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**Commuting decomposition of** $K_{n_1, n_2, \ldots, n_k}$ **through realization of the product** $A(G)A(G_k^P)$

https://doi.org/10.1515/spma-2018-0028
Received November 2, 2017; accepted July 7, 2018

**Abstract:** In this paper, we introduce the notion of perfect matching property for a $k$-partition of vertex set of given graph. We consider nontrivial graphs $G$ and $G_k^P$, the $k$-complement of graph $G$ with respect to a $k$-partition of $V(G)$, to prove that $A(G)A(G_k^P)$ is realizable as a graph if and only if $P$ satisfies perfect matching property. For $A(G)A(G_k^P) = A(\Gamma)$ for some graph $\Gamma$, we obtain graph parameters such as chromatic number, domination number etc., for those graphs and characterization of $P$ is given for which $G_k^P$ and $\Gamma$ are isomorphic. Given a 1-factor graph $G$ with $2n$ vertices, we propose a partition $P$ for which $G_k^P$ is a graph of rank $r$ and $A(G)A(G_k^P)$ is graphical, where $n \leq r \leq 2n$. Motivated by the result of characterizing decomposable $K_{n,n}$ into commuting perfect matchings [2], we characterize complete $k$-partite graph $K_{n_1, n_2, \ldots, n_k}$ which has a commuting decomposition into a perfect matching and its $k$-complement.

**Keywords:** Adjacency matrix, $k$-complement, Graphical, Matrix product, Commutativity, Commuting Decomposition

**MSC:** 05C50; 15A24

### 1 Introduction

Graphs considered in this paper are simple, undirected, and without self-loops. Let $G$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. For any two vertices $v_i$ and $v_j$, $i \neq j$ in the graph $G$, $v_i \sim_G v_j$ denotes that the vertices are adjacent in the graph $G$, and $v_i \not\sim_G v_j$ denotes that the vertices are not adjacent in the graph $G$. The suffix $G$ in the notations $\sim_G$ and $\not\sim_G$ are conveniently ignored if the graph under discussion is clearly understood. The notation $A(G)$ denotes the adjacency matrix of the graph $G$.

A symmetric $(0, 1)$ matrix is said to be **graphical** [2], if its diagonal entries are all equal to zero. In this case, a graph $G$ such that $A(G) = B$ is called the realization of the matrix $B$, and $B$ is said to be realizable as graph $G$.

In the following we provide some useful definitions.

**Definition 1.1.** [6] A graph $\Gamma$ is said to be the matrix product of graphs $G$ and $H$ if $A(\Gamma) = A(G)A(H)$.

**Definition 1.2.** [6] (GH path) Given graphs $G$ and $H$ on the same set of vertices $\{v_1, v_2, \ldots, v_n\}$, two vertices $v_i$ and $v_j$ ($i \neq j$) are said to have a GH path from $v_i$ to $v_j$, if there exists a vertex $v_k$, different from $v_i$ and $v_j$, such that $v_i \sim_G v_k$ and $v_k \sim_H v_j$.

The following theorem characterizes graphs $G$ and $H$ with the property that $A(G)A(H)$ is graphical.
Theorem 1.3. [6, Theorem 1] Let $G$ and $H$ be graphs on the same set of vertices $\{v_1, v_2, \ldots, v_n\}$. Then $C = A(G)A(H)$ is graphical if and only if the following three statements hold.

(i) $G$ and $H$ are edge disjoint.
(ii) For every two vertices $v_i$ and $v_j$, there exists at most one $GH$ path from the vertex $v_i$ to the vertex $v_j$.
(iii) For every two vertices $v_i$ and $v_j$, if there exists a $GH$ path from the vertex $v_i$ to the vertex $v_j$, then there exists a $HG$ path from $v_i$ to $v_j$.

The following two properties regarding degree of vertices of graphs $G$, $H$ and $\Gamma$ satisfying $A(G)A(H) = A(\Gamma)$ are useful.

Lemma 1.4. [6, Corollary 2] Let $\Gamma$ be the matrix product of $G$ and $H$. If the vertices $u$ and $v$ are connected by a path in $G$, then $\deg_H u = \deg_H v$.

Theorem 1.5. [6, Theorem 7] Let $G$, $H$ and $\Gamma$ be the graphs such that $\Gamma$ is the matrix product of $G$ and $H$. Then $\deg_G u = \deg_G u \cdot \deg_H u$ for all vertices $u$.

The notion of $k$-complement of graph $G$ with respect to a partition $P$ of $V(G)$ with $k$ partite sets is a generalization of complement of a graph introduced by E. Sampath Kumar and L. Pushpalatha [3].

Definition 1.6. [3] Given a graph $G$ and a $k$-partition $P = \{V_1, V_2, \ldots, V_k\}$ of $V(G)$, the $k$-complement of $G$ with respect to the partition $P$ is a graph $H$ with $V(H) = V(G)$ in which the adjacency between $v \in V_i$ and $w \in V_j$ is defined by

$$v \sim_H w \text{ if } \begin{cases} v \sim_G w, & \text{for } i \neq j \\ v \sim_G w, & \text{for } i = j. \end{cases}$$

The above graph $H$ is usually denoted by $G_k^p$ and is also known as generalized complement of graph $G$, where it reduces to the well known complement of $G$ when $k$ is equal to $n$, the number of vertices in the graph $G$. From the definition it is clear that the subgraph induced by $V_i$, $1 \leq i \leq k$, in $G$ and $G_k^p$ are same. We refer [3], [4], [5], [7] and [8] for the further properties and results associated with generalized complement of a graph.

Definition 1.7. A collection of subgraphs $H_1, H_2, \ldots, H_k$ of $G$ is said to be a decomposition of $G$ if $\bigcup_{i=1}^k H_i = G$ and $E(H_i) \cap E(H_j) = \emptyset$ for $i \neq j$.

Definition 1.8. A decomposition $H_1, H_2, \ldots, H_k$ of $G$ is said to be a commuting decomposition of $G$ if $H_1, H_2, \ldots, H_k$ are spanning subgraphs and $A(H_i)A(H_j) = A(H_j)A(H_i)$ for all $i$ and $j$ with $1 \leq i, j \leq k$.

Readers are referred to [1] for the results on commuting decomposition of $K_n$ and [2] for the results on the commuting decomposition of $K_{n,n}$ into perfect matchings and Hamiltonian cycles.

In this paper, we focus our study on characterization of the partition $P$ of $V(G)$ for which the matrix product $A(G)A(G_k^p)$ is graphical. We discuss about some parameters of graphs $G$, $G_k^p$ and $\Gamma$, when $A(G)A(G_k^p) = A(\Gamma)$. We comment on rank of $G$, $G_k^p$ and $\Gamma$, whenever $A(G)A(G_k^p) = A(\Gamma)$. We also derive a result on commuting decomposition of the complete $k$-partite graph $K_{n_1,n_2,\ldots,n_k}$ into a perfect matching $G$ and its $k$-complement $G_k^p$ with respect to $k$-partition $P = \{V_1, V_2, \ldots V_k\}$ of $V(K_{n_1,n_2,\ldots,n_k})$.

Readers are referred to [9] for all the elementary notations and definitions not described but used in this paper.
2 Realization of $A(G)A(G^P_k)$ into graph

In this section we explore the characterization of graph $G$ and its $k$-partition $P = \{V_1, V_2, \ldots, V_k\}$ so that $A(G)A(G^P_k)$ is graphical.

**Theorem 2.1.** Let $G$ be a graph, $P = \{V_1, V_2, \ldots, V_k\}$ be a $k$-partition of $V(G)$. The matrix product $A(G)A(G^P_k)$ is graphical if and only if the following statements are true.

(i) The subgraph $(V_i)$ is totally disconnected for every $i$, $1 \leq i \leq k$.

(ii) For every pair of vertices $u \in V_i$ and $v \in V_j$, there exists at most one vertex $w \in V_r$, $r \neq i, j$ and $1 \leq i, j, r \leq k$, such that $u \sim_G w$ and $w \sim_G v$.

(iii) For a pair of vertices $u$ and $v$, if there exists $w$ satisfying (ii), then there exists a unique $w' \in V_l$, $l \neq i, j$, such that $v \sim_G w'$ and $w' \sim_G u$.

**Proof.** Let $A(G)A(G^P_k)$ be graphical. If $u \sim_G v$ for any $u, v \in V_i$, then from the definition of $G^P_k$, we have that $u \sim_{G^P_k} v$. It contradicts the fact that $G^P_k$ is subgraph of $G$. This proves (i).

From (ii) of Theorem 1.3 we have that for every pair of vertices $u$ and $v$, there exists $w$ different from $u$ and $v$ such that $u \sim_G w$ and $w \sim_{G^P_k} v$. Since $G^P_k$ is subgraph of $G$, it is trivial that $w \sim_G v$. Further, if $w \in V_r$ for some $r$, (i), we get that $r$ is neither equal to $i$ nor equal to $j$. Thus (ii) is proved.

Proof of (iii) follows immediately from (iii) of Theorem 1.3.

Conversely, let (i), (ii) and (iii) of the theorem are satisfied. Consider any two vertices $u \in V_i$ and $v \in V_j$. Since $(V_i)$ and $(V_j)$ are totally disconnected in $G$, from the definition of $G^P_k$, and from (ii), we get that there is at most one $GG^P_k$ path between $u$ and $v$. For the same reason, (iii) implies that there exists at most one $GG^P_k$ path between $u$ and $v$ when ever there exists one $GG^P_k$ path between $u$ and $v$. Now, referring to Theorem 1.3, we get that $A(G)A(G^P_k)$ is graphical.

**Note 2.2.** Let $A(G)A(G^P_k)$ be graphical, and let $u \in V_i$ and $v \in V_j$, $1 \leq i, j \leq k$, $i \neq j$. Let $w \in V_r$, $r$ different from $i$ and $j$ such that $u \sim_G w$ and $w \sim_G v$. In other words, there exists a $G^P_k$ path between $u$ and $v$. Now, from the definition of $G^P_k$ and uniqueness of $G^P_k$ path between $u$ and $v$, we have that for $w' \neq v$ in the neighbourhood of $u$, either $w'$ belongs to $V_j$ or $w' \sim_G v$.

We describe the above theorem with the help of the following example.

**Example 2.3.** Consider a partition $P_1 = \{V_1, V_2, V_3, V_4\}$ of $V(G)$ as shown in the Figure 1. $G^P_k$ is constructed and is shown in the same Figure 1. From the figure, it is clear that $(V_1)$, $(V_2)$, $(V_3)$ and $(V_4)$ are totally disconnected and also satisfy conditions (ii) and (iii) given in the Theorem 2.1. Further, by direct computation of $A(G)A(G^P_k)$, which are also shown below, we observe that the product is graphical and the product $I'$ realizing $A(G)A(G^P_k)$ is also shown in the same figure.

The partition $P_2 = \{U_1, U_2, U_3, U_4\}$ of $V(G)$ is such that from the vertex $v_3$ to the vertex $v_4$, there is a $GG^P_k$ path but there is no $G^P_k$ path between $u$ and $v$. That is, $P_2$ does not satisfy the condition (iii) of the Theorem 2.1. Figure 2 shows graph $G$ and its 4-complement $G^P_k$. In $A(G)A(G^P_k)$, the $(i, j)$ entry is different from $(j, i)$ entry for $i = 3$ and $j = 4$. Therefore $A(G)A(G^P_k)$ is not symmetric and hence is not graphical.
Let $G$ be a graph on $n$ vertices and $P = \{V_1, V_2, \ldots, V_n\}$ be a partition of $V(G)$ of size $n$. Then every partite set is a singleton and the $n$-complement $G_n$ is same as the actual complement $\overline{G}$ of $G$. Then, $A(G)A(G_n^P) = A(G)A(\overline{G})$ is graphical if and only if $G$ is either a 1-factor graph or $(n - 2)$ regular graph or else $C_5$, the cycle on five vertices ([6, Theorem 10]). So, while characterizing graph $G$ and partition $P$ of $V(G)$ satisfying the property that $A(G)A(G_n^P)$ is graphical, we consider $k \neq n$.

**Definition 2.4.** Given a graph $G$, a $k$-partition $P = \{V_1, V_2, \ldots, V_k\}$ of $V(G)$ is said to have perfect matching property in $G$ if $V_i$ is an independent set for every $i$, $1 \leq i \leq k$, and for every $V_i$ there exists exactly one $V_j$, $1 \leq i, j \leq k$, and $i \neq j$, such that the graph induced by the set $V_i \cup V_j$ is a perfect matching. In such a case, $V_i$ and $V_j$ are said to be the matching partite sets.
Lemma 2.5. Let $G$ be a 1-factor graph and let $P = \{V_1, V_2, \ldots, V_k\}$ be a $k$-partition of $V(G)$. Then $A(G)A(G_k^P)$ is graphical if and only if the partition $P$ has the perfect matching property.

Proof. Given a 1-factor graph $G$, let $A(G)A(G_k^P)$ be graphical. Since $k \neq n$, there exists at least one $V_i$ with two or more vertices. Consider any two vertices $u$ and $v$ from $V_i$, $1 \leq i \leq k$. Since $A(G)A(G_k^P)$ is graphical, from (i) of Theorem 2.1, we have that $\langle V_i \rangle$ is totally disconnected and therefore $u$ and $v$ are not adjacent with each other. Since $G$ is a 1-factor graph, there exists distinct vertices $w_1$ and $w_2$ such that $u \sim_G w_1$ and $v \sim_G w_2$ and further $w_1$ and $w_2$ are not adjacent in $G$.

Now to prove that $P$ has perfect matching property, it is enough if we prove that both $w_1$ and $w_2$ are in the same partite set in $P$. To the contrary, suppose $w_1$ and $w_2$ are in different partite sets, say $w_1 \in V_r$ and $w_2 \in V_s$, where $1 \leq r, s \leq k, r \neq s$. From the definition of $G_k^P$, we observe that there exists a $G_k^P$ path $(u \sim_G w_1 \sim_G v \sim_G w_2)$. But there is no $G_k^P$ path between $u$ and $w_2$, as $v$ is the only vertex adjacent with $w_2$ in $G$ and $u$ and $v$ remains to be non adjacent in $G_k^P$. This would imply that $A(G)A(G_k^P)$ is not graphical, which is a contradiction. Therefore $w_1$ and $w_2$ must be in the same partite set, say $V_r$. Thus $\langle V_r \cup V_i \rangle$ is a perfect matching and therefore $P$ has perfect matching property.

Conversely, let the partition $P = \{V_1, V_2, \ldots, V_k\}$ of $V(G)$ be with perfect matching property. So, each of partite set $V_i$ is an independent set in the graph $G$ and therefore, from the definition of $G_k^P$, we get that $G_k^P$ is a subgraph of $G$. Since $G$ is a 1-factor graph, there exists at most one $G_k^P$ path between any two vertices. Now let there be a $G_k^P$ path between the vertices $u$ and $v$ given by $u \sim_G x \sim_G v$. Let $u \in V_i, v \in V_j$ and $x \in V_r$ and $1 \leq i, j, r \leq k$. Since $P$ has perfect matching property, we have that $r \neq i$, and $V_i$ and $V_j$ are matching partite sets. Also, $r \neq j$ as $V_j$ is an independent set in the graph $G$ as well as in the graph $G_k^P$. Now for $y \sim_G v$, it is clear that $y$ is not in $V_i$, as $V_i$ and $V_j$ are not matching partite sets. So, $y$ is not adjacent with $u$ in $G$ and therefore we obtain $G_k^P$ path between $u$ and $v$, given by $u \sim_{G_k^P} y \sim_G v$. So, from Theorem 1.3, we get that $A(G)A(G_k^P)$ is graphical.

\[\square\]

Theorem 2.6. Let $G$ be a graph on $n$ vertices and $P = \{V_1, V_2, \ldots, V_k\}$ be a $k$-partition of $V(G)$ with $k \neq n$. Then $A(G)A(G_k^P)$ is graphical if and only if one of the following holds.

(i) Either $G$ or $G_k^P$ is a null graph, in which case the other one is a complete $k$-partite graph.

(ii) The partition $P$ has perfect matching property either in $G$ or in $G_k^P$.

Proof. Let $A(G)A(G_k^P)$ be graphical, in which case each part of partite set is an independent set in $G$. Therefore from the definition of $k$-complement of $G$, it is trivial that $G$ is a null graph (complete $k$-partite graph) if and only if $G_k^P$ is a complete $k$-partite graph (null graph). In this case (i) holds.

Suppose $G$ is neither null graph nor a complete $k$-partite graph. Consider a pair of vertices $u$ and $v$ from some partite set in the partition, say $V_1$, which is possible as $k \neq n$. Since $A(G)A(G_k^P)$ is graphical, by Theorem 2.1, we have one of the following cases.

(a) For all $w \notin V_1$, $u \sim_G w$ and $w \sim_G v$ (equivalently, $u \sim_{G_k^P} w$ and $w \sim_{G_k^P} v$).

(b) There exist unique $r \notin V_1$ such that $u \sim_G r$ and $r \sim_{G_k^P} v$ in which case there exists a vertex $s$ such that, $u \sim_{G_k^P} s$ and $s \sim_G v$.

Case (a): For any $x \notin V_1$, it is not possible that $u \sim_{G_k^P} x$ and $x \sim_{G_k^P} v$, otherwise there would be two $G_k^P$ paths between $w$ and $x$. So, for every $x \notin V_1$, $x$ is adjacent with both of $u$ and $v$ in $G$. This implies $\deg_{G_k^P} u = 0$. By Lemma 1.4, $\deg_{G_k^P} u = \deg_{G_k^P} x = 0$ for all $x \notin V_1$. Further, from the definition of $k$-complement of $G$ and $V_1$ is independent set in $G$, it follows that $\deg_{G_k^P} v = 0$ as $\deg_{G_k^P} x = 0$ for all $x \notin V_1$. Hence $G_k^P$ is a null graph, which is not the case in discussion.

Case (b): Without loss of generality, let there exist unique $r \notin V_1$ such that $u \sim_G r$ and $r \sim_{G_k^P} v$, in which case there exists a vertex $s$ such that, $u \sim_{G_k^P} s$ and $s \sim_G v$. For any $t \neq r, s$ and $t \notin V_1$, both $u$ and $v$ are simultaneously adjacent or not adjacent with $t$ in $G$. Also, note that there is no pair of vertices $p$ and $q$ such...
that \( p \) is adjacent with both \( u \) and \( v \) in \( G \), and \( q \) is not adjacent with both \( u \) and \( v \) in \( G \). Otherwise, there would be two \( GG_k^P \) paths \( (p \sim_G u \sim_G^q v \sim_G q \sim_G p) \) between \( p \) and \( q \).

In case both the vertices \( u \) and \( v \) are not adjacent with \( t \not\in V_1 \) and \( t \not\equiv r, s \), then \( \text{deg}_G u = \text{deg}_G v = 1 \). As \( u \) and \( v \) are adjacent with \( t \) in \( G_k^P \), from Lemma 1A, we have that \( \text{deg}_G t = \text{deg}_G u = \text{deg}_G v = 1 \) for all \( t \not\in V_1 \). Since \( r \) and \( s \) are adjacent with each other in \( G_k^P \), \( \text{deg}_G r = 1 \). For all \( z \in V_1 \), and \( z \not\equiv u \) we have that \( r \sim_G z \), and by Lemma 1A, it implies that \( \text{deg}_G z = 1 \).

Similarly, in the case of both the vertices \( u \) and \( v \) are adjacent with \( t \not\equiv V_1 \) and \( t \not\equiv r, s \), we prove that \( G_k^P \) is 1-factor graph. So in the case \( (ii) \), we have that either \( G \) or \( G_k^P \) is 1-factor graph. Therefore, by Lemma 2.5, it follows that the partition \( P \) satisfies perfect matching property.

Converse part is trivial if \( G \) and \( G_k^P \) satisfy the condition \( (i) \) given in the theorem. If \( G \) and \( G_k^P \) satisfy the condition \( (ii) \), the converse part follows from Lemma 2.5, as graph \( G \) with partition \( P \) satisfying perfect matching property is trivially a 1-factor graph.

We note the following whenever \( A(G)A(G_k^P) \) is graphical.

(i) If \(|V_i| = n_i, 1 \leq i \leq k\), then \( A(G) + A(G_k^P) = A(K_{n_1, n_2, \ldots , n_k}) \), where \( K_{n_1, n_2, \ldots , n_k} \) is the complete \( k \)-partite graph with respect to the same partition \( P \).

(ii) If \( P \) is with perfect matching property in \( G \), then \( G_k^P \) is connected except for the case when \( k = 2 \) and \( n = 4 \), in which case \( G \) and \( G_k^P \) are isomorphic.

(iii) If \( P \) is with perfect matching property in \( G \), then by Theorem 1.5, degree sequences of graphs \( G_k^P \) and \( \Gamma \) are the same, where \( A(\Gamma) = A(G)A(G_k^P) \). In fact, complement of graph \( \Gamma \) is the disjoint union of complete bipartite graph, where each component is of the form \( (V_i \cup V_j) \) in which \( V_i \) and \( V_j \) are the partite sets in this component and are matching partite sets in \( G \).

### 3 Some more results on the graphs \( G, G_k^P \) and \( \Gamma \) satisfying \( A(G)A(G_k^P) = A(\Gamma) \)

In this section, we investigate the graph parameters like chromatic number, domination number etc., of \( G_k^P \) and \( \Gamma \) where, \( A(G)A(G_k^P) = A(\Gamma) \) and \( G \) is a 1-factor graph with perfect matching property.

We recall that, a set \( S \subseteq V(G) \) is a dominating set if every vertex not in \( S \) has a neighbour in \( S \). The domination number \( \gamma(G) \) is the minimum size of a dominating set in \( G \). A dominating set \( S \) in \( G \) is a connected dominating set if \( S \) is connected, an independent dominating set if \( S \) is independent.

A clique in a graph is a set of pairwise adjacent vertices. The clique number of \( G \), denoted by \( \omega(G) \), is the maximum size of a clique in \( G \).

The chromatic number of a graph \( G \), denoted by \( \chi(G) \), is the minimum number of colours needed to colour the vertices so that adjacent vertices receive different colours.

In all the discussions that follow, we consider a graph \( G \) and \( 2k = K \)-partition \( P = \{V_1, V_2, \ldots , V_k, V'_1, V'_2, \ldots , V'_k\} \) of \( V(G) \). Let \( V_i \) and \( V'_j \), \( 1 \leq i \leq k \) be the matching partite sets. The following remarks hold when \( A(G)A(G_k^P) = A(\Gamma) \).

**Remark 3.1.** By the definition of \( \Gamma \), for a vertex \( u \in V_i, 1 \leq i \leq k \), \( u \sim_{\Gamma} v \) for every vertex \( v \in V'_i \) and \( u \sim_{\Gamma} w \) for every vertex \( w \not\in V'_i \). Hence, for a vertex \( u \in V_i \) and a vertex \( v \) in \( V'_i \), \( 1 \leq i \leq k \), \( \{u, v\} \) is an independent dominating set of the graph \( \Gamma \). The set \( \{x, y\} \), where \( x \in V_i \) and \( y \in V'_j, 1 \leq i, j \leq k, i \neq j \), is a connected dominating set of \( \Gamma \). Since no single vertex can dominate all other vertices, both connected domination number and independent domination number are equal to two for the graph \( \Gamma \).

**Remark 3.2.** Similarly, for a vertex \( u \in V_i, u \sim_{G_k^P} v \) for every vertex \( v \in V_i \) and also \( u \sim_{G_k^P} w \) for that vertex \( w \) in \( V'_i \) for which \( u \sim_G w, 1 \leq i \leq k \). Hence the set \( \{u, v\} \) where, \( u \in V_i \) and \( v \in V'_j \) and \( u \sim_G v \) is an independent
Dominating set of $G^p_K$, $1 \leq i \leq k$. Also, the set $\{x, y\}$ forms a connected dominating set of $G^p_K$ in all the following cases.

(i) $x \in V_i$ and $y \in V_j$,
(ii) $x \in V_i$ and $y \in V'_i$,
(iii) $x \in V'_i$ and $y \in V'_j$, $1 \leq i, j \leq k$, $i \neq j$.

As no single vertex dominates all other vertices in $G^p_K$, both connected domination number and independent domination number of $G^p_K$ are equal to two.

**Remark 3.3.** By definition of $\Gamma$, the graphs $G_1 = \bigcup_{i=1}^{k} V_i$ and $G_2 = \bigcup_{i=1}^{k} V'_i$ are complete subgraphs of $\Gamma$. Since no vertex of $G_1$ is adjacent to all the vertices of $G_2$ and no vertex of $G_2$ is adjacent to all the vertices of $G_1$, both $G_1$ and $G_2$ are maximal complete subgraphs of $\Gamma$, with order $\frac{n}{2}$, where $n$ is the order of the graph $\Gamma$. Hence, $\omega(\Gamma) = \frac{n}{2}$.

Now we show that the complete subgraph of $\Gamma$ with maximum size is of order $\frac{n}{2}$. Let $H$ be a complete subgraph of $\Gamma$. For any vertex $x \in V(H)$, $x \in V_i$, $1 \leq i \leq k$, we have that every $y \in V_i$ is also in $V(H)$ and every $z \in V'_i$ is not in $V(H)$. Otherwise, by the structure of $\Gamma$, for $y \in V_i$ and not in $V(H)$, note that for every vertex $v$ with $v \sim \Gamma x$, we have $v \sim \Gamma y$ also. Therefore, the graph $(y \cup V(H))$ is a complete graph with cardinality more than that of $V(H)$, which contradicts the maximality. Similarly, for $z \in V'_i$, if $z \in V(H)$ then by the structure of $\Gamma$, we have that $z \sim \Gamma x$, which contradicts the completeness of $\Gamma$.

Also note that, for every $i$, either $V_i$ is a subset of $V(H)$ or else $V'_i$ is a subset of $V(H)$. Suppose that for an $i$, $1 \leq i \leq k$, $V'_i$ is not a subset of $V(H)$. If $V_i$ is also not a subset of $V(H)$, then for any $x \in V_i$, we have that $x \sim \Gamma y$ for every $y \in (V_i \cup V'_i)$ for $j \neq i$. Therefore, $(x \cup V(H))$ is complete, which contradicts the maximality of $V(H)$. Therefore $V_i$ must be a subset of $V(H)$. This proves that the cardinality of $V(H)$ is $\frac{n}{2}$.

**Remark 3.4.** In $G^p_K$, since every partite set is independent, no clique can have more than one element from same partite set. Hence, $\omega(G^p_K) \leq K$. Consider the case when every partite set has at least two vertices. Let $H$ be a maximum clique in $G^p_K$. Suppose $V(H)$ does not contain any vertex from some partite set $V_i$, $1 \leq i \leq k$. If $V(H)$ contains no vertex of $V'_i$ also, then by definition of $G^p_K$, for every vertex $x$ in $V'_i$, $x \sim G^p_K y$ for every $y$ in $H$. Hence $(x \cup V(H))$ is a clique with cardinality more than that of $H$, which is a contradiction to the maximality of $H$. If $V(H)$ contains a vertex $z$ of $V'_i$, then for any vertex $x \in V_i$, $x \sim G^p_K z$, by definition of $G^p_K$, $x \sim G^p_K y$ for every $y \in V(H)$. This implies that $(x \cup V(H))$ is complete, contradicting the maximality of $H$. Thus $V(H)$ contains exactly one vertex from every partite set and hence $|V(H)| = K$.

Let $V_i$ and $V'_i$ be two matching partite sets which are singletons. Then, maximum complete graph $H$ cannot contain both the vertices $u$ and $v$ where $u \in V_i$ and $v \in V'_i$, since $u \sim G^p_K v$. If $H$ neither contains $u$ nor contains $v$ then both $(u \cup V(H))$ and $(v \cup V(H))$ are complete graphs of order greater than that of $V(H)$, a contradiction to the maximality of $H$. Therefore, $V(H)$ contains exactly one of the two vertices $u$ and $v$. Let there be $k_1$ partite sets, which are singletons. Then any maximum complete graph $H$ can contain exactly one of the two vertices of $(V_i \cup V'_i)$. Thus, when the partition $P$ has $k_1$ singleton partite sets, $\omega(G^p_K) = K - \frac{k_1}{2}$.

**Remark 3.5.** By Theorem 2.1, every partite set $V_i$ and $V'_i$ are independent. Hence $\chi(G^p_K) \leq K$. Suppose that $|V_i| \geq 2$ for every $i$, $1 \leq i \leq k$. By definition of $G^p_K$, it follows that, in any proper colouring of vertices of $G^p_K$, two vertices $u$ and $v$ may receive the same colour if and only if either they are in the same partite set or one is in $V_i$ and the other is in $V'_i$ and are adjacent to each other in $G$, $1 \leq i \leq k$. Suppose that $u \in V_i$ and $v \in V'_i$ with $u \sim G v$ receive same colour say $c$, in a proper colouring of $G^p_K$. Then the colour $c$ cannot be given to any vertex in $V'_i$ other than $u$ (since they are all adjacent to $v$) and any vertex in $V'_i$, other than $v$ (since they are all adjacent to $u$). At least two colours (2 colours when $|V_i| = 2$ and 3 colours when $|V_i| \geq 3$) are needed to colour vertices in $V_i \cup V'_i$, $1 \leq i \leq k$. Thus, optimal colouring results when all the vertices of any partite set receive the same colour and no two vertices of two different partite sets receive the same colour. Thus $\chi(G^p_K) = K$. 

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Now consider the case that there are $k_1$ partite sets which are singletons. Let $u \in V_i$ and $v \in V_j'$ where $|V_i| = |V'_j| = 1$ for some $i, j \leq k$. Then $u$ and $v$ can share the same colour in any proper colouring of $G^K_P$. But, the colour shared by $u$ and $v$, cannot be given to any other vertex because every other vertex is adjacent to both $u$ and $v$ in $G^K_P$. A proper colouring of $G^K_P$ with minimum number of colours can be obtained as follows.

All the vertices of any partite set with cardinality greater than or equal to two receive the same colour so that no two vertices from two different partite sets with cardinality greater than or equal to two share the same colour. This requires $K - k_1$ colours. For the remaining $k_1$ partite sets which are singletons, by sharing the colour between the vertices in the matching partite sets, a proper minimal colouring of $G^K_P$ can be obtained with $(K - k_1) + \frac{k_1}{2} = K - \frac{k_1}{2}$ colours. This proves that $\chi(G^K_P) = K - \frac{k_1}{2}$, where $k_1$ is the number of singleton partite sets in the partition $P$.

**Remark 3.6.** From the Remark 3.3, we have $\omega(\Gamma) = \frac{n}{2}$. By sharing the colour between the vertices $u$ and $v$ such that $u \sim_G v$, we get a proper minimal colouring of $\Gamma$ with $\frac{n}{2}$ colours. Therefore $\chi(\Gamma) = \frac{n}{2}$.

**Remark 3.7.** For any $u \in V_i, 1 \leq i \leq k$, $deg_{G^K_P} u = deg_G u = n - n_i - 1$, where $n_i = |V_i| = |V'_j|$. When $|V_i| = |V'_j|$ for every $i$ and $j$, $1 \leq i, j \leq k$, both $G^K_P$ and $\Gamma$ are regular with regularity $(K - 1)\frac{n}{K} - 1$. If $K = n$ then $G^K_P = G^n_P = \overline{G}$, is regular with regularity $(n - 2)$ and has maximum number of edges, $\frac{n(n - 2)}{2}$. If $K = 2$, then $n_1 = |V_1| = |V'_1| = \frac{n}{2}$ and both $G^K_P$ and $\Gamma$ are regular with regularity $\frac{n - 2}{2}$. In this case, they have minimum number of edges, given by $\frac{n(n - 2)}{4}$.

Using the clique size of the graphs $G^K_P$ and $\Gamma$, we characterize the partition $P$ such that $G^K_P$ is isomorphic to $\Gamma$.

**Theorem 3.8.** Let $G$ be a graph on $n$ vertices and let $P = \{V_1, V_2, \ldots, V_k, V'_1, V'_2, \ldots, V'_k\}$ be a $2k$-partition of $V(G)$ satisfying perfect matching property in $G$. Let $V_i$ and $V'_j$ be the matching partite sets, $1 \leq i \leq k$, and let $A(G)A(G^K_P) = A(\Gamma)$ where $K = 2k$. Then the graphs $G^K_P$ and $\Gamma$ are isomorphic to each other if and only if the partition $P$ satisfies the property that $|V_i| = |V'_j| \leq 2$ for every $i, j \leq k$.

**Proof.** Let $|V_i| = |V'_j| \leq 2$ for every $i, 1 \leq i \leq k$. Define a mapping $\sigma$ from $V(G^K_P)$ to $V(\Gamma)$ as follows. For every vertex $u$ in $V_i$ or $V'_j$ with $|V_i| = |V'_j| = 1$, $\sigma(u) = u$. And, for every $u_1, u_2 \in V_i$ and $u'_1, u'_2 \in V'_j$ with $|V_i| = |V'_j| = 2$, and $u_1 \sim_G u'_1, u_2 \sim_G u'_2$, $\sigma(u_1) = u_1, \sigma(u_2) = u'_2$, $\sigma(u'_1) = u'_1$ and $\sigma(u'_2) = u_2$, i.e., $\sigma$ is given by

$$\sigma = \begin{pmatrix}
  u_1 & u_2 & u'_1 & u'_2 \\
  u_1 & u'_2 & u'_1 & u_2
\end{pmatrix}.$$

The mapping $\sigma$ is a permutation from $V(G^K_P)$ to $V(\Gamma)$ which fixes $u_1, u'_1$ and swaps $u_2$ and $u'_2$. To prove that $\sigma$ is an isomorphism, we show the following.

For every $u \in V(G^K_P)$, $\sigma(u) \in V(\Gamma)$, an element $v \in N_{G^K_P}(u)$ if and only if $\sigma(v) \in N_{\Gamma}(\sigma(u))$.  

Observe that, when $|V_i| = |V'_j| = 1$ for every $i, 1 \leq i \leq k$, both $G^K_P$ and $\Gamma$ are same as $\overline{G}$, the complement of $G$. Hence they are same.

Now consider the case that the partition $P$ has partite sets of cardinality two also. We show that $\sigma$ is an isomorphism from $G^K_P$ to $\Gamma$ using the property in 1.

Consider $V_i$ and $V'_j$ with $u_1, u_2 \in V_i$ and $u'_1, u'_2 \in V'_j$ with $u_1 \sim_G u'_1$ and $u_2 \sim_G u'_2$ (Refer Figure 3).

We consider the following two cases...

Case (i): Consider $u_1 \in V_i$ such that $\sigma(u_1) = u_1$. Then, by definition of $G^K_P$,

$$N_{G^K_P}(u_1) = u'_2 \cup \{x \mid x \notin V_i \cup V'_j\},$$
Figure 3: Adjacencies of $u_1$, $u_2$, $u_i'$ and $u_j'$ in the graphs $G$, $G^p_k$ and $\Gamma$

and

$$N_\Gamma(\sigma(u_1)) = N_\Gamma(u_1) = \{u_2\} \cup \{y \neq V_i \cup V_i'\}$$

$$= \{\sigma(u_2')\} \cup \{y \neq V_i \cup V_i'\}.$$

Case (ii): Consider $u_2 \in V_i$ with $\sigma(u_2) = u_2'$. Now,

$$N_{G^p_k}(u_2) = u_2' \cup \{x \neq V_i \cup V_i'\},$$

and

$$N_\Gamma(\sigma(u_2)) = N_\Gamma(u_2') = \{u_1'\} \cup \{y \neq V_i \cup V_i'\}$$

$$= \{\sigma(u_1')\} \cup \{y \neq V_i \cup V_i'\}.$$

Thus, we observe that Equation 1 is satisfied which implies that $\sigma$ is an isomorphism from $G^p_k$ to $\Gamma$.

Conversely, let $G^p_k \cong \Gamma$. Then $\omega(G^p_k) = \omega(\Gamma)$

$$\Rightarrow K - \frac{k_1}{2} = \frac{n}{2},$$

where $k_1$ is the number of singleton sets in $P$ i.e., $2K = n + k_1$.

If every partite sets with cardinality greater than one contains exactly two elements, then there are $\frac{n-k_1}{2}$ partite sets which are not singletons. Now suppose that there is a partite set with three or more vertices in it, then $K$ satisfies

$$K < \frac{n-k_1}{2} + k_1 = \frac{n + k_1}{2}.$$

$$\Rightarrow 2K < n + k_1,$$

which is a contradiction, since $2K = n + k_1$. Therefore there is no partite set $V_i$ with $|V_i| \geq 3$. \qed

4 Rank of the graphs $G$, $G^p_k$ and $\Gamma$ when $A(G)A(G^p_k) = A(\Gamma)$

Throughout this section we consider the graph $G$ and $G^p_k$ are such that $A(G)A(G^p_k)$ is realizable and let $\Gamma$ be the realization of the matrix product. Since the rank of null graph and complete $k$-partite graph are trivial, we consider the nontrivial case in which the partition $P$ has the perfect matching property in $G$. Also note that the rank of 1-factor graph is same as the number of vertices in the graph. Therefore, we restrict ourselves to find rank of $G^p_k$, which is same as that of rank of $\Gamma$.

Given a 1-factor graph $G$ with $2n$ vertices, we propose a partition $P$ for which $G^p_k$ is a graph of rank $r$ and $A(G)A(G^p_k)$ is graphical, where $n \leq r \leq 2n$.

In the following theorem we obtain relation between rank of $G^p_k$, the number of vertices and the number of partite sets with cardinality one.

**Theorem 4.1.** Let $G$ be a graph and let $P$ be a partition of size $2k = K$ of $V(G)$ satisfying perfect matching property in $G$. Suppose that $A(G)A(G^p_k) = A(\Gamma)$. Then the rank of $G^p_k$ is $n - l$ where $n$ is the number of vertices in $G$ and $2l$ is the number of partite sets of cardinality one in the partition $P$.

**Proof.** Let $P = \{V_1, V_2, \ldots, V_m, V_{m+1}, \ldots, V_k, V'_1, V'_2, \ldots, V'_m, V'_{m+1}, \ldots, V'_k\}$ be a partition of $V(G)$ with $|V_i| = |V'_i| = n_i$, satisfying the following properties.

(i) $n_i \geq 2$ for $1 \leq i \leq m,$
(ii) \( n_i = 1 \) for \( m + 1 \leq i \leq k \), with \( k - m = l \).

Further, let \( V_i = \{ v_{i1}, v_{i2}, \ldots, v_{in} \} \) and \( V'_i = \{ v'_{i1}, v'_{i2}, \ldots, v'_{in} \} \) be the matching partite set with \( v_{ij} \) adjacent with \( v'_{ij} \) in \( G \) for \( 1 \leq i \leq k \), \( 1 \leq j \leq n_i \).

We prove that the rank of \( G'_k \) is \( n - l \) by showing that the dimension of the right null space of \( A(G'_k) \) is equal to \( l \).

We index the rows of \( A(G'_k) \) as follows. The first \( n_1 \) rows correspond to the \( n_1 \) vertices in \( V_1 \), and the next \( n_2 \) rows correspond to \( n_1 \) vertices in \( V'_1 \). Further, the next \( n_2 \) rows correspond to vertices in \( V_2 \) and so on. The last row corresponds to the vertex in \( V'_k \).

Then \( A(G'_k) \) can be viewed as follows:

\[
A(G'_k) = \begin{pmatrix}
0_{n_1 \times n_1}, (J - I)_{n_2 \times n_1}, J_{n_1 \times n_2}, \ldots, J_{n_1 \times n_1} \\
(J - I)_{n_2 \times n_1}, 0_{n_2 \times n_1}, J_{n_2 \times n_2}, \ldots, J_{n_2 \times n_1} \\
J_{n_2 \times n_1}, J_{n_2 \times n_1}, \ldots, J_{n_2 \times n_1} \\
\vdots & \vdots & & \ddots & \vdots \\
J_{n_m \times n_1}, J_{n_m \times n_1}, \ldots, 0_{n_m \times n_m}, (J - I)_{n_m \times n_m}, 0_{n_m \times n_m}, \ldots, J_{n_m \times n_1} \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
J_{1 \times n_1}, J_{1 \times n_1}, \ldots, 0_{1 \times 1}
\end{pmatrix}
\]

Observe that the row in \( A(G'_k) \) corresponding to the vertex \( v_{ij} \) of \( V_i \) has zero entries at column positions corresponding to vertices in \( V_i \) and at one more column position corresponding to the vertex \( v'_{ij} \). All other entries in the row except these \( (n_i + 1) \) entries are equal to one.

Let \( X \) be a vector of order \((n \times 1)\) given by

\[
X = (x_{11}, x_{12}, \ldots, x_{1n_1}, x'_{11}, x'_{12}, \ldots, x'_{1n_1}, x_{21}, x_{22}, \ldots, x_{2n_2}, x'_{21}, x'_{22}, \ldots, x'_{2n_2}, \ldots)
\]

which is in the right null space of \( A(G'_k) \). Then \( A(G'_k)X = 0 \).

For our convenience, we say that component \( x_{ij} \) of the vector \( X \) corresponds to the vertex \( v_{ij} \), \( 1 \leq i \leq k \), \( 1 \leq j \leq n_i \).

Consider two vertices \( v_{ij} \) and \( v_{ir} \) in the same partite set \( V_i \), \( 1 \leq i \leq m \), \( 1 \leq j, r \leq n_i \), \( j \neq r \). Then \( v'_{ij} \) and \( v'_{ir} \) are their corresponding vertices in \( V'_i \), such that \( v_{ij} \sim G v_{ij} \) and \( v_{ir} \sim G v_{ir} \). The rows of \( A(G'_k) \) corresponding to the vertices \( v_{ij} \) and \( v_{ir} \) differ only at the two columns corresponding to the vertices \( v'_{ij} \) and \( v'_{ir} \).

Taking difference of the two equations in \( A(G'_k)X = 0 \) corresponding to the above two rows, we get,

\[
x'_{ij} - x'_{ir} = (J - I)_{n_m \times n_m} x_{ij} - (J - I)_{n_m \times n_m} x_{ir} = \mathbf{0}_{n_m \times 1}.
\]

Observe that \( a_i \) and \( a'_i \) repeat \( n_i \) times for every \( i \), \( 1 \leq i \leq m \).

Now consider the vertices, \( v_{ij} \) from \( V_i \) and \( v_{ir} \) from \( V'_i \), \( 1 \leq i \leq m \), \( 1 \leq j, r \leq n_i \). In \( G'_k \), \( v_{ij} \) and \( v_{ir} \) are commonly adjacent with every vertex outside \( V_i \cup V'_i \) and \( v_{ij} \) is adjacent with every vertex in \( V'_i \), except \( v'_{ij} \) and \( v'_{ir} \) is adjacent with every vertex in \( V_i \), except \( v_{ij} \). Thus the rows corresponding to \( v_{ij} \) and \( v_{ir} \) differ at \((n_i - 1)\) column positions. Taking the difference of the corresponding equations in \( A(G'_k)X = 0 \), we get,

\[
(2n_i - 1) a_i = (n_i - 1) a'_i.
\]

The above implies that \( a_i = a'_i \), provided \( n_i \neq 1 \).

Then \( X \) is as follows,

\[
X = (a_1, a_1, \ldots, a_1, a_2, a_2, \ldots, a_2, a_3, a_3, \ldots, a_k)
\]

where each \( a_i \) repeats \( 2n_i \) times, \( 1 \leq i \leq m \).

Suppose that, for every \( i \), \( 1 \leq i \leq k \), \( n_i \neq 1 \). Then all the entries in \( X \) corresponding to vertices of \( V_i \) and \( V'_i \) are same for each \( i \), \( 1 \leq i \leq k \). Then \( X \) becomes,

\[
X = (a_1, a_1, \ldots, a_1, a_2, a_2, \ldots, a_2, a_3, a_3, \ldots, a_k)
\]

where each \( a_i \) repeats \( 2n_i \) times, \( 1 \leq i \leq k \).

Consider two vertices, \( v_{ij} \in V_i \) and \( v_{ij} \in V'_j \), where \( 1 \leq i, j \leq k \), \( i \neq j \), \( 1 \leq l \leq n_j \), \( 1 \leq r \leq n_i \). The rows of \( A(G'_k) \) corresponding to the above two vertices differ at \((n_i + 1) + (n_j + 1)\) column positions. Taking the difference of the equations in \( A(G'_k)X = 0 \) corresponding to the above two rows, we get,
(n_l + 1)x_{ij} = (n_j + 1)x_{ji}, \text{i.e.,}
\alpha_i = \frac{n_j + 1}{n_i + 1} \alpha_j, \quad 1 \leq i, j \leq k.

Since the above is true for every \( i \) and \( j \), \( 1 \leq i, j \leq k, i \neq j \), every \( \alpha_i \) can be written as a scalar multiple of \( \alpha \) where \( \alpha_1 = \alpha \). Then \( X \) becomes,
\[ X = (a, a, \ldots, a, r_2a, r_2a, \ldots, r_2a, \ldots, r_ka, r_ka, \ldots, r_ka) \text{ where } r_i = \frac{n_t + 1}{n_i + 1}, \quad 2 \leq t \leq k. \]

Now, \( A(G^p_k)X = 0 \) implies \( \alpha = 0 \). Hence \( X \) is a zero vector and the right null space is a zero space with dimension zero. This implies that rank of \( G^p_k \) is \( n \).

Now suppose that there are \( 2l \) partite sets which are singletons, in addition to \( 2m \) partite sets with cardinality greater than or equal to two. Then \( X \) is of the form
\[ X = (a, a, \ldots, a, r_2a, r_2a, \ldots, r_2a, \ldots, r_m a, r_m a, \ldots, r_m a, a_{m+1}, \ldots, a_k, a'_l) \]
where \( r_i = \frac{n_t + 1}{n_i + 1}, \quad 2 \leq t \leq m. \)

To resolve the above case, we consider two vertices, \( v_{1r} \in V_i \) and \( v_{j1} \in V_j \) where \( m + 1 \leq j \leq k, 1 \leq r \leq n_1. \)

The rows of \( A(G^p_k) \) corresponding to these vertices differ at \((n_1 + 1) + (1 + 1)\) column positions. Taking the difference of the equations in \( A(G^p_k)X = 0 \) corresponding to the above two rows, we get,
\[ (n_1 + 1)x_{1r} = x_{j1} + x'_{j1} \]
which implies that \( \alpha_1 + \alpha'_l = (n_1 + 1) \alpha \) for \( m + 1 \leq j \leq k. \)

Observe that the equations in \( A(G^p_K)X = 0 \) corresponding to the vertices \( v_{j1} \) and \( v'_{j1} \) are same for \( m + 1 \leq j \leq k. \)

Now consider two vertices \( v_{1r} \) and \( v_{j1} \) in \( V_i \) and \( V_j \) respectively, with \( m + 1 \leq i, j \leq k, i \neq j. \)

Taking the difference of the equations corresponding to the above two rows, we get,
\[ \alpha_i + \alpha'_j = \alpha_1 + \alpha'_l \]
which is true for arbitrary \( i \) and \( j. \)

Finally, consider the \( l \) equations corresponding to the \( l \) vertices \( v_{j1}, m + 1 \leq j \leq k. \) Since \( \alpha_i + \alpha'_j \) is a scalar multiple of \( \alpha \) for every \( j \), substituting in any of the above \( l \) equations we get \( \alpha = 0. \) But, \( \alpha_i + \alpha'_j = (n_1 + 1) \alpha, \)
\[ \Rightarrow \alpha_i + \alpha'_j = 0 \] for every \( j, m + 1 \leq j \leq k, \)
\[ \Rightarrow \alpha_j = -\alpha'_j. \]

Thus, there are \( l \) linearly independent vectors, say \( \beta_{m+1}, \beta_{m+2}, \ldots, \beta_k \), in the right null space of \( A(G^p_k) \), which are given as follows.

The vertex \( \beta_{m+1} \) has all the components equal to zero except at the positions corresponding to the vertices \( v_{m+1} \) and \( v'_{m+1}, 1 \leq i \leq (k - m), \) and the entries at these positions are \( c \) and \( -c \) where \( c \) is any non zero constant.

Since these vectors form a basis for the right null space of \( A(G^p_k) \), rank of \( G^p_k \) is \( n - l. \)

Since \( G \) is a graph with partition \( P \) having perfect matching property, \( A(G) \) is a permutation matrix. Therefore, \( A(G)A(G^p_k) = A(\Gamma), \) rank of \( \Gamma \) is same as rank of \( G^p_k. \) Hence, the following theorem follows.

**Theorem 4.2.** Let \( G \) be a graph and let \( P \) be a partition of size \( 2k = K \) of \( V(G) \) satisfying perfect matching property in \( G. \) Let \( A(G)A(G^p_k) = A(\Gamma), \) Then the rank of \( \Gamma \) is \( n - l \) where \( n \) is the number of vertices in \( G \) and \( 2l \) is the number of partite sets of cardinality one in the partition \( P. \)

**Remark 4.3.** Given a 1-factor graph, using the Theorems 4.1 and 4.2, we propose a partition \( P \) for which \( G^p_k \) is a graph of rank \( r \) and \( A(G)A(G^p_k) \) is graphical, where \( n \leq r \leq 2n. \)

**Example 4.4.** In the following example, we illustrate a procedure to construct a nontrivial graph \( H \) with the given rank \( r, \ n \leq r \leq 2n. \) We consider a properly chosen 1-factor graph \( G \) of suitable order \( n \) and use the Theorems 4.1 and 4.2, to partition the vertex set of \( G \) so that \( n - l = r \) where, \( 2l \) is the number of singleton sets in the partition and \( H = G^p_k \) is a graph of rank \( r. \)

Let \( r = 7. \) Consider the 1-factor graph \( G \) with 10 vertices and a partition \( P \) of size 8 as shown in the Figure 4.

The adjacent graph shown in the Figure 4, is \( G^p_8 \) with respect to the same partition \( P. \) The graph \( G^p_8 \) is of rank \( n - l = 10 - 3 = 7. \) In this case \( G^p_8 \cong \Gamma_1, \) where \( A(\Gamma_1) = A(G)A(G^p_k). \)

For the 1-factor graph \( H_1 \) with 8 vertices and a partition \( P \) of size 4 as shown in the Figure 5, the graph \( (H_1)^p_k \) is of rank \( n - l = 8 - 1 = 7. \) In this case, \((H_1)^p_k \cong \Gamma_2, \) where \( A(\Gamma_2) = A(H_1)A((H_1)^p_k). \) Therefore \( \Gamma_2 \) is also of rank 7.
For the 1-factor graph $H_2$ with 14 vertices and a partition $P$ of size 14 as shown in the Figure 5, and the graph $(H_2)_P^G = \overline{G}$ is of rank $n - l = 14 - 7 = 7$. In this case, $(H_2)_P^G \cong \Gamma_3$, where $A(G(T)) = A(H_2)A((H_2)_P^G)$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure4.pdf}
\caption{Graphs $G$ and $G_P^G$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure5.pdf}
\caption{Graph $H_1(n = 8, k = 4)$ and graph $H_2(n = 14, k = 14)$}
\end{figure}

5 A Commuting decomposition of $K_{n_1, n_2, \ldots, n_k}$ into a 1-factor subgraph and its $k$-complement

In this section we obtain a commuting decomposition of a complete $k$-partite graph $G$ into a perfect matching and its $k$-complement with respect to the $k$-partition of $G$.

**Note 5.1.** From the result of Theorem 1 of [6], one can conclude that the $(i, j)$ entry of $A(G)A(H)$ represents number of $GH$ paths from $v_i$ to $v_j$. Hence $G$ and $H$ commute with each other if and only if for every two vertices $v_i$ and $v_j$, $i \neq j$, $1 \leq i, j \leq n$, the number of $GH$ paths from $v_i$ to $v_j$ is same as number of $HG$ paths from $v_i$ to $v_j$.

The Theorem 5.2 explains the commuting decomposition of $K_{n_1, n_2, \ldots, n_k}$ into a perfect matching and its $k$-complement. We consider the partition $P$ to be the $k$-partition of the complete $k$-partite graph $K_{n_1, n_2, \ldots, n_k}$.

**Theorem 5.2.** Let $G$ be a complete $k$-partite graph $K_{n_1, n_2, \ldots, n_k}$ on $n$ vertices with the $k$-partition of $V(G)$ given by $P = \{V_1, V_2, \ldots, V_k\}$ and let $|V_i| = n_i$, $1 \leq i \leq k$. Then $G$ is decomposable into two commuting subgraphs, one of which is perfect matching of $G$, say $H_1$, and the other one is the $k$-complement of $H_1$ with respect to the
same partition \( P = \{V_1, V_2, \ldots, V_k\} \), if and only if the number of partite sets in \( P \) with same cardinality \( n_i \) is even for every \( i, 1 \leq i \leq k \).

**Proof.** Consider the complete \( k \)-partite graph \( G = K_{n_1, n_2, \ldots, n_k} \). Suppose that \( G \) has a commuting decomposition into two subgraphs \( H_1 \) and \( H_2 \), one of which, say \( H_1 \), is a perfect matching of \( G \) and the other one is given by \( H_2 = (H_1)^P \), the \( k \)-complement of \( H_1 \) with respect to the partition \( P = \{V_1, V_2, \ldots, V_k\} \).

Since \( G \) has a perfect matching \( H_1 \), the number of vertices in \( G \) is even. If all the partite sets are singletons, then \( k = n \) is also even and assertion of the theorem is obviously true.

Suppose that there is a partite set, say \( V_i \), \( 1 \leq i \leq k \), which has two or more vertices. Let \( u \in V_i \) and \( v \in V_j \) with \( u \sim_{H_1} v \), \( 1 \leq i, j \leq k \), \( i \neq j \). Let \( w \in V_l \) and let \( w \sim_{H_1} x \). If \( x \in V_i \), \( t \neq j \), then there is a \( H_1 \)-2-path from \( w \) to \( v \) but no \( H_2 \) path from \( w \) to \( v \) which by Note 5.1, contradicts the assumption that \( H_1 \) and \( H_2 \) commute with each other. Hence if \( u \in V_i \) and \( v \in V_j \), \( 1 \leq i, j \leq k \), \( i \neq j \), then for each vertex in \( V_i \) the only adjacent vertex of \( H_1 \) is also in \( V_j \). Same is true for \( V_j \) which implies that \(|V_i| = |V_j| \). Similarly, partite sets with same cardinality are paired and since \( H_1 \) is a perfect matching, the number of partite sets with cardinality \( n_i \) is even for every \( i, 1 \leq i \leq k \).

Conversely, assume that the number of partite sets with same cardinality is even. Then we can pair the partite sets having the same cardinality. Without loss of generality, assume that \(|V_1| = |V_2| = n_1, |V_3| = |V_4| = n_2, \ldots, |V_{2r-1}| = |V_{2r}| = n_r \), where \( 2r = k \). Observe that \( n_i \) may be equal to \( n_j \) for some \( i \) and \( j \), \( 1 \leq i, j \leq r \), \( i \neq j \). Since for every \( i \), \( 1 \leq i \leq r \), \( (V_{2i-1} \cup V_{2i}) \) is a complete bipartite graph with \(|V_{2i-1}| = |V_{2i}| \), there is a perfect matching in the induced graph. Union of all these perfect matchings gives a perfect matching in \( G \), denoted by \( H_1 \). In the graph \( G \setminus H_1 \), obtained by deleting the edges of \( H_1 \) from \( G \), two vertices are adjacent if and only if they are in two different partite sets and they are not adjacent in \( H_1 \). Hence \( G \setminus H_1 \) is same as \((H_1)^P \) which implies that \( G = H_1 \cup (H_1)^P \) and \( E(H_1) \cap E((H_1)^P) = \emptyset \). Thus \( H_1 \) and \( H_2 = (H_1)^P \) form a decomposition of \( G \). To show that \( H_1 \) and \( (H_1)^P \) form a commuting decomposition of \( G \), we show that for any two vertices \( u \) and \( v \) in \( G \), the number of \( H_1 \)-2-path from \( u \) to \( v \) is same as the number of \( H_2 \)-2-path from \( u \) to \( v \).

Consider \( u, v \) in the same partite set \( V_i \). Let \( u \sim_{H_1} v \) and \( v \sim_{H_1} v' \). Then, \( u \sim_{H_1} u' \sim_{H_1} v \) is the only \( H_1 \)-2-path and \( u \sim_{H_1} v' \sim_{H_1} v \) is the only \( H_2 \)-2-path from \( u \) to \( v \).

Consider \( u \in V_i \) and let \( u' \in V_j \) with \( u \sim_{H_1} u' \). For any other vertex \( x \) in \( V_j \), there neither exists a \( H_2 \)-2-path nor exists a \( H_2 \)-2-path from \( u' \) to \( x \). Consider a vertex \( x \in V_i, i \neq j \), with \( x \sim_{H_1} x' \). Then \( u \sim_{H_1} u' \sim_{H_1} x \) is the only \( H_1 \)-2-path and \( u \sim_{H_1} x' \sim_{H_1} x \) is the only \( H_2 \)-2-path from \( u \) to \( x \). Hence the decomposition \( \{H_1, H_2 = (H_1)^P \} \) of \( G \) is a commuting decomposition.

**Note 5.3.** Let \( G \) be a complete \( k \)-partite graph \( K_{n_1, n_2, \ldots, n_k} \) with respect to a partition \( P = \{V_1, V_2, \ldots, V_k\} \) of \( V(G) \). Then the graph \( G \) is decomposable into two commuting perfect matchings i.e., both \( H_1 \) and \( H_2 = (H_1)^P \) are perfect matchings only when \( n = 4 \) and \( k = 2 \). Equivalently, \( K_{2,2} \) is the only complete \( k \)-partite graph, with \( k = 2 \) which is decomposable into two commuting perfect matchings as above.

**Note 5.4.** Let \( G \) be a complete \( k \)-partite graph \( K_{n_1, n_2, \ldots, n_k} \) with respect to a partition \( P = \{V_1, V_2, \ldots, V_k\} \) of \( V(G) \). Then the graph \( G \) may be decomposed into two commuting subgraphs, one of them is a perfect matching and the other one is a Hamiltonian cycle i.e., \( H_1 \) is a perfect matching and \( H_2 = (H_1)^P \) is a Hamiltonian cycle when either \( n = 6 \) and \( k = 2 \) or \( n = 4 \) and \( k = 4 \). Equivalently, \( K_{1,1,1,1} \) and \( K_{1,1,1,1} \) are the only complete \( k \)-partite graphs, with \( k = 2 \) and \( k = 4 \) respectively, which are decomposable into two commuting subgraphs, one of them is a perfect matching and the other one is a Hamiltonian cycle.

**Conclusion:** This paper contains characterization of a graph \( G \) and the \( k \)-partition \( P \) of \( V(G) \) satisfying the property that \( A(G)A(G_k) \) is graphical. It also contains characterization of the partition \( P \) such that \( G_k \cong \Gamma \), when \( A(G)A(G_k) = A(\Gamma) \). Given a \( 1 \)-factor graph \( G \) with \( 2n \) vertices, we propose a partition \( P \) for which \( G_k \) is a graph of rank \( r \) and \( A(G)A(G_k) \) is graphical, where \( n < r < 2n \). Finally, a characterization of the existence of commuting decomposition of the complete \( k \)-partite graph \( K_{n_1, n_2, \ldots, n_k} \) into a perfect matching and its \( k \)-complement is obtained.
References


