],
 [0, 1, 2, 6, 8, 9, 11, 15, 16, 18, 20, 24, 28, 29, 31, 41];
 [0, 1, 2, 3, 4, 7, 11, 13, 15, 17, 19, 20, 22, 32, 33, 34, 35, 37, 39, 40, 42],
 [0, 1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 20, 23, 24, 25, 27, 34, 38, 39],
 [2, 3, 4, 10, 12, 14, 15, 20, 23, 28, 29, 31, 33, 39, 40, 41];

(43; 21, 21, 21, 15; 35)

[0, 1, 2, 3, 4, 8, 9, 12, 14, 19, 22, 23, 26, 28, 29, 31, 32, 34, 38, 39, 41],
 [1, 4, 6, 9, 10, 11, 13, 14, 15, 16, 17, 21, 23, 24, 25, 31, 35, 36, 38, 40, 41],
 [0, 7, 9, 13, 14, 15, 17, 18, 25, 26, 28, 29, 30, 34, 36];
 [0, 1, 2, 3, 6, 7, 9, 11, 13, 15, 16, 17, 21, 22, 25, 26, 29, 33, 38, 39, 41],
 [0, 1, 2, 3, 4, 5, 6, 7, 8, 13, 16, 17, 20, 22, 25, 27, 29, 34, 35, 37, 40],
 [0, 5, 6, 12, 13, 14, 16, 20, 23, 27, 29, 30, 31, 37, 38];

The last example consists of a D-optimal design (blocks A and D) and two copies of the Paley difference set in \mathbf{Z}_{43} (blocks $B = C$). It is taken from the paper [3].

(45; 18, 21, 21, 18; 33)

[4, 7, 8, 9, 10, 11, 16, 19, 20, 25, 26, 29, 34, 35, 36, 37, 38, 41],
 [0, 1, 2, 3, 5, 7, 8, 12, 13, 16, 19, 22, 23, 27, 32, 34, 36, 39, 40, 42, 44],
 [0, 1, 2, 3, 10, 12, 13, 15, 17, 19, 24, 25, 32, 34, 37, 38, 39, 41];

(45; 19, 20, 20, 18; 32)

[0, 1, 6, 12, 13, 14, 16, 17, 20, 22, 23, 25, 28, 29, 31, 32, 33, 39, 44],
 [1, 3, 7, 8, 10, 11, 12, 13, 17, 20, 25, 28, 32, 33, 34, 35, 37, 38, 42, 44],
 [1, 6, 12, 13, 14, 16, 17, 20, 22, 23, 25, 28, 29, 31, 32, 33, 39, 44];

(45; 21, 18, 18, 21; 33)

[0, 4, 6, 11, 12, 13, 14, 17, 18, 20, 22, 23, 25, 27, 28, 31, 32, 33, 34, 39, 41],

[0, 1, 2, 5, 6, 8, 9, 11, 13, 20, 21, 23, 31, 32, 34, 37, 38, 41],

[0, 1, 2, 3, 6, 10, 12, 17, 18, 19, 21, 22, 23, 25, 33, 35, 37, 38, 40, 41, 42];

(45; 21, 20, 20, 17; 33)

[0, 2, 3, 4, 5, 10, 14, 15, 17, 19, 22, 23, 26, 28, 30, 31, 35, 40, 41, 42, 43],

[0, 1, 2, 3, 4, 8, 12, 13, 15, 22, 23, 26, 28, 30, 31, 36, 37, 39, 42, 43],

[0, 1, 2, 3, 4, 6, 10, 14, 20, 26, 27, 30, 33, 35, 37, 42, 43];

[0, 1, 2, 3, 8, 9, 15, 16, 18, 22, 23, 25, 28, 32, 33, 34, 36, 38, 39, 42, 43],

[0, 1, 2, 3, 5, 6, 8, 9, 12, 14, 16, 18, 19, 20, 24, 27, 29, 31, 32, 41],

[0, 1, 2, 7, 10, 12, 18, 19, 22, 23, 26, 27, 33, 35, 38, 43, 44];

(45; 21, 22, 22, 16; 36)

[0, 3, 4, 6, 7, 9, 11, 12, 13, 14, 18, 27, 31, 32, 33, 34, 36, 38, 39, 41, 42],

[0, 1, 2, 3, 4, 6, 9, 11, 14, 15, 16, 19, 20, 26, 28, 30, 34, 35, 36, 38, 42, 43],

[0, 1, 2, 3, 10, 11, 13, 17, 22, 23, 24, 27, 30, 34, 39, 42];

[0, 1, 2, 4, 6, 7, 9, 11, 12, 17, 21, 24, 25, 27, 28, 30, 32, 33, 34, 39, 43],

[0, 1, 2, 4, 5, 9, 10, 13, 14, 15, 18, 20, 21, 26, 28, 29, 35, 36, 38, 40, 42, 43],

[3, 9, 13, 15, 16, 17, 18, 19, 26, 27, 28, 29, 30, 32, 36, 42];

(45; 22, 19, 19, 18; 33)

[2, 3, 4, 7, 8, 10, 12, 13, 14, 17, 20, 25, 28, 31, 32, 33, 35, 37, 38, 41, 42, 43],

[0, 1, 2, 5, 9, 11, 14, 18, 19, 20, 22, 24, 26, 27, 30, 31, 32, 33, 34],

[0, 1, 2, 7, 9, 10, 13, 16, 17, 19, 24, 27, 33, 35, 36, 38, 40, 43];

[0, 1, 2, 3, 7, 10, 11, 15, 16, 18, 19, 20, 25, 28, 30, 31, 35, 36, 37, 40, 42, 43],

[0, 1, 2, 4, 6, 12, 19, 20, 21, 24, 25, 29, 31, 32, 33, 35, 40, 42, 43],

[1, 3, 4, 5, 8, 10, 11, 18, 21, 24, 27, 34, 35, 37, 40, 41, 42, 44];

(49; 22, 22, 22, 19; 36)

[1, 3, 5, 8, 9, 11, 12, 15, 16, 18, 19, 30, 31, 33, 34, 37, 38, 40, 41, 44, 46, 48],

[0, 1, 2, 3, 4, 5, 6, 9, 14, 15, 18, 25, 27, 30, 32, 33, 35, 37, 38, 42, 43, 44],

[0, 1, 2, 5, 6, 10, 12, 14, 18, 19, 21, 27, 32, 34, 35, 36, 40, 43, 45];

[0, 1, 2, 3, 4, 7, 10, 14, 15, 18, 19, 24, 26, 30, 31, 32, 33, 35, 37, 40, 41, 47],

[0, 1, 2, 3, 4, 6, 10, 11, 14, 16, 17, 23, 24, 31, 34, 36, 38, 39, 41, 42, 43, 47],

[0, 1, 3, 6, 8, 12, 18, 21, 22, 23, 26, 27, 28, 31, 37, 41, 43, 46, 48];

(49; 22, 24, 24, 18; 39)

[2, 3, 6, 8, 9, 17, 19, 20, 21, 22, 24, 25, 27, 28, 29, 30, 32, 40, 41, 43, 46, 47],

[0, 1, 2, 3, 7, 8, 9, 16, 19, 21, 23, 25, 26, 28, 29, 32, 34, 36, 37, 38, 40, 41, 42, 46],
 [0, 1, 2, 3, 7, 8, 12, 15, 17, 19, 26, 29, 36, 37, 39, 42, 43, 46];
 [0, 1, 2, 3, 7, 10, 12, 13, 14, 17, 19, 20, 22, 23, 25, 26, 27, 28, 32, 37, 40, 46],
 [0, 1, 2, 3, 4, 5, 6, 9, 11, 13, 14, 15, 17, 19, 22, 27, 28, 31, 33, 34, 35, 38, 43, 45],
 [2, 5, 6, 10, 16, 17, 19, 23, 24, 25, 26, 30, 32, 33, 39, 43, 44, 47];

(49; 23, 20, 20, 22; 36)

[0, 1, 2, 4, 6, 8, 15, 16, 17, 20, 21, 23, 26, 28, 29, 32, 33, 34, 41, 43, 45, 47, 48],
 [3, 6, 10, 13, 14, 15, 20, 21, 23, 24, 25, 26, 28, 29, 34, 35, 36, 39, 43, 46],
 [1, 2, 4, 6, 8, 15, 16, 17, 20, 21, 23, 26, 28, 29, 32, 33, 34, 41, 43, 45, 47, 48];

(49; 23, 23, 23, 18; 38)

[0, 3, 4, 7, 9, 10, 11, 13, 15, 17, 18, 23, 26, 31, 32, 34, 36, 38, 39, 40, 42, 45, 46],
 [0, 1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 15, 16, 21, 23, 25, 27, 28, 32, 35, 40, 41, 43],
 [0, 1, 2, 5, 6, 13, 15, 18, 21, 22, 27, 32, 34, 36, 37, 39, 46, 47];
 [0, 1, 2, 3, 4, 5, 6, 7, 8, 11, 15, 17, 20, 24, 27, 29, 33, 36, 38, 41, 44, 45, 47],
 [0, 1, 2, 3, 5, 7, 8, 10, 12, 14, 20, 22, 23, 24, 30, 31, 32, 35, 36, 37, 38, 41, 46],
 [3, 4, 10, 13, 14, 16, 20, 21, 24, 25, 28, 29, 33, 35, 36, 39, 45, 46];

(51; 23, 22, 22, 21; 37)

[0, 2, 4, 10, 11, 13, 16, 20, 21, 23, 24, 25, 26, 27, 28, 30, 31, 35, 38, 40, 41, 47, 49],
 [0, 1, 2, 3, 4, 5, 9, 13, 15, 20, 22, 23, 27, 31, 32, 33, 38, 39, 41, 44, 47, 48],
 [0, 1, 2, 3, 6, 8, 9, 12, 13, 14, 17, 19, 22, 31, 34, 36, 37, 38, 40, 44, 49];
 [0, 1, 2, 3, 4, 5, 10, 12, 13, 14, 15, 19, 21, 22, 28, 30, 34, 37, 39, 41, 42, 47, 49],
 [0, 1, 2, 4, 5, 8, 10, 13, 18, 19, 21, 24, 25, 28, 29, 31, 33, 35, 38, 39, 40, 43],
 [0, 2, 3, 4, 5, 6, 9, 13, 19, 20, 25, 26, 31, 32, 38, 42, 45, 46, 47, 48, 49];

(51; 25, 25, 21, 20; 40)

[0, 1, 4, 5, 7, 9, 15, 16, 17, 18, 22, 29, 33, 34, 35, 36, 42, 44, 46, 47, 50],
 [0, 1, 2, 4, 5, 7, 8, 11, 15, 16, 21, 23, 25, 26, 28, 30, 35, 36, 40, 43, 44, 46, 47, 49, 50],
 [1, 4, 5, 7, 9, 15, 16, 17, 18, 22, 29, 33, 34, 35, 36, 42, 44, 46, 47, 50];

(53; 26, 22, 22, 23; 40)

[1, 5, 6, 10, 11, 12, 15, 18, 22, 27, 28, 29, 30, 32, 33, 34, 36, 37, 39, 40, 44, 45, 46, 49, 50, 51],
 [0, 1, 2, 3, 9, 11, 18, 21, 24, 25, 29, 33, 34, 35, 36, 41, 44, 46, 48, 49, 50, 52],
 [0, 1, 3, 9, 10, 12, 14, 16, 17, 20, 23, 25, 28, 30, 33, 36, 37, 39, 41, 43, 44, 50, 52];

(55; 23, 26, 26, 22; 42)

[0, 6, 7, 10, 11, 15, 17, 18, 19, 21, 24, 26, 29, 31, 34, 36, 37, 38, 40, 44, 45, 48, 49],

[1, 2, 4, 8, 14, 16, 17, 18, 19, 23, 24, 25, 27, 28, 30, 31, 32, 36, 37, 38, 39, 41, 47, 51, 53, 54],

[6, 7, 10, 11, 15, 17, 18, 19, 21, 24, 26, 29, 31, 34, 36, 37, 38, 40, 44, 45, 48, 49];

(57; 28, 28, 28, 21; 48)

[2, 4, 12, 13, 15, 21, 23, 24, 25, 27, 28, 31, 35, 37, 38, 39, 40, 41, 43, 46, 47, 48, 49, 50, 51, 52, 54, 56],

[0, 1, 2, 3, 4, 6, 9, 11, 13, 16, 17, 20, 23, 28, 31, 32, 34, 35, 37, 39, 40, 41, 43, 44, 45, 49, 50, 53],

[0, 1, 4, 6, 13, 14, 15, 19, 20, 21, 26, 31, 36, 37, 38, 42, 43, 44, 51, 53, 56];

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