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Bordering method to compute Core-EP inverse

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Abstract: Following the work of Kentaro Nomakuchi[10] and Manjunatha Prasad et.al., [7] which relate various generalized inverses of a given matrix with suitable bordering, we describe the explicit bordering required to obtain core-EP inverse, core-EP generalized inverse. The main result of the paper also leads to provide a characterization of Drazin index in terms of bordering.

Keywords: core-EP inverse, core-EP generalized inverse, outer inverse, bordered matrix

MSC: 15A09; 15A18

1 Introduction

Bordering technique to find the generalized inverses of a given matrix has already been studied by many authors and literature is available, e.g., Goldman and Zelen[6], Ben-Israel and Greville[3], Blatter[4], Kentaro Nomakuchi[10], Manjunatha Prasad[7] are a few to name. In [10], Nomakuchi had shown that any generalized inverse of matrix \( A \) of rank \( r \) over a complex field can be obtained by looking into the inverse of a suitable bordered matrix \( T = \begin{bmatrix} A & P \\ Q & R \end{bmatrix} \), where \( P \), \( Q \) and \( R \) are of size \( m \times (m - r) \), \( (n - r) \times n \) and \( (n - r) \times (m - r) \), respectively. In [7], Prasad and Rao have considered the matrices over a commutative ring and related the existence of such bordering with the rank factorizations of \( I - AG \) and \( I - GA \), where \( G \) is a generalized inverse of \( A \). In [5], Eagambaram had shown that given any outer inverse \( G \) of \( A \), there exists a suitable bordering \( T \) such that \( G \) is the block correspond to \( A \) of \( T \) in the inverse of \( T^{-1} \).

Since core-EP inverse is a particular type of outer inverse, the authors had a natural motivation from [5] to use the bordering technique to find the core-EP inverse of \( A \). In this paper, we identify a suitable bordering to find the core-EP inverse.

2 Preliminaries

Matrices considered in this article are with entries from the field \( \mathbb{K} \) of reals (\( \mathbb{R} \)) or the field \( \mathbb{K} \) of complex numbers (\( \mathbb{C} \)). \( \mathbb{K}^n \) denotes \( n \)-dimensional Euclidean or Hermitian space depending on whether \( \mathbb{K} \) is \( \mathbb{R} \) or \( \mathbb{C} \). \( \mathbb{K}^{m \times n} \) denotes set of all \( m \times n \) matrices over \( \mathbb{K} \). Transpose (conjugate transpose) of a matrix \( A \) is denoted by \( A^T \). Let \( A \) and \( G \) be any matrices over \( \mathbb{K} \) and of sizes \( m \times n \) and \( n \times m \) respectively. The column space of \( A \) is denoted by \( \mathbb{C}(A) \) and the row space of \( A \) is denoted by \( \mathbb{R}(A) \). The index (also known as Drazin index) of a square matrix \( A \) is the smallest positive integer \( d \) such that \( \text{rank}(A^d) = \text{rank}(A^{d+1}) \), where \( \text{rank}(A) \) denotes the rank of \( A \) and the index of matrix \( A \) is denoted by \( \delta(A) \).

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Consider the following matrix equations, which are popularly known as Penrose condition or Moore-Penrose conditions:

\[(1) \quad AGA = A \quad (2) \quad GAG = G\]
\[(3) \quad (AG)^\dagger = AG \quad (4) \quad (GA)^\dagger = GA.\]

If \(A\) and \(G\) are square matrices, we consider the following equations

\[(1^k) \quad GA^{k+1} = A^k \quad (k^1) \quad A^{k+1}G = A^k\]

where \(k\) is some positive integer. In addition to Penrose conditions given above, the following condition (5) is useful in defining the group inverse of a square matrix, whenever it exists.

\[(5) \quad AG = GA.\]

A matrix \(G\) satisfying condition (1) is called a generalized inverse (g-inverse) of \(A\) and an arbitrary g-inverse of \(A\) is denoted by \(A^-\). If \(G\) satisfies the condition (2), then \(G\) is known as an outer inverse of \(A\) and an arbitrary outer inverse of \(A\) is denoted by \(A^\circ\). A matrix \(G\) satisfying conditions (1) and (2) is called a reflexive generalized inverse of \(A\) and an arbitrary reflexive generalized inverse of \(A\) is denoted by \(A^-\). The matrix satisfying all the four conditions of Moore-Penrose equations is known as Moore-Penrose inverse of \(A\) (denoted by \(A^\dagger\)). The group inverse of \(A\), denoted by \(A^\#\) whenever it exists, is the matrix \(G\) satisfying (1), (2) and (5). If a matrix \(G\) satisfies the properties \((i, j, \ldots, k)\), then such a \(G\) is termed as \((i, j, \ldots, k)\)-inverse.

The notion of core-EP inverse defined in [9] is the generalization of core inverse defined in [1] which exists for matrices of index one. In fact, core inverse has been termed as core-EP generalized inverse in [9], attributing to the EP (row space and column space are identical) characteristic of the core inverse.

**Definition 1 (Core-EP inverse).** An outer inverse \(G\) of \(A\) is called a core-EP inverse if

\[c(G) = \mathcal{R}(G) = c(A^d),\]

for some positive integer \(d\).

We use the notation \(A^\circ\) to represent core-EP inverse of \(A\). The existence, uniqueness, and some properties of \(A^\circ\) were discussed in [9]. In fact, the smallest integer \(d\) for which there exists an outer inverse satisfying Eq. 1 is nothing but the Drazin index of \(A\). The *core-EP inverse is defined similar to the above core-EP inverse where the above Eq. 1 is replaced by

\[c(G) = \mathcal{R}(G) = c(A^d).\]

Note that core-EP inverse (similarly *core-EP inverse) is a \((i, j, \ldots, k)\)-inverse unless the matrix is of index one, in which case it is termed as core-EP generalized inverse (*core-EP generalized inverse) and given by \(A^gAA^\dagger (A^\dagger AA^g)\). For the details, readers are referred to [1, 9]. In fact, we have the following result from [9] in the general case.

**Theorem 1 (Lemma 3.3, [9]).** Given a square matrix \(A\) with index \(d\), the following statements are equivalent:

(i) \(G\) is a core–EP inverse of \(A\)
(ii) \(G\) is a matrix satisfying the conditions (2), (3), (1^k) for some positive integer \(k\) and \(c(G) \subseteq c(A^k)\).

Similar result holds in the case of *core-EP inverse, replacing (3) by (4) and (1^k) by \((k^1)\) in the above theorem.

The notion of bordered matrix is well known in the literature and the readers are referred to any basic books on matrix theory or generalized inverses for the details. In the following definition, for our convenience, we introduce the terminology ‘\(k\)-invertible bordering’ of given matrix.

**Definition 2 (\(k\)-invertible bordering).** Given an \(m \times n\) matrix \(A\) over \(\mathbb{K}\) and \(0 \leq k \leq \min(m, n)\), matrix \(A\) is said to have \(k\)-invertible bordering if there exists a matrix \(P\) of size \(m \times (m - k)\), \(Q\) of size \((n - k) \times n\) and \(R\) of size
We denote the set of all \( k \)-invertible bordered matrices by \( \mathcal{B}_k(A) \). Note that the study of bordered matrix of order \( m + n \) has no relevance in the study of generalized inverse of given matrix, as we observe that with \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \), the bordered matrix \( T = \begin{bmatrix} 1 & 2 \\ -5 & 1 \end{bmatrix} \) is not revealing any information on any generalized inverse or any outer inverse of \( A \) from \( T^{-1} = \begin{bmatrix} 1 & 2 \\ -5 & 1 \end{bmatrix} \). So, we restrict ourselves to the bordering of given matrix as in the above definition. It is also noteworthy to observe that for \( r < k \leq \min(m, n) \), where \( r \) is the rank of \( A \), the matrix \( \begin{bmatrix} A & P \\ Q & R \end{bmatrix} \) could be of rank strictly less than \( m \) and hence the matrix \( T \) as in the above definition is not invertible. In fact, for such \( k \), \( \mathcal{B}_k(A) \) is empty.

In case that \( k \leq r \), the rank of matrix \( A \), then we find that \( \mathcal{B}_k(A) \) is nonempty with \( T = \begin{bmatrix} A & P \\ Q & 0 \end{bmatrix} \) in it for some \( P, Q \). For example, consider an outer inverse \( G \) of rank \( k \) (reflexive generalized inverse if \( k = r \)) and the rank factorizations \( I - AG = PY \) and \( I - GA = XQ \). Now, \( T = \begin{bmatrix} A & P \\ Q & 0 \end{bmatrix} \) is invertible with \( T^{-1} = \begin{bmatrix} G & X \\ Y & -YAX \end{bmatrix} \) is easily proved by verification. In case \( k = r \) and \( G \) is a reflexive generalized inverse, then \( AGA = A \) implies \( YAX \) in the \( T^{-1} \) is a null matrix. Conversely, if \( T = \begin{bmatrix} A & P \\ Q & 0 \end{bmatrix} \in \mathcal{B}_k(A) \) with \( T^{-1} = \begin{bmatrix} G & X \\ Y & Z \end{bmatrix} \), then by considering

\[
\begin{bmatrix} A & P \\ Q & 0 \end{bmatrix} \begin{bmatrix} G & X \\ Y & Z \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} G & X \\ Y & Z \end{bmatrix} \begin{bmatrix} A & P \\ Q & 0 \end{bmatrix}
\]

we get that \( GP = 0, YP = I \) and \( AG + PY = I \). This in turn implies \( GAG = G \) and \( \text{rank}(G) = \text{rank}(AG) = \text{rank}(I - PY) = n - \text{rank}(P) = k \) (because, \( PY \) is an idempotent as \( YP = I \)). Also, noting that \( \mathcal{C}(G) = \mathcal{C}(GA) \) and \( \mathcal{R}(G) = \mathcal{R}(AG) \) as \( G \) is an outer inverse \( A \), we get

\[
AG = I - PY \Rightarrow \mathcal{R}(G) = \mathcal{C}(P)^{\perp}
\]

and

\[
GA = I - XQ \Rightarrow \mathcal{C}(G) = \mathcal{R}(Q)^{\perp}.
\]

We will summarize this discussion in the following lemma.

**Lemma 2.** Given an \( m \times n \) matrix \( A \) of rank \( r \) and \( k \leq (m, n) \), we have the following:

(i) For \( k > r \), \( \mathcal{B}_k(A) \) is empty

(ii) For \( 0 < k \leq r \), \( \mathcal{B}_k(A) \) is nonempty and there exist suitable \( P, Q \) such that \( \begin{bmatrix} A & P \\ Q & 0 \end{bmatrix} \in \mathcal{B}_k(A) \)

(iii) For \( 0 < k \leq r \), if \( T = \begin{bmatrix} A & P \\ Q & 0 \end{bmatrix} \in \mathcal{B}_k(A) \) with \( T^{-1} = \begin{bmatrix} G & X \\ Y & Z \end{bmatrix} \) then \( G \) is an outer inverse of \( A \) with rank \( k \) and

\[
\mathcal{C}(G) = \mathcal{R}(Q)^{\perp}; \quad \mathcal{R}(G) = \mathcal{C}(P)^{\perp}.
\]

An objective of the present article is to identify suitable \( k \), matrices \( P, Q \) in \( T \) to obtain core-EP inverse of given square matrix in the \( T^{-1} \). The following theorem provides a necessary and sufficient condition for the existence of an outer inverse \( G \) of \( A \) with the property:

\[
\mathcal{C}(G) = \mathcal{C}(Y), \quad \mathcal{R}(G) = \mathcal{R}(X).
\]
Theorem 3 (Lemma 4.3 (iv) [8]). Let \( A \in \mathbb{C}^{m \times n}, \ X \in \mathbb{C}^{p \times m} \) and \( Y \in \mathbb{C}^{n \times q} \). If \( XAY \) is nonnull, there exists an outer inverse \( G \) of \( A \) such that \( \mathcal{C}(G) = \mathcal{C}(Y), \mathcal{R}(G) = \mathcal{R}(X) \) if and only if \( \text{rank}(XAY) = \text{rank}(X) = \text{rank}(Y) \). Further, such a \( G \) is unique.

The above theorem is the matrix version of Theorem 5 discussed in the article [2]. The existence of core-EP inverse follows as corollary for the above theorem, as \( \text{rank}(A^{d}A^{d+1}) = \text{rank}(A^{d}) = \text{rank}(A^{-d}). \) For the details the readers are referred to [9]. Now we shall proceed to relate the core-EP inverse with suitable bordering in the next section.

3 Core-EP inverse and Bordering

In this section, we will discuss two main results of the paper of which the first narrates existence of suitable bordered matrix when we have prior knowledge of core-EP inverse and the index of given matrix. The theorem and proof are in the lines of discussion we had prior to Lemma 2. The second describes a method of obtaining core-EP inverse by investigating a suitable bordered matrix, in step by step method, with the construction of a matrix \( P \) with the help of basis for \( \mathcal{C}(A^{k})^{\perp} \). This eventually provide a new characterization of Drazin index of a square matrix with reference to bordering.

Theorem 4. Given an \( n \times n \) matrix \( A \), there exist positive integers \( d, k \) and a matrix \( P \) of size \( n \times (n - k) \) such that \( \mathcal{C}(P) = \mathcal{C}(A^{d})^{\perp} \) and

\[
T = \begin{bmatrix} A & P \\ P' & 0 \end{bmatrix}
\]

is invertible. In other words, above \( T \in \mathcal{B}_{k}(A) \).

Proof. Let \( G \) be the core-EP inverse of \( A \). From the properties of core-EP inverse as discussed in [9], the least positive integer \( k \) for which \( \mathcal{C}(G) = \mathcal{R}(G) = \mathcal{C}(A^{k}) \) is the Drazin index of matrix, say \( d \). Now from the definition of core-EP inverse, we have that \( AG \) is a Hermitian projector onto \( \mathcal{C}(A^{d}) \) and therefore

\[
\mathcal{C}(I - AG) = \mathcal{R}(I - AG) = \mathcal{C}(A^{d})^\perp.
\]

If \( \text{rank}(A^{d}) = k \), then the dimension of \( \mathcal{C}(A^{d})^{\perp} \) is \( n - k \). Now consider a matrix \( P \) of size \( n \times (n - k) \) which consists of basis vectors of \( \mathcal{C}(A^{d})^{\perp} \) as its columns. Consider a matrix \( X \) such that

\[
I - AG = PX,
\]

which in fact a rank factorization of \( I - AG \). Since \( PX \) is a rank factorization of an idempotent matrix, we get \( XP = I \).

Similarly, \( \mathcal{C}(GA) = \mathcal{C}(G) = \mathcal{C}(A^{d}) \) implies that \( GA^{d}A^{d} = A^{d} \) and therefore

\[
(I - GA)A^{d} = 0.
\]

So, we get

\[
\mathcal{R}(I - GA) \subset \mathcal{C}(A^{d})^{\perp}.
\]

Now, rank\((GA) = \text{rank}(A^{d}) \) implies that

\[
\text{rank}(I - GA) = n - \text{rank}(A^{d}) = n - k.
\]

Therefore, \( \mathcal{R}(I - GA) = \mathcal{C}(A^{d})^{\perp} = \mathcal{R}(P) = \mathcal{R}(P'). \) Therefore, there exists a matrix \( Y \) such that

\[
I - GA = YP'.
\]

Clearly, \( YP' \) is a rank factorization of \( I - GA \) and therefore \( P'Y = I \). Now, consider the matrix

\[
\begin{bmatrix} G & Y \\ X & -XAY \end{bmatrix}.
\]
which could be verified easily that it is the inverse of
\[ T = \begin{bmatrix} A & P \\ P^* & 0 \end{bmatrix}. \]

Analogues to Theorem 4, we have the following:

**Theorem 5.** Given an \( n \times n \) matrix \( A \), there exist positive integers \( d, k \) and a matrix \( Q \) of size \( n \times (n - k) \) such that \( C(Q) = R(A^d)^\dagger \) and
\[ T = \begin{bmatrix} A & Q \\ Q^* & 0 \end{bmatrix} \]
is invertible. In other words, above \( T \in \mathcal{B}_k(A) \).

The following theorem provides a method to compute core-EP inverse.

**Theorem 6.** Given an \( n \times n \) matrix \( A \) and for \( i = 1, 2, \ldots \), let \( k(i) = \text{rank}(A^i) \) and \( T_i = \begin{bmatrix} A & P \\ P^* & 0 \end{bmatrix} \), where \( P \) is an \( n \times (n - k(i)) \) matrix such that \( C(P) = C(A^i)^\dagger \). Then the following assertions hold.

(i) If \( q \) is the smallest integer for which \( T_q \in \mathcal{B}_{k(q)}(A) \), the matrix corresponds to \( A \) in \( T_q^{-1} \) is the core-EP inverse of \( A \).

(ii) If \( q \) is as in (i), then \( q \) is the Drazin index of \( A \).

**Proof.** Let \( q \) be the smallest integer such that \( T_q \in \mathcal{B}_{k(q)}(A) \). So, \( T = \begin{bmatrix} A & P \\ P^* & 0 \end{bmatrix} \) is invertible, where \( P \) is an \( n \times (n - k) \) matrix such that \( C(P) = C(A^q)^\dagger \). If \( \begin{bmatrix} G & S \\ T & R \end{bmatrix} \) is the inverse of \( T \), then we have
\[ \begin{bmatrix} AG + PT & AS + PR \\ P^*G & P^*S \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} GA + SP^* & GP \\ TA + RP^* & TP \end{bmatrix}. \]

From the above, we have \( GP = 0 \) and \( GAG = G(I - PT) \), which in turn gives \( GAG = G - GPT = G \). So, \( G \) is an outer inverse of \( A \). From Eq. 2, we have \( P^*G = 0 \) implies that
\[ C(G) \subseteq R(P^*)^\dagger = C(P)^\dagger = C(A^q). \]

Again from Eq. 2, we have that \( P^*S = I \) and therefore \( SP^* \) is an idempotent matrix. So, \( GA + SP^* = I \) implies that \( \text{rank}(G) = \text{rank}(GA) = \text{rank}(I - SP^*) = n - \text{rank}(P) = \text{rank}(A^q) \). Hence the equality
\[ C(G) = C(A^q). \]

(3)

Similarly, \( GP = 0 \) implies that \( R(G) \subseteq C(P)^\dagger \), and \( \text{rank}(G) = n - \text{rank}(P) \) implies that \( R(G) = C(P)^\dagger = C(A^q) \). Thus, \( G \) is the core-EP inverse of \( A \) proving (i).

To prove (ii), consider the core-EP inverse obtained in (i) satisfying Eq. 3. \( C(G) = C(A^q) \) implies that \( GAA^q = A^q \) and therefore \( \text{rank}(A^{q+1}) = \text{rank}(A^q) \). So from the definition of Drazin index, we get that \( q \in \delta(A) \). The reverse inequality follows immediately from Theorem 4.

Analogues to Theorem 6, the following theorem computes *core-EP inverse.

**Theorem 7.** Given an \( n \times n \) matrix \( A \) and for \( i = 1, 2, \ldots \), let \( k(i) = \text{rank}(A^i) \) and \( T_i = \begin{bmatrix} A & Q \\ Q^* & 0 \end{bmatrix} \), where \( Q \) is an \( n \times (n - k(i)) \) matrix such that \( C(Q) = R(A^i)^\dagger \). Then the following assertions hold.

(i) If \( q \) is the smallest integer for which \( T_q \in \mathcal{B}_{k(q)}(A) \), the matrix corresponds to \( A \) in \( T_q^{-1} \) is the *core-EP inverse of \( A \).

(ii) If \( q \) is as in (i), then \( q \) is the Drazin index of \( A \).
From the Theorem 6, we could get a new characteristic of Drazin index in terms of bordering. We give an alternative definition for the Drazin index in the following.

**Definition 3** (Drazin index of a matrix). Given an $n \times n$ matrix $A$, the Drazin index of $A$ is the smallest integer $q$ for which there exists an invertible bordered matrix

$$T_q = \begin{bmatrix} A & P \\ p' & 0 \end{bmatrix},$$

where $P$ is an $n \times (n - k(q))$ matrix such that $C(P) = C(A^d) \perp$ and $k(q) = \text{rank}(A^d)$.

### 4 Numerical Example

Now we will demonstrate the results of Theorem 6 in the following numerical example. Consider the matrix

$$A = \begin{pmatrix} 3 & -1 & -1 & -1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \\ -1 & 3 & -1 & -1 \end{pmatrix}$$

whose rank is three. Now for $q = 1, 2, 3$, we will find $k(q)$ given by $\text{rank}(A^q)$, corresponding matrix $T_q$ with the help of matrix $P$ of size $4 \times (4 - k(q))$ given by the basis for null space of $A^q$. For $q = 1$, we have

$$A^1 = \begin{pmatrix} 3 & -1 & -1 & -1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \\ -1 & 3 & -1 & -1 \end{pmatrix}$$

for which $k(1) = 3$ and $P = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$. Now by augmenting $P$, we get that

$$T_1 = \begin{pmatrix} 3 & -1 & -1 & -1 & 1 \\ 1 & 1 & -3 & 1 & -1 \\ 1 & 1 & 1 & -3 & -1 \\ -1 & 3 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 0 \end{pmatrix}$$

is singular as $\det(T_1) = 0$.

For $q = 2$, we have

$$A^2 = \begin{pmatrix} 8 & -8 & 0 & 0 \\ 0 & 0 & -8 & 8 \\ 8 & -8 & 0 & 0 \\ 0 & 0 & -8 & 8 \end{pmatrix}$$

and corresponding $k(2) = 2$ and $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$. So, we get that

$$T_2 = \begin{pmatrix} 3 & -1 & -1 & -1 & 1 & 0 \\ 1 & 1 & -3 & 1 & 0 & 1 \\ 1 & 1 & 1 & -3 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \end{pmatrix}$$

is also singular as $\det(T_2) = 0$. 

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Now, for \( q = 3 \), we have

\[
A^3 = \begin{pmatrix}
-16 & -16 & 16 & -16 \\
-16 & 16 & 16 & -16 \\
16 & -16 & 16 & -16 \\
-16 & 16 & 16 & -16 \\
\end{pmatrix}
\]

and corresponding \( k(3) = 3 \) and \( P = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & -1 & 1 \\
\end{pmatrix} \). Further, \( T_3 = \begin{pmatrix}
3 & -1 & -1 & -1 & 1 & 0 & 0 \\
1 & 1 & -3 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & -3 & 0 & 0 & 1 \\
-1 & 3 & -1 & -1 & 1 & -1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
\end{pmatrix} \)

and find that \( T_3 \) is nonsingular.

So, \( q = 3 \) is the least positive integer such that \( T_q \) is nonsingular and \( T_3 \in \mathcal{B}_3(A) \). So, the index of \( A \) is three which could be verified by noting \( \text{rank}(A^3) = \text{rank}(A^4) = 1 \) and \( \text{rank}(A^2) = 2 \). Now we get the core-EP inverse of \( A \) by looking into the block corresponding to \( A \) in

\[
T_3^{-1} = \frac{1}{16} \begin{pmatrix}
1 & -1 & 1 & -1 & 12 & 4 & -4 \\
-1 & 1 & -1 & 1 & 4 & 12 & 4 \\
1 & -1 & 1 & -1 & -4 & 4 & 12 \\
-1 & 1 & -1 & 1 & 4 & -4 & 4 \\
12 & 4 & -4 & 4 & -32 & 0 & 32 \\
4 & 12 & 4 & -4 & 0 & 32 & 32 \\
-4 & 4 & 12 & 4 & 0 & -32 & 0 \\
\end{pmatrix}.
\]

So, the core-EP inverse of \( A \) is

\[
G = \frac{1}{16} \begin{pmatrix}
1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 \\
\end{pmatrix}.
\]

It could be verified by direct computation that \( GAG = G \) and observe that \( \mathcal{C}(G) = \mathcal{C}(A^3) = \text{span} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \).

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