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Extensions of pseudo-Perron-Frobenius splitting related to generalized inverse $A^{(2)}_{T,S}$

https://doi.org/10.1515/spma-2018-0005
Received November 2, 2017; accepted January 8, 2018

Abstract: We in this paper define the outer-Perron-Frobenius splitting, which is an extension of the pseudo-Perron-Frobenius splitting defined in [A.N. Sushama, K. Premakumari, K.C. Sivakumar, Extensions of Perron-Frobenius splittings and relationships with nonnegative Moore-Penrose inverse, Linear and Multilinear Algebra 63 (2015) 1–11]. We present some criteria for the convergence of the outer-Perron-Frobenius splitting. The findings of this paper generalize some known results in the literatures.

Keywords: proper splitting, Perron-Frobenius splitting, outer-Perron-Frobenius splitting, generalized inverse, cones of matrices

MSC: 15A09, 15B48

1 Introduction and preliminaries

Let $\mathbb{C}^{m\times n}$ ($\mathbb{R}^{m\times n}$) denote the set of all $m \times n$ complex (real) matrices with rank $r$. $\mathbb{R}^n$ stands for the set of all vectors $x \in \mathbb{R}^n$ such that $x \geq 0$ ($x \geq 0$ means that all the coordinates of $x$ are nonnegative). For $A \in \mathbb{C}^{m\times n}$, the symbols $A^*$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ denote the conjugate and transpose, the range space and the null space of $A$, respectively. When $m = n$, $\rho(A)$, $\sigma(A)$, and $||A||$ denote the spectral radius, the spectrum, and the multiplicative norm of $A$, respectively.

An outer (or (2)) inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^{(2)}$, is defined to an arbitrary solution $X$ of the matrix equation $XAX = X$. The Moore-Penrose inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^\dagger$, is defined to be the unique solution $X$ to the following four matrix equations

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad (XA)^* =XA.$$  

The Drazin inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A_d$, is defined to be the unique solution $X$ to the following matrix equations

$$XAX = X, \quad AX =XA, \quad A^{k+1}X = A^k$$

for some nonnegative integer $k$. The smallest positive exponent $k$ for which the last equation holds is called Drazin index and denoted by $\text{Ind}(A)$.

The fact is well known [1, p.72, Theorem 14] that let $A \in \mathbb{C}^{m\times n}$, $T$ be a subspace of $\mathbb{C}^n$ of dimension $s \leq r$, and let $S$ be a subspace of $\mathbb{C}^m$ of dimension $m - s$. Then $A$ has an out inverse $X$ such that $\mathcal{R}(X) = T$ and...
\[N(X) = S \text{ if and only if } AT \oplus S = \mathbb{C}^m,\]
in which case \(X\) is unique and called the generalized inverse with prescribed range \(T\) and null space \(S\), and denoted by \(A_{T,S}^{(2)}\). Clearly, the Moore-Penrose inverse and the Drazin inverse are the special cases of the generalized inverse \(A_{T,S}^{(2)}\).

Recall that for \(A \in \mathbb{R}^{n \times n}\), \(A = U - V\) is called a splitting of \(A\) if \(U\) is invertible. A splitting of \(A\) is said to be weak (nonnegative) splitting if \(U^{-1}V \geq 0\); weak regular splitting if \(U^{-1}V > 0\) and \(V \geq 0\); regular splitting if \(U^{-1} \geq 0\) and \(V \geq 0\). For any regular splitting \(A = U - V\), it has been shown that \(A\) is invertible and \(A^{-1}\) is positive if and only if \(\rho(U^{-1}V) < 1\), which ensures the convergence of the iterative scheme by the splitting (see [10]). In [7], a new type of splitting, called a Perron-Frobenius splitting, was proposed as follows.

**Definition 1.1.** [7, Definition 2.1] A matrix \(A \in \mathbb{R}^{n \times n}\) is said to have the

(i) Perron-Frobenius property if \(\rho(A) \geq 0\), \(\rho(A) \in \sigma(A)\) and there exists a nonnegative eigenvector corresponding to \(\rho(A)\).

(ii) Strong Perron-Frobenius property if, in addition to having the Perron-Frobenius property, \(\rho(A)\) is an eigenvalue of \(A\), \(\rho(A) > |\lambda|\) for all \(\lambda \in \sigma(A)\), \(\lambda \neq \rho(A)\) and the corresponding eigenvector is strictly positive.

**Definition 1.2.** [7] Let \(A \in \mathbb{R}^{n \times n}\). The splitting \(A = U - V\) is said to be a Perron-Frobenius splitting if \(U^{-1}V\) possesses the Perron-Frobenius property.

Recently, Suhama, Premakumari and Sivakumar [9] extended the Perron-Frobenius splitting to any matrix (square or rectangular) using the Moore-Penrose inverse, and called it pseudo-Perron-Frobenius splitting. They also gave a criterion for the convergence of the splitting. Noting that the Moore-Penrose inverse and the Drazin inverse are the special cases of the generalized inverse \(A_{T,S}^{(2)}\), we in this paper use the generalized inverse \(A_{T,S}^{(2)}\) to define a new and more general type splitting of a matrix and investigate the criteria for the convergence of this splitting.

This work is organized as follows. In the rest of this section, we will give some notations and results for later discussion. In Section 2, we define the outer-Perron-Frobenius splitting and obtain several necessary conditions and sufficient conditions for convergence of an outer-Perron-Frobenius splitting. Moreover, we present the existence of solutions or positive solutions of the equation \(Ax = 0\) under the condition \(\rho(U_{T,S}^{(2)}V) = 1\).

The notion of a \((T, S)\) splitting plays a crucial role in characterizing various types of monotone matrices. Let us recall its definition.

**Definition 1.3.** [4, 6] Let \(A \in \mathbb{C}^{m \times n}\), and \(T\) and \(S\) be subspaces of \(\mathbb{C}^n\) and \(\mathbb{C}^m\), respectively. Then the splitting \(A = U - V\) is called a proper splitting of \(A\) with respect to \((2)\)-inverse with prescribed range \(T\) and null space \(S\), a \((T, S)\) splitting for short, if \(A_{T,S}^{(2)}\) and \(U_{T,S}^{(2)}\) exist.

**Remark 1.1:** (i) When discussing any topic related to the definition, any generalized inverse of \(A\) is required to have the same range and null space as those of \(U_{T,S}^{(2)}\). It guarantees that there is a relationship between the generalized inverse chosen and \(U_{T,S}^{(2)}\), e.g. Lemma 1.1(ii) and Remark 1.2, and makes the definition cover the proper splitting in [2]. Here we concern \(A_{T,S}^{(2)}\), which is the same type as \(U_{T,S}^{(2)}\). For convenience, we therefore add the condition \(A_{T,S}^{(2)}\) exists to the definition.

(ii) For such known generalized inverses as the Moore-Penrose inverse and the Drazin inverse, range \(T\) and null space \(S\) are implicit, and then the equivalent conditions of the same ranges and null spaces of \(U\) and of \(A\), respectively, need to be explicitly added in their definitions in what follows.

So the proper splitting in [2] is a proper splitting of \(A\) with respect to \((2)\)-inverse with prescribed range \(\mathcal{R}(A)\) and null space \(\mathcal{N}(A)\), where \(\mathcal{R}(A) = \mathcal{R}(U)\) and \(\mathcal{N}(A) = \mathcal{N}(U)\) which are equivalent to \(\mathcal{R}(A^*) = \mathcal{R}(U)\) and \(\mathcal{N}(A^*) = \mathcal{N}(U)\), respectively. Note that \(\mathcal{R}(A) = \mathcal{R}(U)\) and \(\mathcal{N}(A) = \mathcal{N}(U)\) in [2] are equivalent to \(\mathcal{R}(A^*) = \mathcal{R}(U)\) and \(\mathcal{N}(A^*) = \mathcal{N}(U)\), respectively.

With regard to the proper splitting, we need the following lemma.
Lemma 1.1. [4, Theorem 2.1] Let $A \in \mathbb{C}^{m \times n}$ with a $(T, S)$ splitting $A = U - V$, where $T$ and $S$ are subspaces of $\mathbb{C}^n$ and of $\mathbb{C}^m$, respectively. Then the following statements hold.
(i) $I - U^{(2)}_{T,S} V$ is invertible;
(ii) $A^{(2)}_{T,S} = (I - U^{(2)}_{T,S} V)^{-1} U^{(2)}_{T,S}$.

Remark 1.1. When $\mathcal{R}(A') = T$ and $\mathcal{N}(A') = S$ (equivalently, $\mathcal{R}(A') = T$ and $\mathcal{N}(A') = S$), Statement (ii) in Lemma 1.1 becomes $A^T = (I - U^{(2)}_{T,S} V)^{-1} U^{(2)}_{T,S}$ since $A^T = A^{(2)}_{\mathcal{R}(A'), \mathcal{N}(A')}$.

Next, we recall the notions concerning eventually nonnegative (positive) matrix.

Definition 1.4. [8, Definition 2.2] A matrix $A \in \mathbb{R}^{n \times n}$ is called eventually nonnegative (positive), denoted by $A \geq 0$ ($A > 0$), if there exists a nonnegative integer $k_0$ such that $A^k \geq 0$ ($A^k > 0$) for all $k \geq k_0 \geq 0$. In this case, we denote the smallest such nonnegative integer by $k_0 = k_0(A)$ and refer to it as the power index of $A$ with respect to eventual nonnegativity (positivity).

The following known result, needed later in the paper, show the relation of the above definitions.

Lemma 1.2. [7, Theorem 2.2] Let $A \in \mathbb{R}^{n \times n}$, the following are equivalent:
(i) Both $A$ and $A^*$ have the strong Perron-Frobenius property;
(ii) $A$ is eventually positive;
(iii) $A$ is eventually positive.

Now, we introduce new notion. Here we substitute an outer inverse for the inverse in the Perron-Frobenius splitting proposed in [7]. The notion extends that of the pseudo-Perron-Frobenius splitting presented in [9], in which the inverse was replaced by the Moore-Penrose inverse in the Perron-Frobenius splitting.

Definition 1.5. Let $A \in \mathbb{C}^{m \times n}$, and $T$ and $S$ be subspaces of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. A splitting $A = U - V$ of $A$ is said to be an outer-Perron-Frobenius splitting (with prescribed range $T$ and null space $S$) if both $A^{(2)}_{T,S}$ and $U^{(2)}_{T,S}$ exist and $U^{(2)}_{T,S} V = 0$ has the Perron-Frobenius property.

Definition 1.6. Let $A \in \mathbb{R}^{m \times n}$, and $T$ and $S$ be subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. A splitting $A = U - V$ of $A$ is said to be an outer-nonnegative splitting (with prescribed range $T$ and null space $S$) if both $A^{(2)}_{T,S}$ and $U^{(2)}_{T,S}$ exist, and $U^{(2)}_{T,S} V \geq 0$ and is not nilpotent. A splitting $A = U - V$ is said to be an outer-positive splitting with prescribed range $T$ and null space $S$ if both $A^{(2)}_{T,S}$ and $U^{(2)}_{T,S}$ exist, and $U^{(2)}_{T,S} V > 0$.

Remark 1.3. Obviously, outer-Perron-Frobenius splittings, outer-nonnegative splittings and outer-positive splittings are $(T, S)$ splittings.

The following definitions are degenerate forms of Definitions 1.5 and 1.6.

Definition 1.7. Let $A \in \mathbb{C}^{m \times n}$. A splitting $A = U - V$ of $A$ is said to be
(i) a Moore-Penrose-Perron-Frobenius splitting if $\mathcal{R}(A) = \mathcal{R}(U)$ and $\mathcal{N}(A) = \mathcal{N}(U)$, and $U^T V$ has the Perron-Frobenius property.
(ii) a Moore-Penrose-nonnegative splitting if $\mathcal{R}(A) = \mathcal{R}(U)$ and $\mathcal{N}(A) = \mathcal{N}(U)$, and $U^T V \geq 0$ and is not nilpotent.
(iii) a Moore-Penrose-positive splitting if $\mathcal{R}(A) = \mathcal{R}(U)$ and $\mathcal{N}(A) = \mathcal{N}(U)$, and $U^T V > 0$.

Remark 1.4. The above definitions are combination of the proper splitting in [2] and “pseudo-" in [9]

Definition 1.8. Let $A \in \mathbb{C}^{m \times n}$ and $k = \max \{\text{Ind}(A), \text{Ind}(U)\}$. A splitting $A = U - V$ of $A$ is said to be
(i) a Drazin-Perron-Frobenius splitting if $\mathcal{R}(A^k) = \mathcal{R}(U^k)$ and $\mathcal{N}(A^k) = \mathcal{N}(U^k)$, and $U_d V$ has the Perron-Frobenius property.

(ii) a Drazin-nonnegative splitting if $\mathcal{R}(A^k) = \mathcal{R}(U^k)$ and $\mathcal{N}(A^k) = \mathcal{N}(U^k)$, and $U_d V \geq 0$ and is not nilpotent.

(iii) a Drazin-positive splitting if $\mathcal{R}(A^k) = \mathcal{R}(U^k)$ and $\mathcal{N}(A^k) = \mathcal{N}(U^k)$, and $U_d V > 0$.

Finally, we use the following lemmas to end the section.

Lemma 1.3. [1, Ex. 0.45 and 0.46] Let $A \in \mathbb{C}^{m \times m}$. Then $\rho(A) < 1$ if and only if $(I - A)^{-1}$ exists and $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$.

Lemma 1.4. Let $u_i \in \mathbb{C} \setminus \{1\}$, $u_i > 0$, and $u_i = \max\{|u_i| : i = 1, \ldots, n\}$. Then

$$u_i < 1 \text{ if and only if } \frac{u_i}{1-u_i} = \max\left\{\frac{|u_i|}{1-u_i} : i = 1, \ldots, n\right\}.$$ 

Proof. To show the necessity, owing to $u_i = \max\{|u_i|, i = 1, \ldots, n\}$, it follows that $|u_i| \leq u_i < 1$, and then $1 - |u_i| > 0$. Thus,

$$\frac{|u_i|}{1-u_i} = \frac{|u_i|}{1-u_i} \leq \frac{|u_i|}{1-|u_i|} \leq \frac{u_i}{1-u_i}.$$

For the sufficiency, since

$$\frac{u_i}{1-u_i} = \max\left\{\frac{|u_i|}{1-u_i} : i = 1, \ldots, n\right\} \geq 0, \ u_i < 1.$$ 

The proof is complete. \hfill $\square$

2 Main results

Now we characterize results regarding the Perron-Frobenius splitting and the relationship with outer inverses. We formulate the following auxiliary result.

Theorem 2.1. Let $A \in \mathbb{C}^{m \times n}$, $T$ and $S$ be subspaces of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, and $A = U - V$ be an outer-Perron-Frobenius splitting of $A$. If $A_{T,S}^{(2)} V$ has the Perron-Frobenius property, then

$$\rho(A_{T,S}^{(2)} V) = \frac{\rho(U_{T,S}^{(2)} V)}{1 - \rho(U_{T,S}^{(2)} V)} \quad \text{and} \quad \rho(U_{T,S}^{(2)} V) < 1 \quad (2.1)$$

hold.

Proof. By Lemma 1.1,

$$A_{T,S}^{(2)} V = (I - U_{T,S}^{(2)} V)^{-1} U_{T,S}^{(2)} V. \quad (2.2)$$

So $(I - U_{T,S}^{(2)} V)(I + A_{T,S}^{(2)} V) = I$ and then $I + A_{T,S}^{(2)} V$ is invertible and

$$(I + A_{T,S}^{(2)} V)^{-1} = I - U_{T,S}^{(2)} V.$$ 

Thus, by the above equation and (2.2),

$$U_{T,S}^{(2)} V = (I + A_{T,S}^{(2)} V)^{-1} A_{T,S}^{(2)} V. \quad (2.3)$$

Equations (2.2) and (2.3) imply that $\lambda_j = \frac{u_j}{1-u_j}$, where $u_j \in \sigma(U_{T,S}^{(2)} V)$ and $\lambda_j \in \sigma(A_{T,S}^{(2)} V)$. Actually, if $u \in \sigma(U_{T,S}^{(2)} V)$ with an eigenvector $p$, then $A_{T,S}^{(2)} V p = (I - U_{T,S}^{(2)} V)^{-1} U_{T,S}^{(2)} V p = \frac{u}{1-u} p$, i.e.,
\[ \frac{u}{1-u} \in \sigma(A^{(2)}_{T,S})V. \] Conversely, we can obtain similarly that if \( \lambda \in \sigma(A^{(2)}_{T,S}) \), then \( \frac{\lambda}{1+\lambda} \in \sigma(U^{(2)}_{T,S}) \). Hence \( \sigma(A^{(2)}_{T,S}) = \left\{ \frac{u}{1-u} : u_j \in \sigma(U^{(2)}_{T,S}) \right\} \) and \( \rho(A^{(2)}_{T,S}) = \max \left\{ \frac{|u_j|}{1-u_j} : u_j \in \sigma(U^{(2)}_{T,S}) \right\} \).

Since \( A^{(2)}_{T,S}V \) has the Perron-Frobenius property, \( \rho(A^{(2)}_{T,S}) \) \( \sigma(A^{(2)}_{T,S}) \) and there exists \( u_{\lambda} \in \sigma(U^{(2)}_{T,S}) \) such that \( \rho(A^{(2)}_{T,S}) = \frac{u_{\lambda}}{1-u_{\lambda}} \geq 0 \) by the argument above. So
\[
\frac{u_{\lambda}}{1-u_{\lambda}} = \max \left\{ \frac{|u_j|}{1-u_j} : u_j \in \sigma(U^{(2)}_{T,S}) \right\}. \tag{2.4}
\]

In order to finish the proof, it suffices to show that \( u_{\lambda} = \rho(U^{(2)}_{T,S}) \). The reason for this is that relation (2.4) implies \( \rho(U^{(2)}_{T,S}) < 1 \) by Lemma 1.4.

Now we will prove the equality. If \( \frac{u_{\lambda}}{1-u_{\lambda}} = 0 \), then we obtain \( u_{\lambda} = 0 \). It follows from (2.4) that \( u_j = 0, \forall u_j \in \sigma(U^{(2)}_{T,S}) \). Then \( u_{\lambda} = \rho(U^{(2)}_{T,S}) \).

When \( \frac{u_{\lambda}}{1-u_{\lambda}} > 0 \), \( 1-u_{\lambda} > 0 \), namely \( \frac{1}{u_{\lambda}} > 1 \), and then \( 0 < u_{\lambda} < 1 \). Since \( U^{(2)}_{T,S} \) has the Perron-Frobenius property, \( \rho(U^{(2)}_{T,S}) \in \sigma(U^{(2)}_{T,S}) \). Write \( u_m = \rho(U^{(2)}_{T,S}) \). Then, by (2.4),
\[
\frac{u_{\lambda}}{1-u_{\lambda}} \geq \frac{u_m}{1-u_m} \geq \frac{u_m}{1-u_m} = 1,
\]

Since \( u_m \geq u_{\lambda} > 0 \), it follows from the above inequality
\[
\frac{u_{\lambda}}{u_m} \geq \frac{1-u_{\lambda}}{1-u_m} \geq \frac{1-u_m}{1-u_m} = 1,
\]

which implies \( u_{\lambda} \geq u_m \). Therefore we obtain \( u_{\lambda} = u_m = \rho(U^{(2)}_{T,S}) \).

In fact, either in (2.1) implies that \( A^{(2)}_{T,S} \) has the Perron-Frobenius property, and they are also equivalent to each other.

Now we state main result, which extends [9, Theorem 2.1] to an outer inverse and includes the necessary and sufficient conditions for the convergence of an outer-Perron-Frobenius splitting.

**Theorem 2.2.** Let \( A \in \mathbb{C}^{m,n} \), and \( T \) and \( S \) be subspaces of \( \mathbb{C}^n \) and \( \mathbb{C}^m \), respectively. If \( A = U - V \) is an outer-Perron-Frobenius splitting of \( A \), then the following conditions are equivalent:

(i) \( \rho(U^{(2)}_{T,S}) < 1 \);

(ii) \( A^{(2)}_{T,S} \) has the Perron-Frobenius property;

(iii) \( \rho(A^{(2)}_{T,S}) = \frac{\rho(U^{(2)}_{T,S})}{1 - \rho(U^{(2)}_{T,S})} \).

Suppose that \( x \geq 0 \) is an eigenvector corresponding to the eigenvalue \( \rho(U^{(2)}_{T,S}) \). Then the conditions above, are further equivalent to:

(iv) \( A^{(2)}_{T,S}Ux \geq x \);

(v) \( A^{(2)}_{T,S}Vx \geq U^{(2)}_{T,S}Vx \).

Moreover, \( A^{(2)}_{T,S}b \) is the unique solution of the equation \( x = U^{(2)}_{T,S}Vx + U^{(2)}_{T,S}b \). Therefore the conditions (i) \( \sim \) (iii) above, are further equivalent to

(vi) The iteration \( x_{i+1} = U^{(2)}_{T,S}Vx_i + U^{(2)}_{T,S}b \) converges to \( A^{(2)}_{T,S}b \) for every \( x_0 \in \mathbb{C}^n \);

(vii) There exists some multiplicative norm \( \| \cdot \| \) such that \( \| U^{(2)}_{T,S}V \| < 1 \).

**Proof.** (i) \( \Rightarrow \) (ii): There exists a nonzero vector \( p \geq 0 \) such that \( U^{(2)}_{T,S}Vp = u_m p \) where \( u_m = \rho(U^{(2)}_{T,S}) \in \sigma(U^{(2)}_{T,S}) \). Since \( u_m = \max \{ |u_j| : u_j \in \sigma(U^{(2)}_{T,S}) \} < 1 \), we have, by Lemma 1.4,
\[
\frac{u_m}{1-u_m} = \max \left\{ \frac{|u_j|}{1-u_j} : u_j \in \sigma(U^{(2)}_{T,S}) \right\}.
\]
By the proof of Theorem 2.1, $\rho(A^{(2)}_{T,S} V) = \max \left\{ \frac{u_j}{1 - u_j} : u_j \in \sigma(U^{(2)}_{T,S} V) \right\} = \frac{\rho \left( A^{(2)}_{T,S} V \right)}{1 + \rho \left( A^{(2)}_{T,S} V \right)}$, namely, (ii) holds.

(iii) $\Rightarrow$ (i): If $\rho(A^{(2)}_{T,S} V) = 0$, then $\rho(U^{(2)}_{T,S} V) = 0 < 1$. When $\rho(A^{(2)}_{T,S} V) \neq 0, \rho(U^{(2)}_{T,S} V) \neq 0$. Then from Statement (iii), we can easily find out $\rho(U^{(2)}_{T,S} V) = \frac{\rho \left( A^{(2)}_{T,S} V \right)}{1 + \rho \left( A^{(2)}_{T,S} V \right)} < 1$.

(ii) $\Rightarrow$ (i) and (iii): By Theorem 2.1.

(i) $\iff$ (iv): Since $U^{(2)}_{T,S} Vx = \rho(U^{(2)}_{T,S} V)x(x = 0, x \neq 0), x \in \mathcal{R}(U^{(2)}_{T,S} V) \subseteq \mathcal{R}(U^{(2)}_{T,S})$ and then $U^{(2)}_{T,S} Ux = x$. Note that

$$A^{(2)}_{T,S} Ux = (I - U^{(2)}_{T,S} V)^{-1} U^{(2)}_{T,S} Ux = (I - U^{(2)}_{T,S} V)^{-1} x = \frac{1}{1 - \rho(U^{(2)}_{T,S} V)} x.$$ 

If $\rho(U^{(2)}_{T,S} V) < 1$, then $A^{(2)}_{T,S} Ux \geq x$. On the other hand, $A^{(2)}_{T,S} Ux \geq x$, i.e., $\frac{1}{1 - \rho(U^{(2)}_{T,S} V)} x \geq x$. Thus, $0 \leq \rho(U^{(2)}_{T,S} V) < 1$.

(i) $\iff$ (v): Since, for some $x \geq 0$, $U^{(2)}_{T,S} Vx = \rho(U^{(2)}_{T,S} V)x$, and then

$$A^{(2)}_{T,S} Vx = \frac{\rho(U^{(2)}_{T,S} V)}{1 - \rho(U^{(2)}_{T,S} V)} x.$$ 

Thus

$$A^{(2)}_{T,S} Vx - U^{(2)}_{T,S} Vx = \frac{(\rho(U^{(2)}_{T,S} V))^2 + \rho(U^{(2)}_{T,S} V)}{1 - \rho(U^{(2)}_{T,S} V)} x.$$ 

As a result, $A^{(2)}_{T,S} Vx \geq U^{(2)}_{T,S} Vx$ if and only if $\rho(U^{(2)}_{T,S} V) < 1$.

Now let $x = A^{(2)}_{T,S} b$. Then

$$U^{(2)}_{T,S} Vx + U^{(2)}_{T,S} b = U^{(2)}_{T,S} V A^{(2)}_{T,S} b + U^{(2)}_{T,S} b = U^{(2)}_{T,S} (U - A) A^{(2)}_{T,S} b + U^{(2)}_{T,S} b = A^{(2)}_{T,S} b.$$ 

Thus, $x = A^{(2)}_{T,S} b$ is a solution of the equation.

If $y$ is also a solution of the equation, then $x - y = U^{(2)}_{T,S} V(x - y)$, namely, $(I - U^{(2)}_{T,S} V)(x - y) = 0$. By Lemma 1.1(i), $y = x$, namely, $A^{(2)}_{T,S} b$ is the unique solution of the equation.

(i) $\iff$ (vi) See the proof of Theorem 3.1 in [5].

(vii) $\Rightarrow$ (i) For any multiplicative norm $\| \cdot \|, \rho(U^{(2)}_{T,S} V) \leq \|U^{(2)}_{T,S} V\| < 1$, then (i) holds.

(i) $\Rightarrow$ (vii) By $\rho(U^{(2)}_{T,S} V) < 1$, there exists a suitable $\varepsilon > 0$ such that $\rho(U^{(2)}_{T,S} V) + \varepsilon < 1$. Thus, for this $\varepsilon$, there exists a multiplicative norm $\| \cdot \|$ such that $\|U^{(2)}_{T,S} V\| < \rho(U^{(2)}_{T,S} V) + \varepsilon < 1$. \hfill \Box

Since $A^T = A^{(2)}_{\mathcal{X}(A'), \mathcal{N}(A')}$, we can gain immediately [9, Theorem 2.1] as follows.

**Corollary 2.1.** Let $A \in \mathbb{C}^{m \times n}$, and $A = U - V$ be a Moore-Penrose-Perron-Frobenius splitting of $A$. Then the following conditions are equivalent:

(i) $\rho(U^T V) < 1$;

(ii) $A^T V$ has the Perron-Frobenius property;

(iii) $\rho(A^T V) = \frac{\rho(U^T V)}{1 - \rho(U^T V)}$.

Suppose that $x \geq 0$ is an eigenvector corresponding to the eigenvalue $\rho(U^T V)$. Then the conditions above, are further equivalent to:

(iv) $A^T Ux \geq x$;

(v) $A^T Vx \geq U^T Vx$.

Moreover, $A^T b$ is the unique solution of the equation $x = U^T Vx + U^T b$. Therefore the conditions (i) $\sim$ (iii) above, are further equivalent to:

(vi) The iteration $x_{i+1} = U^T Vx_i + U^T b$ converges to $A^T b$ for every $x_0 \in \mathbb{C}^n$;

(vii) There exists some multiplicative norm $\| \cdot \|$, such that $\|U^T V\| < 1$. 
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Also, \( U_d = U^{(2)}_{\mathcal{R}(U^+), N(U)} \), we have the following corollary.

**Corollary 2.2.** Let \( A \in \mathbb{C}^{m \times m} \), and \( A = U - V \) be a Drazin-Perron-Frobenius splitting of \( A \). Then the following conditions are equivalent:

(i) \( \rho(U_d V) < 1 \);

(ii) \( A_d V \) has the Perron-Frobenius property;

(iii) \( \rho(A_d V) = \frac{\rho(U_d V)}{1 - \rho(U_d V)} \).

Suppose that \( x \geq 0 \) is an eigenvector corresponding to the eigenvalue \( \rho(U_d V) \). Then the conditions above, are further equivalent to:

(iv) \( A_d Ux \geq x \);

(v) \( A_d Vx \succeq U_d Vx \).

Moreover, \( A_d b \) is the unique solution of the equation \( x = U_d Vx + U_d b \). Therefore the conditions (i) \( \sim \) (iii) above, are further equivalent to:

(vi) The iteration \( x_{t+1} = U_d Vx_t + U_d b \) converges to \( A_d b \) for every \( x_0 \in \mathbb{C}^n \);

(vii) There exists some multiplicative norm \( \| \cdot \| \), such that \( \| U_d V \| < 1 \).

The following is regarding the outer-nonnegative splitting.

**Theorem 2.3.** Let \( A \in \mathbb{C}^{m \times n} \), \( T \) and \( S \) be subspaces of \( \mathbb{C}^n \) and \( \mathbb{C}^m \), respectively, and \( A = U - V \) be an outer-nonnegative splitting of \( A \). Write \( K = (U^{(2)}_{T,S} V)^k \mathbb{R}^n_+ \), where \( k_0 = k_0(U^{(2)}_{T,S} V) \). Assume \( U(K) \subseteq K \) and \( U^{(2)}_{T,S} K \subseteq K \).

Then the following statements are equivalent:

(i) \( A^{(2)}_{T,S}(K) \subseteq \mathbb{R}^n_+ \);

(ii) \( \rho(U^{(2)}_{T,S} V) < 1 \);

(iii) \( Ax \in K + S, x \in T \Rightarrow x \geq 0 \).

**Proof.** (i) \( \Rightarrow \) (ii): Let \( x \in \mathbb{R}^n_+ \) be the Perron eigenvector of \( U^{(2)}_{T,S} V \). Since \( U^{(2)}_{T,S} Vx = \rho(U^{(2)}_{T,S} V)x(x \geq 0, x \neq 0) \), \( x \in \mathbb{R}(U^{(2)}_{T,S}) \). Using \( U(K) \subseteq K \),

\[
A^{(2)}_{T,S} Ux = (I - U^{(2)}_{T,S} V)^{-1} U^{(2)}_{T,S} Ux = (I - U^{(2)}_{T,S} V)^{-1} x = \frac{1}{1 - \rho(U^{(2)}_{T,S} V)} x \in \mathbb{R}^n_+.
\]

It follows that \( 0 \leq \rho(U^{(2)}_{T,S} V) < 1 \).

(ii) \( \Rightarrow \) (iii): Let \( \rho(U^{(2)}_{T,S} V) < 1 \) and \( Ax = y + s \), where \( y \in K, s \in N(A^{(2)}_{T,S}) \) and \( x \in \mathbb{R}(A^{(2)}_{T,S}) \). Then

\[
x = A^{(2)}_{T,S} Ax = A^{(2)}_{T,S} (y + s) = A^{(2)}_{T,S} y = (I - U^{(2)}_{T,S} V)^{-1} U^{(2)}_{T,S} y = (I - U^{(2)}_{T,S} V)^{-1} z,
\]

where \( z = U^{(2)}_{T,S} y \in K \). Since \( U^{(2)}_{T,S} K \subseteq K \), there exists some \( w \geq 0 \) such that \( z = U^{(2)}_{T,S} y = (U^{(2)}_{T,S} V)^k w \). Then, by Lemma 1.3,

\[
x = (I - U^{(2)}_{T,S} V)^{-1} z = (I - U^{(2)}_{T,S} V)^{-1} (U^{(2)}_{T,S} V)^k w = \sum_{j=0}^{\infty} (U^{(2)}_{T,S} V)^{k+j} w \geq 0.
\]

(iii) \( \Rightarrow \) (i): For any \( x \in A^{(2)}_{T,S}(K), x = A^{(2)}_{T,S} y, \) where \( y \in K \). Since \( y = At + s \in AT \oplus S \) where \( t \in T \) and \( s \in S = N(A^{(2)}_{T,S}), x \in A^{(2)}_{T,S} y = t, \) namely, \( Ax = y - s \). So \( x \geq 0 \). Consequently, Statement (i) is true. \( \square \)

Similarly, using the relations \( U^+ = U^{(2)}_{\mathcal{R}(A^+), N(A^+)} \) and \( U_d = U^{(2)}_{\mathcal{R}(U^+), N(U^+)} \), we get the following results from Theorem 2.3.

**Corollary 2.3.** [9, Theorem 2.2] Let \( A \in \mathbb{C}^{m \times n} \), and \( A = U - V \) be a Moore-Penrose-nonnegative splitting of \( A \). Write \( K = (U^+ V)^k \mathbb{R}^n_+ \), where \( k_0 = k_0(U^+ V) \). Assume \( U(K) \subseteq K \) and \( U^+ K \subseteq K \).

Then the following statements are equivalent:

(i) \( A^+(K) \subseteq \mathbb{R}^n_+ \);
(ii) \( \rho(U^T V) < 1 \);
(iii) \( Ax \in K + \mathcal{N}(A^*) \), \( x \in \mathcal{R}(A^*) \) \( \Rightarrow x \geq 0 \).

**Corollary 2.4.** Let \( A \in \mathbb{C}^{m \times n} \), and \( A = U - V \) be a Drazin-nonnegative splitting of \( A \). Write \( K = (U_d V)^{k_0} \mathbb{R}^n \), where \( k_0 = k_0(U_d V) \). Assume \( U(K) \subseteq K \) and \( U_d K \subseteq K \). Then the following statements are equivalent:
(i) \( A_d(K) \subseteq \mathbb{R}^n \);
(ii) \( \rho(U_d V) < 1 \);
(iii) \( Ax \in K + \mathcal{N}(A^k) \), \( x \in \mathcal{R}(A^k) \), where \( k = \max\{\text{Ind}(U), \text{Ind}(A)\} \) \( \Rightarrow x \geq 0 \).

Successively, we formulate other necessary conditions and sufficient conditions for \( \rho(U^{(2)}_{T,S} V) < 1 \).

**Theorem 2.4.** Let \( A \in \mathbb{C}^{m \times n} \), \( T \) and \( S \) be subspaces of \( \mathbb{C}^n \) and \( \mathbb{C}^m \), respectively, and \( A = U - V \) be an outer-nonnegative splitting of \( A \). Write \( L = \mathbb{R}^m \) and \( K = (U^{(2)}_{T,S} V)^{k_0} \mathbb{R}^n \), where \( k_0 = k_0(U^{(2)}_{T,S} V) \). Assume \( U^{(2)}_{T,S} L \subseteq K \) and \( U^{(2)}_{T,S} VK \subseteq K \). Consider the following statements:
(i) \( A^{(2)}_{T,S} L \subseteq K \);
(ii) \( A^{(2)}_{T,S} VK \subseteq K \);
(iii) \( \rho(U^{(2)}_{T,S} V) < 1 \).
If \( \rho(U^{(2)}_{T,S} V) \neq 0 \), then (ii) \( \iff \) (iii) \( \iff \) (i); If \( \rho(U^{(2)}_{T,S} V) = 0 \), then (i) and (ii) hold.

**Proof.** (ii) \( \iff \) (iii): Assume \( \rho(U^{(2)}_{T,S} V) \neq 0 \). Since \( U^{(2)}_{T,S} V \) possesses the Perron-Frobenius property. There exists an eigenvector \( z_0 \geq 0 \) such that
\[
U^{(2)}_{T,S} V z_0 = \rho(U^{(2)}_{T,S} V) z_0
\]
\[
(U^{(2)}_{T,S} V)^{k_0} z_0 = \rho^{k_0}(U^{(2)}_{T,S} V) z_0
\]
\[
z_0 = \frac{z_0}{\rho^{k_0}(U^{(2)}_{T,S} V)}.
\]
Thus, \( z_0 \in K \) and then \( A^{(2)}_{T,S} V z_0 \in K \), by Statement (ii), that is
\[
A^{(2)}_{T,S} V z_0 = (I - U^{(2)}_{T,S} V)^{-1} U^{(2)}_{T,S} V z_0 = \frac{\rho(U^{(2)}_{T,S} V)}{1 - \rho(U^{(2)}_{T,S} V)} z_0 \in K.
\]
It follows that \( \rho(U^{(2)}_{T,S} V) < 1 \).

(iii) \( \iff \) (ii): Since \( \rho(U^{(2)}_{T,S} V) < 1 \) and \( U^{(2)}_{T,S} VK \subseteq K \), we have \( (U^{(2)}_{T,S} V)^i K \subseteq K(i \geq 1) \), which implies, by Lemma 1.3, that
\[
A^{(2)}_{T,S} VK = (I - U^{(2)}_{T,S} V)^{-1} U^{(2)}_{T,S} VK = \sum_{j=0}^{\infty} (U^{(2)}_{T,S} V)^{i+1} K \subseteq K + K + \cdots + K + \cdots \subseteq K.
\]

(iii) \( \iff \) (i): Assume that (iii) holds, then \( \rho(U^{(2)}_{T,S} V) < 1 \). Since \( U^{(2)}_{T,S} L \subseteq K \), we have
\[
A^{(2)}_{T,S} L = (I - U^{(2)}_{T,S} V)^{-1} U^{(2)}_{T,S} L \subseteq \sum_{j=0}^{\infty} (U^{(2)}_{T,S} V)^{i} K \subseteq K.
\]
(2.5)
If \( \rho(U^{(2)}_{T,S} V) = 0 \), we have \( \rho(U^{(2)}_{T,S} V) < 1 \), and then (i) and (ii) hold by the argument above.

The next two corollaries follow from the above theorem.

**Corollary 2.5.** Let \( A \in \mathbb{R}^{m \times n} \), and \( A = U - V \) be a Moore-Penrose-Perron-Frobenius splitting of \( A \). Write \( L = \mathbb{R}^m \), \( K = (U^T V)^{k_0} \mathbb{R}^n \), where \( k_0 = k_0(U^T V) \). Assume \( U^T L \subseteq K \) and \( U^T VK \subseteq K \). Consider the following statements:
(i) \( A^T L \subseteq K \);
(ii) \( A^T VK \subseteq K \);
(iii) \( \rho(U^†V) < 1 \).
If \( \rho(U^†V) \neq 0 \), then (ii) \( \iff \) (iii) \( \iff \) (i); If \( \rho(U^†V) = 0 \), then (i) and (ii) hold.

**Corollary 2.6.** Let \( A \in \mathbb{R}^{m \times n} \), and \( A = U - V \) be a Drazin-Pennon-Frobenius splitting of \( A \). Write \( L = R^n_k \), \( K = (U_d V)^k R^n_k \), where \( k_0 = k_0(U_d V) \). Assume \( U_d L \subseteq K \) and \( U_d VK \subseteq K \). Consider the following statements:

(i) \( A_d L \subseteq K \);
(ii) \( A_d VK \subseteq K \);
(iii) \( \rho(U_d V) < 1 \).
If \( \rho(U_d V) \neq 0 \), then (ii) \( \iff \) (iii) \( \iff \) (i); If \( \rho(U_d V) = 0 \), then (i) and (ii) hold.

A singular matrix \( A \), which allows an outer-positive splitting \( A = U - V \), has properties which are analogous to those of pseudo-positive splitting given in [9]. This is shown below.

**Theorem 2.5.** Let \( A \in \mathbb{R}^{n \times n} \) be a singular matrix, \( T \) and \( S \) be subspaces of \( \mathbb{R}^n \), and \( A = U - V \) be an outer-positive splitting of \( A \). Assume \( \rho(U^{(2)}_{T,S} V) = 1 \). Then

(i) \( \text{rank}(A) = n - 1 \).
(ii) There exists a vector \( x > 0 \) such that \( Ax = 0 \).

In addition, if \( (VU^{(2)}_{T,S})^* \) has a positive eigenvector corresponding to the eigenvalue \( \rho(VU^{(2)}_{T,S}) = \rho(U^{(2)}_{T,S} V) = 1 \), then we have the implication:

(iii) \( Au \geq 0 \Rightarrow Au = 0 \).

**Proof.** Since \( U^{(2)}_{T,S} V \) \( > 0 \), it has the strong Perron-Frobenius property by Lemma 1.2. Then there exists \( w \) \( > 0 \) such that \( (U^{(2)}_{T,S} V)w = \rho(U^{(2)}_{T,S} V)w = w \). Obviously \( w \in R(U^{(2)}_{T,S}) = T \).

(i) \( U^{(2)}_{T,S} Aw = U^{(2)}_{T,S} (U - V)w = U^{(2)}_{T,S} Uw - U^{(2)}_{T,S} Vw = w - w = 0 \), i.e., \( 0 \) is an eigenvalue of \( U^{(2)}_{T,S} A \). Since \( \rho(U^{(2)}_{T,S} V) \) is a simple eigenvalue of \( U^{(2)}_{T,S} V \), \( 0 \) is a simple eigenvalue of \( U^{(2)}_{T,S} A \). As a result, \( \text{rank}(U^{(2)}_{T,S} A) = n - 1 \). On the other hand, \( \text{rank}(U^{(2)}_{T,S} A) \leq \text{rank}(A) \leq n - 1 \) since \( A \) is singular. Therefore, \( \text{rank}(A) = n - 1 \).

(ii) Since \( U^{(2)}_{T,S} Aw = 0 \) by the argument above, \( Aw \in N(U^{(2)}_{T,S}) = S \) and therefore \( Aw \in AT \cap S = \{0\} \), namely, \( Aw = 0 \).

(iii) Let \( u \) satisfy \( Au \geq 0 \). Suppose that \( Au \neq 0 \). Thus, \( x^* Au > 0 \) for any \( x > 0 \). Since \( VU^{(2)}_{T,S} \) \( > 0 \), there exists a vector \( z \geq 0 \) such that \( z^* (VU^{(2)}_{T,S})^* = \rho(VU^{(2)}_{T,S})z^* = z^* \) and \( z^* Au^{(2)}_{T,S} = 0 \) by the argument in (i). Thus for this \( z \), \( z^* Au = 0 \) by the argument in (ii), a contradiction. So \( Au = 0 \).  

**Corollary 2.7.** [9, Theorem 2.4] Let \( A \in \mathbb{R}^{n \times n} \) be a singular matrix and \( A = U - V \) be a Moore-Penrose-positive splitting of \( A \). Assume that \( \rho(U^†V) = 1 \). Then

(i) \( \text{rank}(A) = n - 1 \).
(ii) There exists a vector \( x > 0 \) such that \( Ax = 0 \).

In addition, if \( (VU^†) \) \( ^* \) has a positive eigenvector corresponding to the eigenvalue \( \rho(VU^†) = \rho(U^†V) = 1 \), then we have the implication:

(iii) \( Au \geq 0 \Rightarrow Au = 0 \).

**Corollary 2.8.** Let \( A \in \mathbb{R}^{n \times n} \) be a singular matrix, and \( A = U - V \) be a Drazin-positive splitting of \( A \). Assume \( \rho(U_d V) = 1 \). Then

(i) \( \text{rank}(A) = n - 1 \).
(ii) There exists a vector \( x > 0 \) such that \( Ax = 0 \).

In addition, if \( (VU_d) \) \( ^* \) has a positive eigenvector corresponding to the eigenvalue \( \rho(VU_d) = \rho(U_d V) = 1 \), then we have the implication:

(iii) \( Au \geq 0 \Rightarrow Au = 0 \).

**Acknowledgement:** This research is supported by the National Natural Science Foundation of China (11571220).
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