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Spectra universally realizable by doubly stochastic matrices

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Abstract: A list of complex numbers \( \Lambda = \{ \lambda_1, \ldots, \lambda_n \} \) is said to be realizable if it is the spectrum of an entrywise nonnegative matrix, and universally realizable if there exists a nonnegative matrix with spectrum \( \Lambda \) for each Jordan canonical form associated with \( \Lambda \). The problem of characterizing the lists which are universally realizable is called the nonnegative inverse elementary divisors problem (NIEDP). This is a hard problem, which remains unsolved. A complete solution, if any, is still far from the current state of the art in the problem. In particular, in this paper we consider the NIEDP for generalized doubly stochastic matrices, and give new sufficient conditions for the existence and construction of a solution matrix. These conditions improve those given in [ELA 30 (2015) 704-720].

Keywords: Elementary divisor, nonnegative matrix, nonnegative inverse elementary divisors problem

MSC: 15A18, 15A51

1 Introduction

Let \( A \in \mathbb{C}^{n \times n} \) and let \( J(A) = S^{-1}AS = \oplus_{i=1}^{k} J_{n_i}(\lambda_i) \) be a Jordan canonical form of \( A \) (hereafter, the JCF of \( A \)). The \( n_i \times n_i \) submatrices

\[
J_{n_i}(\lambda_i) = \begin{bmatrix}
\lambda_i & 1 \\
& \ddots \\
& & \ddots & 1 \\
& & & \lambda_i
\end{bmatrix}, \quad i = 1, 2, \ldots, k,
\]

are the Jordan blocks of \( J(A) \). The elementary divisors of \( A \) are the polynomials \( (\lambda - \lambda_i)^{n_i} \), that is, the characteristic polynomials of \( J_{n_i}(\lambda_i) \), \( i = 1, \ldots, k \).

The nonnegative inverse elementary divisors problem (hereafter, the NIEDP) is the problem of determining necessary and sufficient conditions under which there exists a nonnegative matrix with prescribed elementary divisors (see [5, 8, 9, 12, 14–16]). The NIEDP is closely related to the nonnegative inverse eigenvalue problem (hereafter, the NIEP), which is the problem of determining necessary and sufficient conditions for a list of complex numbers \( \Lambda = \{ \lambda_1, \lambda_2, \ldots, \lambda_n \} \) to be the spectrum of an \( n \times n \) entrywise nonnegative matrix. If there exists a nonnegative matrix \( A \) with spectrum \( \Lambda \), we say that \( A \) is realizable and that \( A \) is the realizing matrix.

Following definition and notation in [4], we shall say that \( A \) is universally realizable (UR) if there exists a non-
negative matrix with spectrum $\Lambda$ for each possible JCF associated with $\Lambda$. Both problems, the \textit{NIEDP} and the \textit{NIEP}, remain unsolved.

In [12] it has been proved that lists of real numbers $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ with

\begin{align*}
  i) & \quad \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \\
  \text{and} & \\
  ii) & \quad \lambda_1 > 0 \times \lambda_2 \geq \cdots \geq \lambda_n
\end{align*}

are \textit{UR}. In [5, 14] it has been proved, respectively, that lists of complex numbers $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ with

\begin{align*}
  iii) & \quad \lambda_1 > 0, \quad \text{Re} \lambda_i < 0, \quad |\text{Re} \lambda_i| > |\text{Im} \lambda_i|, \quad i = 2, \ldots, n, \\
  \text{and} & \\
  iv) & \quad \lambda_1 > 0, \quad \text{Re} \lambda_i < 0, \quad \sqrt{3} |\text{Re} \lambda_i| > |\text{Im} \lambda_i|, \quad i = 2, \ldots, n
\end{align*}

are also \textit{UR}. In (2), (3) and (4), the necessary and sufficient condition is $\sum_{i=1}^{n} \lambda_i \geq 0$. In [4] the authors prove that real spectra with two positive eigenvalues of the form $(p, q, -r, -r, \ldots, -r)$, with $p, q, r > 0$, $p+q+(n-2)r = 0$, $p > q$, and $q \leq r$, are also \textit{UR}. In all these cases the spectra are diagonalizable realizable (DR). Thus, it is natural to ask if all DR lists are also \textit{UR}. This is not true in the case of doubly stochastic matrices. In fact, Minc [9] has showed that the doubly stochastic

\begin{equation}
\begin{bmatrix}
1 & 0 & 1 \\
\frac{1}{2} & 1 & 0 \\
1 & \frac{1}{2} & 0
\end{bmatrix}
\end{equation}

has eigenvalues $1, \frac{1}{2}, \frac{1}{2},$ but no $3 \times 3$ doubly stochastic matrix has the elementary divisor $(\lambda + \frac{1}{2})^2$. For more general lists, sufficient conditions have been obtained in [3, 8, 9, 12, 14].

The set of all matrices with constant row sums equal to $\gamma$ is denoted by $\mathbb{CS}_{\gamma}$. It is clear that $e = [1, 1, \ldots, 1]^T$ is an eigenvector of any matrix $A \in \mathbb{CS}_{\gamma}$, corresponding to the eigenvalue $\gamma$. The relevance of the real matrices with constant row sums is due to the well known fact that the problem of finding a nonnegative matrix with spectrum $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, $\lambda_1$ being the Perron eigenvalue, is equivalent to the problem of finding a nonnegative matrix in $\mathbb{CS}_{\lambda_1}$ with spectrum $\Lambda$.

We mention the following perturbation result due to Brauer [2, Theorem 27], which shows how to change a single eigenvalue of an $n \times n$ matrix, via a rank-one perturbation, without changing any of the remaining $n - 1$ eigenvalues. This result has been employed, in connection with the \textit{NIEP} and the \textit{NIEDP}, to derive sufficient conditions for the existence and construction of nonnegative matrices with prescribed spectrum and prescribed elementary divisors (see [3, 10–14] and the references therein).

\textbf{Theorem 1.1.} \cite{2} Let $A$ be an $n \times n$ arbitrary matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $v = [v_1, v_2, \ldots, v_n]^T$ be an eigenvector of $A$ associated with the eigenvalue $\lambda_k$ and let $q$ be any $n$-dimensional vector. Then the matrix $A + vq^T$ has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}, \lambda_k + v^Tq, \lambda_{k+1}, \ldots, \lambda_n$.

We shall denote by $E_{i,j}$ the $n \times n$ matrix with 1 in the $(i, j)$ position and zeros elsewhere. The following lemma in [12], shows the JCF of Brauer’s perturbation $A + \mathbf{eq}^T$.

\textbf{Lemma 1.1.} \cite{12} Let $A \in \mathbb{CS}_{\lambda_1}$ be with JCF

$$ J(A) = S^{-1}AS = \text{diag}(J_1(\lambda_1), J_{n_2}(\lambda_2), \ldots, J_{n_k}(\lambda_k)) $$

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Let \( q^T = [q_1, \ldots, q_n] \) and \( \lambda_1 + \sum_{i=1}^{n} q_i \neq \lambda_i, i = 2, \ldots, n \). Then \( A + eq^T \) has JCF \( J(A) + \left( \sum_{i=1}^{n} q_i \right) E_{11} \). In particular, if \( \sum_{i=1}^{n} q_i = 0 \), then \( A \) and \( A + eq^T \) are similar.

In [8, Theorem 1], Minc proved the following result:

**Theorem 1.2.** [8] Let \( A = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) be a list of complex numbers, which is realizable by a diagonalizable positive (diagonalizable positive doubly stochastic) matrix \( A \). Then, for each JCF \( J_A \) associated with \( A \), there exists a positive (doubly stochastic) matrix \( B \), with the same spectrum as \( A \), and with JCF \( J(B) = J_A \).

The problem of finding a nonnegative matrix with spectrum \( A = \{\lambda_1, \ldots, \lambda_n\} \) for each possible JCF associated with \( A \) (NIEDP) is a very hard problem. As with the NIEP, it remains unsolved. A complete solution, if any, is still far from the current state of the art about the problem. For that reason, we identified some open topics to work on, for which we have a reasonable chance of success. In particular, the NIEDP for generalized doubly stochastic matrices. In Section 2, we discuss certain results relating nonnegative with positive matrices. In particular, on nonnegative matrices which are cospectral (similar) to positive matrices. In Section 3 we consider the NIEDP for generalized doubly stochastic matrices and give new sufficient conditions for the existence of a solution. These conditions improve the sufficient conditions in [16]. Moreover, we show that, under certain restrictions, Theorem 1.2 of Minc holds for nonnegative matrices and that the diagonalizability condition can be removed.

### 2 On nonnegative matrices similar to positive matrices

In [1, 7] the authors prove that if \( A \) is a nonnegative irreducible matrix with a positive row or column, then \( A \) is similar to a positive matrix. Next, we show, in a simple way, that this similarity preserves the positive row or column and that the irreducibility condition is not necessary if \( A \in \mathbb{CS}_{\lambda_1} \). Moreover, if \( A \in \mathbb{CS}_{\lambda_1} \) is diagonalizable, it is similar to a diagonalizable positive matrix. As a consequence, if \( A = \{\lambda_1, \ldots, \lambda_n\} \) is the spectrum of a nonnegative diagonalizable matrix \( A \in \mathbb{CS}_{\lambda_1} \), with a positive row or column, then \( A \) is universally realizable. First we consider the following result, which is often attributed to Johnson [6].

**Lemma 2.1.** Let \( A \) be a nonnegative matrix with spectrum \( A = \{\lambda_1, \ldots, \lambda_n\} \), \( \lambda_1 \) being the Perron eigenvalue. Then \( A \) is cospectral to a nonnegative matrix \( B \in \mathbb{CS}_{\lambda_1} \).

**Theorem 2.1.** Let \( A \) be a nonnegative matrix with spectrum \( A = \{\lambda_1, \ldots, \lambda_n\} \), \( \lambda_1 > |\lambda_i|, i = 2, \ldots, n \) and a positive row or column. If \( A \) is irreducible (respectively reducible), then \( A \) is similar (respectively cospectral) to a nonnegative matrix \( B \in \mathbb{CS}_{\lambda_1} \) with a positive row or column.

**Proof.** If \( A = [a_{i,j}] \) is irreducible nonnegative with a positive row or column, it has a positive Perron eigenvector \( x^T = [x_1, \ldots, x_n] \). Let \( D = diag\{x_1, \ldots, x_n\} \). Then \( B = D^{-1}AD = \left[ \frac{1}{x_i} a_{i,j} \right] \in \mathbb{CS}_{\lambda_1} \) is nonnegative, similar to \( A \), and with a positive row or column.

If \( A \) is an arbitrary nonnegative matrix with a positive column, then our proof reproduces Johnson’s proof of Lemma 2.1 for the case \( \lambda_1 > |\lambda_i|, i = 2, \ldots, n \), and a positive column. It is clear that the nonnegative matrix \( B \in \mathbb{CS}_{\lambda_1} \) has a positive column. If \( A \) is nonnegative with a positive row, we take \( A^T \) and the result follows. \( \square \)

**Corollary 2.1.** Let \( A \) be an nonnegative matrix with spectrum \( A = \{\lambda_1, \ldots, \lambda_n\} \), \( \lambda_1 > |\lambda_i|, i = 2, \ldots, n \), and a positive row or column. Then \( A \) is cospectral to a positive matrix. If \( A \in \mathbb{CS}_{\lambda_1} \), then \( A \) is similar to a positive matrix. Moreover, if \( A \) is diagonalizable, then it is similar to a diagonalizable positive matrix.
Proof. Let $A$ be nonnegative with a positive column. Then, from Theorem 2.1, $A$ is cospectral to a nonnegative matrix $B \in \mathcal{CS}_{\lambda_1}$ with a positive column, say the last column $[b_{1n}, b_{2n}, \ldots, b_{mn}]^T$. Let

$$q^T = [q_1, \ldots, q_n], \quad q_i > 0, \quad i = 1, \ldots, n - 1,$$

$$\sum_{i=1}^n q_i = 0, \quad \sum_{i=1}^{n-1} q_i < \min b_{in}.$$  \hspace{1cm} (5)

Then $M = B + eq^T$ is positive and from Lemma 1.1 it is similar to $B$ and cospectral to $A$. If $A$ is nonnegative with a positive row, then we take $A^T$, which is nonnegative with a positive column. Then $A^T$ is cospectral to a nonnegative matrix $B \in \mathcal{CS}_{\lambda_1}$ with a positive column, and the result follows. If the last column of $A \in \mathcal{CS}_{\lambda_1}$ is positive, then $M = A + eq^T$, where $q^T$ is as in (5), is positive and from Lemma 1.1, it is similar to $A$. If $A$ has a positive row, we take $A^T$ and the result follows as before. If $A$ is diagonalizable, then from Lemma 1.1 $M$ is also diagonalizable. \hfill \Box

As an example, we have that the reducible nonnegative matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} \in \mathcal{CS}_4$$

is similar to the positive matrix $M = A + eq^T$, where $q^T = [\frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, -\frac{3}{7}]$.

Corollary 2.2. Let $A$ $(A \in \mathcal{CS}_{\lambda_1})$ be a nonnegative matrix with spectrum $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, $\lambda_1 > |\lambda_i|$, $i = 2, \ldots, n$, and having fewer than $n$ zero entries. Then $A$ is cospectral (similar) to a positive matrix.

Proof. It is clear that if $A$ has fewer than $n$ zero entries, then $A$ has a positive row or column. Thus, from Corollary 2.1, $A$ is cospectral to a positive matrix, and if $A \in \mathcal{CS}_{\lambda_1}$, then it is similar to a positive matrix. \hfill \Box

In [7] the authors define a Perron extreme spectrum as follows:

Definition 2.1. A realizable list of complex numbers $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, with the Perron eigenvalue $\lambda_1$, is Perron extreme if the list $\Lambda_{-\epsilon} = \{\lambda_1 - \epsilon, \lambda_2, \ldots, \lambda_n\}$ is not realizable for every $\epsilon > 0$.

We shall say that a matrix is Perron extreme if its spectrum is Perron extreme. All nonnegative matrices with trace zero are Perron extreme.

Lemma 2.2. Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a realizable list of complex numbers. Then $\Lambda$ is not Perron extreme if and only if $\Lambda$ is the spectrum of a positive matrix.

Proof. If $\Lambda$ is not Perron extreme, then there exists $\epsilon_0 > 0$ such that $\{\lambda_1 - \epsilon, \lambda_2, \ldots, \lambda_n\}$, with $0 < \epsilon \leq \epsilon_0$, is realizable by a nonnegative matrix $B$, which from Lemma 2.1, is cospectral to a matrix $B' \in \mathcal{CS}_{\lambda_1-\epsilon}$. Then for $q^T = [\frac{1}{\epsilon}, \frac{1}{\epsilon}, \ldots, \frac{1}{\epsilon}]$, $A = B' + eq^T$ is positive with spectrum $\Lambda$.

If $\Lambda$ is the spectrum of a positive matrix $A$, then $A$ is similar to a positive matrix $A' = (a'_{ij}) \in \mathcal{CS}_{\lambda_1}$. Then for $q^T = [\epsilon, 0, \ldots, 0]$, with $0 < \epsilon \leq \min_{1 \leq i \leq n} a'_{ii}$, we have that $A' + eq^T$ is nonnegative with spectrum $\Lambda_{-\epsilon} = \{\lambda_1 - \epsilon, \lambda_2, \ldots, \lambda_n\}$. Thus, $A$ is not Perron extreme. \hfill \Box
3 Lists universally realizable by nonnegative generalized doubly stochastic matrices

In [3] the authors introduce the $n \times n$ nonsingular matrix

$$S = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & 1 & \cdots & 1 & -1 \\
1 & 1 & 1 & \cdots & -2 & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
1 & 1 & -(n-2) & \cdots & 0 & 0 \\
1 & -(n-1) & 0 & \cdots & 0 & 0 \\
\end{bmatrix} \tag{6}$$

with rows

$$R_1 = (1, 1, \ldots, 1)$$

$$R_k = \begin{pmatrix} 1, 1, \ldots, 1, -(k-1), 0, \ldots, 0 \end{pmatrix}, \quad k = 2, \ldots, n,$$

and $S^{-1}$ with columns

$$C_1^T = \frac{1}{n} \begin{pmatrix} 1, \frac{1}{n(n-1)}, \frac{1}{(n-1)(n-2)}, \cdots, \frac{1}{3 \cdot 2}, \frac{1}{2 \cdot 1} \end{pmatrix}$$

$$C_k^T = \frac{1}{n} \begin{pmatrix} 1, \frac{1}{n(n-1)}, \frac{1}{(n-1)(n-2)}, \cdots, \frac{1}{(k+1)k}, \frac{1}{k}, 0, \ldots, 0 \end{pmatrix}, \quad k = 2, \ldots, n.$$ 

In terms of a real list $A = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, the authors in [3] prove that if

$$-\frac{n-r+2}{n-r+1} T_{r-1} < \lambda_r < (n-r+2) T_{r-1}, \quad r = 2, \ldots, n, \tag{7}$$

where

$$T_{r-1} = \frac{\lambda_1}{n} + \sum_{p=2}^{r-1} \frac{\lambda_p}{(n-p+2)(n-p+1)}, \tag{8}$$

$r = 3, \ldots, n$, with $T_1 = \frac{\lambda_1}{n}$, and $D = \text{diag} \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, then $B = SDS^{-1}$ is a positive symmetric generalized doubly stochastic matrix with spectrum $A$. In [3] it is also shown that if $D = \text{diag} \{\lambda_1, \ldots, \lambda_n\}$ with $\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, then $\lambda_r$, $r = 2, \ldots, n$, satisfies (7) and $SDS^{-1}$ is a positive symmetric generalized doubly stochastic matrix. Thus, from Theorem 1.2, $A$ is UR. Then, we have:

**Theorem 3.1.** Let $A = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a list of real numbers with

$$\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n \geq 0.$$

Then $A$ is UR by positive generalized doubly stochastic matrices.

Now, we show that if $A = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is a list of real numbers of “Suleimanova” type, then under certain conditions, $A$ is the spectrum of a positive diagonalizable generalized doubly stochastic matrix, and as a consequence $A$ is UR by positive generalized doubly stochastic matrices.

**Theorem 3.2.** Let $A = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a list of real numbers with $\lambda_1 > 0 > \lambda_2 \geq \cdots \geq \lambda_n$ and let $D = \text{diag} \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.

If

$$\lambda_1 > -n \sum_{p=2}^{r-1} \frac{\lambda_p}{(n-p+2)(n-p+1)} - \frac{n(n-r+1)}{n-r+2} \lambda_r \tag{9}$$
for \( r = 2, 3, \ldots, n \), where the sum on the right side is zero for \( r = 2 \), then SDS\(^{-1} \) is a positive symmetric generalized doubly stochastic matrix with spectrum \( \Lambda \).

**Proof.** Observe that inequality (9) is the first inequality in (7), in terms of \( \lambda_1 \). Then

\[
-\frac{n-r+2}{n-r+1} T_{r-1} < \lambda_r.
\]

Since \( n \lambda_r < 0 \) and \( n - r + 2 > 0 \), then

\[
\frac{n}{n-r+2} \lambda_r < -\frac{n(n-r+1)}{n-r+2} \lambda_r.
\]

Therefore, from (9) we have

\[
-\frac{n}{n-r+2} \lambda_r < \frac{n}{n-r+2} \frac{\sum_{p=2}^{r-1} \frac{\lambda_p}{(n-p+1)(n-p+2)}}{\frac{n}{n-r+2} \lambda_r} < \lambda_1
\]

and then

\[
\lambda_r < (n-r+2) T_{r-1}.
\]

Thus, \( \lambda_r \) satisfies (7), \( r = 2, \ldots, n \), and the result follows.

The following corollary simplifies the condition (9):

**Corollary 3.1.** Let \( \Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) be a list of real numbers with \( \lambda_1 > 0 > \lambda_2 \geq \cdots \geq \lambda_n \) and let \( D = \text{diag} \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \).

If

\[
\lambda_1 > -n \sum_{p=2}^{n} \frac{\lambda_p}{(n-p+1)(n-p+2)},
\]

then SDS\(^{-1} \) is a positive symmetric generalized doubly stochastic matrix with spectrum \( \Lambda \).

**Proof.** Let

\[
f(r) = -n \sum_{p=2}^{r-1} \frac{\lambda_p}{(n-p+1)(n-p+2)} - \frac{n(n-r+1)}{n-r+2} \lambda_r,
\]

\( r = 2, \ldots, n \), with the first term on the right side being zero for \( r = 2 \). We shall prove that if \( r_1 < r_2 \), then \( f(r_1) < f(r_2) \).

\[
f(r) - f(r-1) = \sum_{p=2}^{r-2} \frac{\lambda_p}{(n-p+1)(n-p+2)} - \frac{n-r+2}{n-r+3} \lambda_r - \frac{n-r+1}{n-r+2} \lambda_r
\]

\[
= -n(\lambda_r - \lambda_{r-1}) \frac{n-r+1}{n-r+2} > 0.
\]

Thus, \( f(n) > f(n-1) > \cdots > f(3) > f(2) \), and the result follows.

The following lemma shows that, for a list of real numbers of "Suleimanova" type, Corollary 3.1 gives a better result than Corollary 2.5 in [16]:

**Lemma 3.1.** The lower bound (10) of Corollary 3.1 is less than or equal to the lower bound of Corollary 3.5 in [16].
Proof.
\[-n \sum_{p=2}^{n} \frac{\lambda_p}{(n-p+1)(n-p+2)} \leq -n \sum_{p=2}^{n} \frac{\lambda_p}{(n-p+1)(n-p+2)} = -(n-1)(\lambda_n).\]

There are many examples which show that Theorem 1.2 holds for diagonalizable nonnegative (diagonalizable nonnegative generalized doubly stochastic) matrices. For instance, the lists in (1), (2), (3), and (4) are all UR. Moreover, if \(\Lambda = \{\lambda_1, \ldots, \lambda_n\}\) is a list of complex numbers, which is realizable by a diagonalizable nonnegative matrix \(A\) satisfying the hypothesis of Corollaries 2.1 or 2.2, then \(A\) is UR. In the same way, the realizable lists of Theorem 3.1 and Theorem 3.2 are also UR. However, we do not know if Theorem 1.2 holds for a general diagonalizable nonnegative (diagonalizable nonnegative generalized doubly stochastic) matrix. Here we introduce other results on universal realizability and discuss the positivity and diagonalizability conditions of Theorem 1.2, and we find, under certain restrictions, some extensions of the Minc result to nonnegative matrices. We start with the following result:

**Theorem 3.3.** Let \(\Lambda = \{\lambda_1, \ldots, \lambda_n\}, \lambda_1 > |\lambda_i|, i = 2, \ldots, n,\) be a list of complex numbers which is the spectrum of an \(n \times n\) diagonalizable nonnegative matrix \(A \in \mathcal{CS}_{\lambda_1}\) with a positive row or column. Then \(\Lambda\) is UR.

**Proof.** From Corollary 2.1, \(A\) is similar to a diagonalizable positive matrix \(\tilde{A}\). Then, from Theorem 1.2, there exists an \(n \times n\) nonnegative matrix \(B\) with the same spectrum as \(A\) for each JCF associated with \(\Lambda\).

Observe that if \(A\) is nonnegative with spectrum \(\Lambda = \{\lambda_1, \ldots, \lambda_n\}\), then we may assume that \(A\) has at least one zero entry in each column, otherwise \(A\) has a positive column. The following result generates a sequence of nonnegative matrices with the desired JCF, and with the Perron eigenvalue being \(\lambda_1 + \epsilon\) for all \(\epsilon > 0\).

**Theorem 3.4.** Let \(\Lambda = \{\lambda_1, \ldots, \lambda_n\}\) be a list of complex numbers which is the spectrum of an \(n \times n\) diagonalizable nonnegative generalized doubly stochastic matrix \(A\). Then, for all \(\epsilon > 0\), \(A_\epsilon = \{\lambda_1 + \epsilon, \lambda_2, \ldots, \lambda_n\}\) is UR.

**Proof.** Let \(\epsilon > 0\). Since \(A \in \mathcal{CS}_{\lambda_1}\), there exists a positive vector \(q^T = (q_1, \ldots, q_n)\) with \(q_i = \frac{\epsilon}{n}, i = 1, \ldots, n,\) such that \(B = A + \epsilon q q^T\) is a positive generalized doubly stochastic matrix with spectrum \(A_\epsilon = \{\lambda_1 + \epsilon, \lambda_2, \ldots, \lambda_n\}\) and \(B \in \mathcal{CS}_{\lambda_1+\epsilon}\).

From Lemma 1.1, \(B\) is diagonalizable and from Theorem 1.2, \(A_\epsilon\) is UR.

**Example 3.1.** Consider the matrix
\[
A = \begin{bmatrix}
0 & 2 & 2 & 1 & 1 & 1 & 2 \\
2 & 0 & 2 & 1 & 1 & 1 & 2 \\
2 & 2 & 0 & 1 & 1 & 1 & 2 \\
2 & 2 & 2 & 0 & 0 & 1 & 2 \\
0 & 2 & 2 & 2 & 0 & 1 & 2 \\
2 & 2 & 2 & 1 & 1 & 0 & 1 \\
0 & 2 & 2 & 1 & 1 & 2 & 1 \\
\end{bmatrix} \in \mathcal{CS}_2,
\]
which is diagonalizable nonnegative with a positive column and spectrum \(\Lambda = \{9, -2, -2, -1 \pm i, -1 \pm i\}\). Then, from Theorem 3.3, \(\Lambda\) is UR.
Remark 3.1. Observe that if we want that a realizable list \( A = \{\lambda_1, \ldots, \lambda_n\} \) to be the spectrum of a nonnegative matrix \( A \in \mathcal{E} \mathcal{S}_{\lambda_1} \), with a positive column, then the Perron eigenvalue \( \lambda_1 \) must be simple. In fact, if \( \lambda_1 \) has algebraic multiplicity \( m_A \geq 2 \), for instance, then \( A \) has a reducible nonnegative realization \( A \), and therefore \( A \) cannot have a positive column.

Now, we show that under certain restrictions, the requirement that \( A \) is a diagonalizable matrix can be removed from the statement of Theorem 1.2. However, our results do not allow us to realize, universally, the list \( A \).

Theorem 3.5. Let \( A \) be an \( n \times n \) nonnegative matrix with spectrum \( \Lambda = \{\lambda_1, \ldots, \lambda_n\} \) and a positive row or column. Let \( J(\Lambda) \) be a JCF of \( A \). Then there exists a nonnegative matrix \( C \) with spectrum \( \Lambda \) for each possible modification of \( J(\Lambda) \), which does not include elementary divisors of smaller size than the size of the elementary divisors \( A \).

Proof. We assume that \( A = [a_{i,j}] \in \mathcal{E} \mathcal{S}_{\lambda_1} \) with its last column being positive and that

\[
J(\Lambda) = \text{diag}(J_n(\lambda_1), J_n(\lambda_2), \ldots, J_n(\lambda_k)), \quad n_1 + \cdots + n_k = n,
\]

is not a diagonal matrix. Let \( q^T = [q_1, q_2, \ldots, q_n] \) be as in (5). Then \( B = (A + eq^T) \in \mathcal{E} \mathcal{S}_{\lambda_1} \) is positive, and from Lemma 1.1 \( J(B) = J(\Lambda) \). Let \( S \) be a nonsingular matrix such that \( S^{-1}BS = J(B) \). Let \( E = \sum_{i \in K} E_{i,i+1} \), \( K \subset \{2, \ldots, n-1\} \) be such that \( J(\Lambda) + E \) is the desired JCF. Let \( \delta > 0 \) be small enough such that \( B + \delta SES^{-1} \geq 0 \). Then the matrix

\[
C = B + \delta SES^{-1} = SJ(B)S^{-1} + \delta SES^{-1} = S(J(B) + \delta E)S^{-1}
\]

is nonnegative with spectrum \( \Lambda \), and with the desired elementary divisors. Besides, since \( SES^{-1} \in \mathcal{E} \mathcal{S}_0 \), \( C \in \mathcal{E} \mathcal{S}_{\lambda_1} \).

Corollary 3.2. Let \( A \in \mathcal{E} \mathcal{S}_{\lambda_1} \) be an \( n \times n \) nonnegative matrix with spectrum \( \Lambda = \{\lambda_1, \ldots, \lambda_n\} \) and JCF \( J(\Lambda) \). Then for all \( \epsilon > 0 \) there exists a nonnegative matrix \( C \) with spectrum \( \Lambda_\epsilon = \{\lambda_1 + \epsilon, \ldots, \lambda_n\} \), for each possible modification of \( J(\Lambda) \), which does not include elementary divisors of smaller size than the size of the elementary divisors \( A \).

Example 3.2. The matrix

\[
A = \begin{bmatrix}
0 & 5 & 0 & 0 & 0 & 1 \\
5 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 5 & 0 \\
1 & 0 & 5 & 0 & 0 \\
0 & 0 & 4 & 0 & 2
\end{bmatrix}
\]

has JCF \( J(\Lambda) = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 \\
0 & 3 & 1 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & -5 & 0 \\
0 & 0 & 0 & 0 & -5
\end{bmatrix} \). Let \( \epsilon = 0.001 \). We shall use Corollary 3.2 to construct a nonnegative matrix \( C \) with JCF \( J(C) = \text{diag}(J_1(6.001), J_2(3), J_2(-5)) \). First we compute a positive matrix \( B \in \mathcal{E} \mathcal{S}_{6.001} : \)

\[
B = A + \frac{0.001}{5} \epsilon e^T = \begin{bmatrix}
\frac{1}{5000} & \frac{25001}{5000} & \frac{1}{5000} & \frac{1}{5000} & \frac{5001}{5000} \\
\frac{25001}{5000} & \frac{1}{5000} & \frac{1}{5000} & \frac{1}{5000} & \frac{5001}{5000} \\
\frac{5001}{5000} & \frac{1}{5000} & \frac{25001}{5000} & \frac{1}{5000} & \frac{1}{5000} \\
\frac{5001}{5000} & \frac{1}{5000} & \frac{25001}{5000} & \frac{1}{5000} & \frac{1}{5000} \\
\frac{1}{5000} & \frac{1}{5000} & \frac{1}{5000} & \frac{20001}{5000} & \frac{10001}{5000}
\end{bmatrix}
\]

Next, we compute the matrix \( S \) such that \( S^{-1}BS = J(B) \). Then, for \( E_{4,5} \) and \( \delta \) small enough, \( C = B + \delta E_{4,5}S^{-1} \) will be nonnegative in \( \mathcal{E} \mathcal{S}_{6.001} \) with the desired JCF. Obviously the matrix \( S \) is necessary to choose an appropriate \( \delta \), which in this example is \( \delta = 0.000001 \).
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References