Research Article

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Uniform entropy vs topological entropy

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Abstract: We discuss the connection between the topological entropy and the uniform entropy and answer several open questions from [10, 15]. We also correct several erroneous statements given in [10, 18] without proof.

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The topological entropy and the uniform entropy have been largely studied in the last fifty years. The topological entropy $h(f)$ of a continuous selfmap $f : X \to X$ of a compact space $X$ was defined by Adler, Konheim and McAndrew [1]. In order to extend the notion of topological entropy for non-compact spaces, Bowen [4] defined uniform entropy $h_U(f)$ of a uniformly continuous selfmap $f : X \to X$ of a metric space $X$, that coincides with the topological entropy when the metric space $X$ is compact. This approach was later extended by Hood [19] and Hofer [18] for uniform spaces. A handicap of the uniform entropy $h_U$ is the fact that its definition heavily relies on the compact subspaces of the space, so that in uniform spaces with few compact sets (e.g., only finite ones) the uniform entropy of a map is zero regardless of how complicated the map is. Motivated by this fact, Kimura [20] proposed a modified version $h_{UK}$ of the uniform entropy replacing the compact sets by totally bounded ones.

Topological entropy functions for non-compact topological spaces were discussed by Hofer [18], who outlined the fact that the entropy $h$ has a quite natural extension to continuous selfmaps of arbitrary topological spaces, that here will be denoted by $h_{\text{fin}}$, by simply replacing the open covers used in the definition of $h$ by finite open covers. For continuous selfmaps of compact spaces obviously $h_{\text{fin}} = h$, as every open cover of a compact space has a finite open subcover.

The above mentioned paper of Hofer proposes also an alternative way to define entropy of continuous selfmaps in the full subcategory $\text{Tych}$ of the category $\text{Top}$ having as objects all Tychonov spaces. Namely, he made use of the functor $\text{Tych} \to \text{Comp}$ of the Stone-\v{C}ech compactification $X \mapsto \beta X$. Then he defined the entropy $h_{\beta}(f)$ for a selfmap $f : X \to X$ in $\text{Tych}$ as $h(\beta f)$, where $\beta f$ is the continuous extension of $f$ to $\beta X$. Clearly, also the entropy $h_{\beta}$ can be defined for arbitrary topological spaces $X$, noting that in general $X$ need not be a subspace of $\beta X$. Shortly speaking, the entropy $h_{\beta}$ is obtained from $h$ by pulling back along the functor $\beta : \text{Top} \to \text{Tych}$.

Recently, a new topological entropy function $h_{T}$ for Tychonov spaces was proposed in [15], making use of a similar idea. Namely, by using the fine uniformity of a Tychonov space. Since the fine uniformity of a Tychonov space $X$ is a functorial uniformity, i.e., the assignment $X \mapsto \mathcal{F}(X)$, where $\mathcal{F}(X)$ denotes the uniform space with underlying set $X$ and the fine uniformity of $X$, defines a functor $\mathcal{F} : \text{Tych} \to \text{Unif}$. This motivated the conjecture, formulated explicitly in [10], that the further use of functorial uniformities may give rise to new topological entropy functions for Tychonov spaces.
In this paper we discuss in detail the various notions of topological entropy as well as the notion of uniform entropy trying to establish the precise connection between the topological entropy and the uniform entropy. Answering a question from [15] we establish the precise relation between the three entropies $h_{fin}$, $h_\beta$ and $h_\gamma$. To do this we show first that all functorial uniformities on Tych give the same uniform entropy. Among others, we correct also several erroneous statements given in [10, 18] without proof.

In a forthcoming paper [6] we explore various possibilities to define entropy functions in the category $QUnif$ of quasi uniform spaces and the use of functorial quasi uniformities (i.e., sections of the forgetful functor $T : QUnif \to \text{Top}$) in order to define topological entropy functions in not necessarily Tychonov spaces.

1 The topological entropy

1.1 Topological entropy in compact spaces

We now recall the definition of topological entropy given in [1]. Let $X$ be a compact topological space and let $\text{cov}(X) = \{U : U$ is an open cover of $X\}$. We allow open covers to have empty members. Given $U, V \in \text{cov}(X)$, the join of $U$ and $V$ is the open cover $U \vee V = \{U \cap V : U \in U, V \in V\}$.

For $U \in \mathcal{O}(X)$, let $N(U)$ denote the number of elements of a finite subcover of $U$ with minimal cardinality.

If $\phi : X \to X$ is a continuous selfmap and $U$ is a cover of $X$, then $\phi^{-i}(U)$, for $i \in \mathbb{N}$, will stand for the open cover $(\phi^{-i}(U) : U \in U)$. Clearly,

$$C_n(U, \phi) = U \vee \phi^{-1}(U) \vee \phi^{-2}(U) \vee \cdots \vee \phi^{-n+1}(U)$$

is a cover of $X$. The topological entropy of $\phi$ with respect to the cover $U$ is defined as

$$h(\phi, U) = \lim_{n \to \infty} \frac{\log N(C_n(U, \phi))}{n}.$$ 

The topological entropy of $\phi$ is

$$h(\phi) = \sup \{ h(\phi, U) : U \in \text{cov}(X) \}.$$ 

(1)

1.2 The topological entropies $h_{fin}$ and $h_\beta$ in non-compact spaces

In the sequel $h_{fin}$ will denote the topological entropy function for continuous selfmaps of spaces defined as above by making recourse only to finite open covers. In the case of compact spaces, this obviously coincides with the classical definition of topological entropy $h$ given by (1).

The advantage of the entropy function $h_{fin}$ is that its definition does not require any specific separation property for the underlying space.

For a topological space $X$ the complete lattice $\mathcal{O}(X)$ of all open sets satisfies the distributive law

$$\left( \bigvee_{i \in I} U_i \right) \wedge V = \bigvee_{i \in I} (U_i \wedge V),$$

where the join $\bigvee$ is simply the union (of arbitrary families of open sets) and the meet $\wedge$ is the intersection of (two) open sets, i.e., $(\mathcal{O}(X), \bigvee, \wedge)$ is a frame. The top element of $\mathcal{O}(X)$ is $X$, the bottom element is $\emptyset$. Every continuous selfmap $f : X \to X$ gives rise to a frame-endomorphism $\mathcal{O}(f) : \mathcal{O}(X) \to \mathcal{O}(X)$ defined by $\mathcal{O}(f)(U) = \bigvee_{\phi \in \mathcal{O}(X)} f(U)$.
f^{-1}(U) for U ∈ O(X). Every (finite) open cover U of X gives rise to a (finite) cover of O(X) and O(f) : O(X) → O(X) takes (finite) covers of O(X) to (finite) covers of O(X). At this point we can notice that the topological entropy, using in its definition only open covers of the space X, is practically entirely defined in terms of O(X). Hence, if two spaces X and Y have isomorphic frames ξ : O(X) → O(Y), and if f : X → X and g : Y → Y are continuous selfmaps, such that O(f) and O(g) are conjugated via the isomorphism ξ, then the maps f and g have equal entropies. Applying this argument to the T₀ reflection we obtain:

**Theorem 1.1.** Let rX be the T₀-reflection of a space X. Then for every continuous selfmap f : X → X the reflection \( f^* : rX \rightarrow rX \) in Top₀ has the same entropy as f.

This result allows us to restrain the study of topological entropy within the category Top₀. The counterpart of this theorem for the category Top₁ fails [6]. On the other hand, examples given in [6], show that every (compact) Hausdorff space X can be densely embedded into a one-point T₀ extension \( a₀X \), so that every continuous selfmap \( f : X \rightarrow X \) extends to a continuous selfmap \( a₀f : a₀X \rightarrow a₀X \) with \( h_{a₀}(a₀f) = 0 \). In other words, allowing T₀ extensions, one may loose monotonicity of the entropy \( h_{a₀} \) (with respect to taking restrictions to invariant subspaces) in a spectacular way.

Hofer [18], who defined the entropy \( h_{a₀} \) with respect to the family of all finite open covers, defined also another entropy \( h_β \) for continuous selfmaps of Tychonov spaces, using the extension \( βf \) of the selfmap \( f \) to the Stone-Čech compactifications and letting \( h_β(f) = h(βf) \).

As noted independently in [16, Corollary 2] and [15, Theorem 2.20], for general Tychonov spaces one has \( h_β(f) \leq h_{a₀}(f) \). Equality holds for selfmaps of normal spaces [18].

The problem posed in [18] on whether \( h_β = h_{a₀} \) in all Tychonov spaces remained open for about 25 years. Fedeli [16] showed that these two entropies do not coincide for non-normal Tychonov spaces. For the sake of completeness we recall below his example since it can be used also to disprove monotonicity of \( h_β \) with respect to taking restrictions to closed invariant subspaces (see Example 1.5).

The following example shows the behaviour of these two entropies in discrete spaces:

**Example 1.2.**

(a) [18] Let \( T : \mathbb{Z} \rightarrow \mathbb{Z} \) be the shift \( n \mapsto n + 1 \) of the discrete space \( \mathbb{Z} \). Then \( βT : β\mathbb{Z} \rightarrow β\mathbb{Z} \) has entropy \( ∞ \), so \( h_β(T) = h_{a₀}(T) = ∞ \), even if \( \mathbb{Z} \) is discrete.

(b) An easy modification of item (a) shows that every infinite discrete space admits a selfmap of infinite entropy.

So for a discrete space X the following are equivalent:

(b₁) X has a selfmap of positive entropy;

(b₂) X is infinite;

(b₃) X has a selfmap of infinite entropy.

### 1.3 Properties of the topological entropy

We give now the basic properties of the topological entropy for continuous selfmaps of topological spaces.

**Lemma 1.3.** Let X and Y be topological spaces and \( ψ : X \rightarrow X \) and \( φ : Y \rightarrow Y \) be continuous selfmaps. Assume that there exists a continuous map \( α : X \rightarrow Y \), with \( αψ = φα \), i.e., the following diagram commutes

\[
\begin{array}{ccc}
X & \xrightarrow{ψ} & X \\
\downarrow & & \downarrow \\
Y & \xrightarrow{φ} & Y
\end{array}
\]

\((\ast)\)

(a) (Monotonicity of \( h_{a₀} \) and \( h_β \) for continuous images) If \( α(X) \) is dense in Y, then \( h_β(ψ) \geq h_β(φ) \). If \( α \) is surjective, then \( h_{a₀}(ψ) \geq h_{a₀}(φ) \).

(b) (Monotonicity of \( h_{a₀} \) for invariant subspaces) If \( α \) is injective and \( α(X) \) is closed in Y, then \( h_{a₀}(ψ) \leq h_{a₀}(φ) \).
(c) Monotonicity of $h_\beta$ for invariant subspaces) If $\alpha$ is injective, $a(X)$ is closed in $Y$ and $X$ is normal, then $h_\beta(\psi) \leq h_\beta(\phi)$.

Proof. (a) The case $h_\beta$ was proved in [18] for surjective $\alpha$. Here we argue in the weaker assumption $a(X)$ is dense in $Y$. In the following diagram we denote by $i : X \to \beta X$ and $j : Y \to \beta Y$ the immersions and we note that $\beta a$ is surjective by the density of $a(X)$ in $Y$ (so in $\beta Y$ as well).

Since monotonicity for continuous images is well known in the compact case, one can deduce that

$$h_\beta(\psi) = h(\beta \psi) \geq h(\beta \phi) = h_\beta(\phi).$$

The case of $h_{\text{fin}}$ was announced without proof in [18], this is why we provide a proof here. Assume that $\alpha$ is surjective and let $U$ be a finite open cover of $Y$. Then $V = \alpha^{-1}(U)$ is a finite open cover of $X$. One can immediately see from diagram (*) that $C_n(V, \psi) = \alpha^{-1}(C_n(U, \phi))$, in particular, $N(C_n(U, \phi)) = N(C_n(V, \psi))$. Hence,

$$\log\left(\frac{N(C_n(U, \phi))}{n}\right) = \log\left(\frac{N(C_n(V, \psi))}{n}\right) \leq h_{\text{fin}}(\psi).$$

This proves that $h_{\text{fin}}(\phi) \leq h_{\text{fin}}(\phi)$.

(b) Since this property was announced without proof in [18], we provide a proof here. According to (a), we can assume wlog that $\alpha$ is a subspace embedding of $X$ as a closed subspace of $Y$. Let $U$ be a finite open cover of $X$. Then there exists an open cover $V$ of $Y$, such that $\alpha^{-1}(V) = U$ (for every $U \in U$ find an open $V$ in $Y$ with $\alpha^{-1}(V) = U$ and add to this family of open sets of $Y$ also $Y \setminus X$ to obtain a finite cover $V$ of $Y$). As above, $C_n(V, \psi) = \alpha^{-1}(C_n(U, \phi))$, so $N(C_n(U, \phi)) \leq N(C_n(V, \phi))$. Hence,

$$\log\left(\frac{N(C_n(U, \phi))}{n}\right) \leq \log\left(\frac{N(C_n(V, \phi))}{n}\right) \leq h_{\text{fin}}(\phi).$$

This gives $h_{\text{fin}}(\phi) \leq h_{\text{fin}}(\phi)$.

(c) If $X$ is normal, then $h_\beta(\psi) \leq h_{\text{fin}}(\psi) \leq h_{\text{fin}}(\phi) = h_\beta(\phi)$, where the second inequality comes from item (b) and the equality is due to the normality of $X$. 

Corollary 1.4. In the hypotheses of Lemma 1.3, if $a(X)$ is dense in $Y$ and $h_\beta(\phi) = \infty$, then $h_\beta(\psi) = h_\beta(\phi) = \infty$.

In the next example we show that if $X$ is not normal, then (c) may fail. This shows that Proposition 2 from [18], asserting that (c) is true for all Tychonov spaces, is wrong.

Example 1.5. Here we use the example built by Fedeli [16] to show that the entropies $h_{\text{fin}}$ and $h_\beta$ do not coincide for non-normal Tychonov spaces.

Let $X$ be the Tychonov plank, presented in the following way. Let $\omega_1 + 1$ be the compact ordinal space and $P$ be the product $(\omega_1 + 1) \times \beta \omega_1$, where $\beta \omega_1 = \omega_1 \cup \{\infty\}$ is the Alexandroff one-point compactification of the discrete space $\omega_1$ of the integers. Then $X = P \setminus (\omega_1, \infty)$. It is well known that $P = \beta X$.

Consider the homeomorphism $T : P \to P$ defined by

$$T(\alpha, \gamma) = \begin{cases} (\alpha, \gamma + 1) & \text{if } \gamma \in \mathbb{Z} \\ (\alpha, \gamma) & \text{if } \gamma = \infty \end{cases}$$
for all \( a \leq \omega_1 \) and \( \gamma \in \mathbb{Z}_{\omega} \). As \( T(\omega_1, \infty) = (\omega_1, \infty) \) and \( T(X) = X \), the subspace \( X \) is \( T \)-invariant and the restriction
\[
T_1 = T|_X : X \to X
\]
is a homeomorphism that will play a prominent role in this example.

Furthermore, \( \text{Fix}(T) = (\omega_1 + 1) \times \{ \infty \} \) is the set of all fixed points of \( T \), so \( h(T|_{[\omega_1 + 1] \times \{ \infty \}}) = 0 \). On the other hand, \((\omega_1 + 1) \times \{ \infty \}\) coincides also with the set \( \Omega(T) \) of non-wandering points of \( T \), as every \( x = (y, n) \in P \setminus (\omega_1 + 1) \times \{ \infty \} \) has an open neighborhood, say \( U = P \times \{ n \} \), such that \( T^k(U) \cap U = \emptyset \) for every \( k \neq 0 \). Since \( h(T) = h(T|_{\Omega(T)}) \) for every compact space, one can deduce that \( h(T) = h(T|_{\Omega(T)}) = 0 \). Therefore, \( h_\beta(T_1) = h(T) = 0 \), as \( \beta X = P \) and \( \beta T_1 = T \).

Consider the closed \( T_1 \)-invariant subspace
\[
M = ((\omega_1) \times \mathbb{Z}_{\omega}) \cap X = \{ \omega_1 \} \times \mathbb{Z}
\]
of \( X \). As \( T_1|_M : M \to M \) is conjugated to the shift \( n \mapsto n + 1 \) of the discrete space \( \mathbb{Z} \), \( T_1|_M \) has entropy
\[
h_{\text{fin}}(T_1|_M) = h_\beta(T_1|_M) = \infty
\]
(see Example 1.2(a)). As \( M \) is a closed \( T_1 \)-invariant subspace of \( X \), we deduce from Lemma 1.3(b) that also \( h_{\text{fin}}(T_1) = \infty \). This simultaneously shows that the non-normal space \( X \) has a homeomorphism \( T_1 \) such that
(a) \( h_\beta(T_1) = 0 \), while its restriction to \( M \) has \( h_\beta(T_1|_M) = \infty \);
(b) \( h_\beta(T_1) = 0 < \infty = h_{\text{fin}}(T_1) \).

An easy consequence of monotonicity for continuous images is the invariance of \( h_{\text{fin}} \) and \( h_\beta \) under topological conjugation.

**Corollary 1.6** (Invariance under conjugation). Let \( X \) be a topological space and let \( \psi : X \to X \) be a continuous selfmap. If \( \alpha : X \to Y \) is a homeomorphism between spaces, then \( h_{\text{fin}}(\alpha \psi \alpha^{-1}) = h_{\text{fin}}(\psi) \) and \( h_\beta(\alpha \psi \alpha^{-1}) = h_\beta(\psi) \).

**Proof.** Let \( \phi = \alpha \psi \alpha^{-1} \). The case of \( h_{\text{fin}} \) follows from items (a) and (b) of Lemma 1.3. For the case of \( h_\beta \) note that from the commutative diagram (*) one obtains the following commutative diagram
\[
\begin{array}{ccc}
\beta X & \xrightarrow{\beta \psi} & \beta X \\
\beta \alpha \downarrow & & \beta \alpha \\
\beta Y & \xrightarrow{\beta \psi} & \beta Y
\end{array}
\]
\[
(**)
\]
Now the invariance of the topological entropy for selfmaps of compact spaces gives
\[
h_\beta(\psi) = h(\beta \psi) = h(\beta \phi) = h_\beta(\phi).
\]

**Theorem 1.7** (Logarithmic Law). Let \( X \) be a Tychonov space and \( \psi : X \to X \) a continuous selfmap. Then:
(a) \( h_\beta(\psi^k) = kh_\beta(\psi) \) and \( h_{\text{fin}}(\psi^k) = kh_{\text{fin}}(\psi) \) for every \( k \in \mathbb{N} \);
(b) if \( \psi : X \to X \) is a homeomorphism, then \( h_\beta(\psi^{-1}) = h_\beta(\psi) \) and \( h_{\text{fin}}(\psi^{-1}) = h_{\text{fin}}(\psi) \), so
\[
h_\beta(\psi^k) = |k|h_\beta(\psi) \quad \text{and} \quad h_{\text{fin}}(\psi^k) = |k|h_{\text{fin}}(\psi) \quad \text{for every} \ k \in \mathbb{Z}.
\]

**Proof.** (a) was proved in [18].

To prove (b) for \( h_\beta \) it suffices to note that for a homeomorphism \( \psi : X \to X \) one has \( \beta \psi^{-1} = (\beta \psi)^{-1} \). So \( h_\beta(\psi^{-1}) = h((\beta \psi)^{-1}) = h(\beta \psi) = h_\beta(\psi) \). The proof in the case of \( h_{\text{fin}} \) goes exactly as in the case of compact spaces.

For compact \( X \) the next proposition was proved in [10].
**Proposition 1.8.** Let $X$ be a topological space and $\psi : X \to X$ a continuous selfmap. Let $X_1, X_2$ be $\psi$-invariant subspaces of $X$ such that $X = X_1 \cup X_2$. Then

(a) $h_{\text{fin}}(\psi) = \max \{ h_{\text{fin}}(\psi|_{X_1}), h_{\text{fin}}(\psi|_{X_2}) \}$;

(b) $h_{\beta}(\psi) \leq \max \{ h_{\beta}(\psi|_{X_1}), h_{\beta}(\psi|_{X_2}) \}$.

**Proof.** Let $\psi_i = \psi|_{X_i}$ for $i = 1, 2$.

(a) By the monotonicity from Lemma 1.3(b) we have the inequality $h_{\text{fin}}(\psi) \geq \max \{ h_{\text{fin}}(\psi_1), h_{\text{fin}}(\psi_2) \}$. To prove the converse inequality, we first fix a notation, namely for $U \in \text{cov}(X)$, let $U_i = U \mid_{X_i} \in \text{cov}(X_i)$ for $i = 1, 2$. Then, for $i = 1, 2$, $U_i, V_i \in \text{cov}(X)$ and $n \in \mathbb{N}$,

$$\text{(1)} U_i \cup V_i = \text{cov}(X_i),$$

$$\text{(2)} \psi^{-n}(U_i) = \psi_i^{-n}(U_i).$$

Moreover

$$N_X(U) \leq N_{X_1}(U_1) + N_{X_2}(U_2).$$

Therefore for every $n \in \mathbb{N}$, we have

$$N_X(U \cup \psi^{-1}(U) \cup \ldots \cup \psi^{-n+1}(U)) \leq N_X(U_1 \cup \psi_1^{-1}(U_1) \cup \ldots \cup \psi_1^{-n+1}(U_1)) + N_X(U_2 \cup \psi_2^{-1}(U_2) \cup \ldots \cup \psi_2^{-n+1}(U_2)).$$

Hence $h(\psi, U) \leq \max \{ h(\psi_1, U_1), h(\psi_2, U_2) \}$; in particular $h_{\text{fin}}(\psi) \leq \max \{ h_{\text{fin}}(\psi_1), h_{\text{fin}}(\psi_2) \}$, and this concludes the proof.

(b) For $i = 1, 2$ let $f_i : X_i \hookrightarrow X$ be the inclusion map and let $g_i = \beta f_i : \beta X_i \to \beta X$ be the canonical extension map. Let $Y_i = g_i(X_i)$. Then $Y_i$ is a $\beta\psi$ invariant subspace of $\beta X$ and $\beta\psi|_{Y_i} \circ g_i = g_i \circ \beta\psi_i$. So, $h(\beta\psi_1, Y_i) \leq h(\beta\psi_2, Y_i)$. Since the space $\beta X = Y_1 \cup Y_2$ is compact, $h(\beta\psi) = h(\beta\psi_1) = \max \{ h(\beta\psi_1, Y_1), h(\beta\psi_1, Y_2) \}$. So, $h_{\beta}(\psi) \leq \max \{ h(\beta\psi_1), h(\beta\psi_2) \} = \max \{ h_{\beta}(\psi_1), h_{\beta}(\psi_2) \}$.

**Remark 1.9.** Every locally compact group $G$ is normal, so $h_{\text{fin}}(f) = h_{\beta}(f)$ for any continuous selfmap $f : G \to G$.

Following [3], for a given abelian group $G$ and an integer $k$ we denote by $m_k$ the endomorphism $G \to G$ defined by the multiplication by $k$. The next theorem is inspired by the final example of [18] where the equalities $h_{\text{fin}}(m_2) = h_{\beta}(m_2) = \infty$ for the group $G = \mathbb{R}$ were established by a different argument.

**Proposition 1.10.** For a non-compact locally compact abelian group $G$ having no proper open compact subgroups $h_{\text{fin}}(m_k) = h_{\beta}(m_k) = \infty$ holds for every integer $k > 1$.

**Proof.** We intend to exploit the following fact: if $K$ is a compact infinite dimensional group, then for every $k > 1$ the endomorphism $m_k : K \to K$ has infinite entropy [3, Lemma 8.3]. To this end we show that the Bohr compactification $bG$ of $G$ is infinite dimensional, so the endomorphism $m_k : bG \to bG$ has infinite entropy. Now Lemma 1.3(a) implies that the endomorphism $m_k : G \to G$ has infinite entropy too.

According to Remark 1.9 and to Lemma 1.3(c), and due to the fact that the endomorphisms $m_k$ leave all subgroups of $G$ invariant, it suffices to prove that for some closed subgroup $H$ of $G$ the endomorphism $m_k : H \to H$ has infinite entropy.

Being locally compact and abelian, $G$ has the form $G = \mathbb{R}^n \times G_0$, where $n \in \mathbb{N}$ and $G_0$ contains an open compact subgroup. Our hypothesis on $G$ yields $G \neq G_0$ so $n > 0$. Then $H = \mathbb{R}^n$ is a non-trivial connected locally compact group. Moreover, $H$ is self dual, i.e., the Pontryagin dual $\hat{H}$ of $H$ is isomorphic to $H$. Hence, the Bohr compactification $bH$ of $H$ is isomorphic to $\hat{H}$, where $H_\text{d}$ denotes $H$ equipped with the discrete topology. As $H_\text{d} \cong \mathbb{R}^n \cong \bigoplus_{\mathbb{Q}} \mathbb{Q}$, we deduce that $bH \cong \hat{\mathbb{C}}^n$. In particular, $bH$ is infinite-dimensional. Hence, the endomorphism $m_k : H \to H$ has infinite entropy.

One has to rule out the (locally) compact abelian groups with proper open compact subgroups as this property strongly fails for such groups. Indeed, if $p$ is a prime and $G$ is the locally compact group of $p$-adic numbers, then $h_{\text{fin}}(f) = h_{\beta}(f) = 0$ as $G$ has a local base of 0 formed by open subgroups which are $m_k$-invariant. Hence,
the uniform entropy $h_U$ of $m_k$ is zero. In this case all three entropies $h_{\text{fin}}, h_\beta$ and $h_U$ coincide. This remains true for all totally disconnected locally compact abelian groups.

2 The uniform entropy

The notion of uniform entropy was extended in a natural way by Hofer [18] and Hood [19] to the case of uniformly continuous selfmaps of uniform spaces.

To extend the definition of uniform entropy $h_U$ to the case of a uniformly continuous selfmap $\psi : X \to X$ of a uniform space $(X, U)$, fix an entourage $V$ of the uniform structure, $x \in X$ and $n \in \mathbb{N}$. Let

$$D_n(x, V, \psi) = \bigcup_{k=0}^{n-1} \psi^{-k}(V(\psi^k(x))).$$

Let $K$ be a compact subset of $X$. One can find a finite set $F$ of points of $X$, such that $\bigcup_{x \in F} D_n(x, V, \psi)$ covers $K$. Let $r_n(V, K, \psi)$ denote the minimum cardinality of such a subset $F$. Hence for every entourage $V \in U$ as above we can define:

$$r(V, K, \psi) = \limsup_{n \to \infty} \frac{\log r_n(V, K, \psi)}{n}.$$

Then, $h_r(K, \psi) = \sup \{r(V, K, \psi) : V \in U\}$ is the uniform entropy of $\psi$ with respect to $K$. Then the notion of uniform entropy $h_U(\psi)$ of $\psi$ is obtained by setting:

$$h_U(\psi) = \sup \{h_r(K, \psi) : K \in \mathcal{K}(X)\}. \quad (2)$$

When $(X, d)$ is a metric space and $U_d$ is the metric uniformity of $X$, we obtain Bowen's definition, which inspired the above definition for uniform spaces.

It is not difficult to prove that $h_U$ is monotone under some continuous images.

Let us first recall the concept of compact-covering map introduced in [21].

**Definition 2.1.** [21] A continuous map $\phi : E \to F$ of topological spaces is compact-covering if each compact subset of $E$ is the image of some compact subset of $F$.

The following lemma gives some monotonicity properties of the uniform entropy.

**Lemma 2.2.** Let $X$ be a uniform space and let $\phi : X \to X$ be a uniformly continuous map of $X$. Then:

(i) if $Y$ is a $\phi$-invariant subspace of $X$, then $h_U(\phi) \geq h_U(\phi |_Y)$.

(ii) $h_U$ satisfies the Logarithmic Law.

**Proof.** These properties are announced without proof in [18, Proposition 5]. To prove (i) one has to note that the compact subsets $K$ of $Y$, used to compute the entropy $h_U(\phi |_Y)$ are also compact subsets of $X$, so can be used for the computation of the entropy $h_U(\phi)$. Moreover, the numbers $h_r(K, \phi)$ and $h_r(\phi |_Y)$ coincide. 

It is proved in [19, Theorem 5] that if $\pi : X \to Z$ is a compact-covering uniformly continuous surjective map satisfying a special property (connecting the uniformity of $X$ to the equivalence relation generated by $\pi$) and $\bar{\phi} : Z \to Z$ is a uniformly continuous map satisfying $\pi \circ \bar{\phi} = \phi \circ \pi$, then $h_U(\phi) \geq h_U(\bar{\phi})$. Therefore, one can deduce that $h_U$ is invariant under uniform conjugation, as uniform isomorphisms are obviously compact-covering maps and satisfy the special property used in the proof of [19, Theorem 5].

It can be shown that this extended notion of entropy, in the case of a compact space and its unique compatible uniform structure, coincides with the topological entropy $h$ (see [15] for details).

Finally, we mention the approach to the uniform entropy $h_U$ by means of covers proposed in [18], similar to the one in the definition of the topological entropies $h$ and $h_{\text{fin}}$. The equivalence of these two approaches was pointed out in [15].
2.1 A modification of the definition of uniform entropy

Here we propose an important modification of the definition of uniform entropy that will be used in the sequel. Shortly speaking, the finite subset $F$ of $X$ used for the definition of the number $r_n(V, K, \psi)$ can be chosen a subset of $K$. In order to rigorously check this fact we first define a parallel entropy function $h^*_U$ that will be proved to coincide with $h_U$.

To define $h^*_U$, for $n \in \mathbb{N}$, $V \in \mathcal{U}$ and a compact set of $X$ let $r^*_n(V, K, \psi)$ be the minimum cardinality of a subset $F'$ of $K$ such that $K \subseteq \bigcup_{x \in F} D_n(x, V, \psi)$. Note that it exists since $K$ is compact. From this point on, one defines $r'(V, K, \psi), h'_U(K, \psi)$ and $h'_U$ as before. Let us see that this variation gives nothing new.

**Theorem 2.3.** $h^*_U = h_U$.

**Proof.** It suffices to prove $h'_U(K, \psi) = h_U(K, \psi)$ for every compact set $K$.

To this end we intend to estimate $r'(V, K, \psi)$ via the values of $r(V, K, \psi)$ and $r(V \circ V, K, \psi)$. Obviously

$$r_n(V, K, \psi) \leq r^*_n(V, K, \psi).$$

(3)

Our aim is to show that

$$r_n(V \circ V, K, \psi) \leq r^*_n(V \circ V, K, \psi) \leq r_n(V, K, \psi)$$

(4)

for a fixed triple $n, K, V$. The first inequality follows from (3). To check the second fix a finite set $F \subseteq X$ with

$$K \subseteq \bigcup_{x \in F} D_n(x, V, \psi).$$

(5)

It suffices to find now a finite set $F' \subseteq K$ with $K \subseteq \bigcup_{x \in F'} D_n(x, V \circ V, \psi)$ and $|F'| \leq |F|$. We can assume without loss of generality that the set $F$ is minimal (with respect to inclusion) with (5).

Before doing this, we show that $D_n(x, V, \psi)$ can be given the more convenient form (7). In the sequel, for a selfmap $f : X \to X$ and $V \in \mathcal{U}$ we denote briefly by $f(V)$ the image of $V$ under the map $f \times f : X \times X \to X \times X$. By a similar *abus de language*, we shortly denote by $f^{-1}$ the inverse image of $V$ under the map $f \times f$. Under this notation, one has

$$f^{-1}(V) \circ f^{-1}(V) \subseteq f^{-1}(V \circ V),$$

(5)

equality is available if $f$ is surjective (but we shall not use that).

For $V \in \mathcal{U}$ and $n \in X$ let

$$C_n(V, \psi) := V \cap f^{-1}(V) \cap f^{-2}(V) \cap \ldots \cap f^{-n+1}(V),$$

the *n*-th cotrajectory of $V$ under $\psi$. From (5) one can easily deduce

$$C_n(V, \psi) \cap C_n(V, \psi) \subseteq C_n(V \circ V, \psi)$$

(6)

The motivation to introduce the *n*-th cotrajectory $C_n(V, \psi)$ is the following useful equality

$$D_n(x, V, \psi) = C_n(V, \psi)[x].$$

(7)

Coming back to our argument, we notice that due to the choice of $F$, $K \cap C_n(V, \psi)[x] \neq \emptyset$ for each $x \in F$. So we can pick an $y_x \in K \cap C_n(V, \psi)[x]$ and notice that $x \in C_n(V, \psi)[y_x]$ whenever $V$ is chosen to be symmetric. Then

$$C_n(V, \psi)[x] \subseteq (C_n(V, \psi) \cap C_n(V, \psi))[y_x] \subseteq C_n(V \circ V, \psi)[y_x]$$

(8)

in view of (6). Let $F' = \{y_x : x \in F\}$. Then (8) yields

$$K \subseteq \bigcup_{x \in F} D_n(x, V, \psi) = \bigcup_{x \in F} C_n(V, \psi)[x] \subseteq \bigcup_{y \in F} C_n(V \circ V, \psi)[y].$$

Since obviously $|F'| \leq |F|$, this proves (4).

From (4) we obtain

$$r(V \circ V, K, \psi) \leq r'(V \circ V, K, \psi) \leq r'(V, K, \psi) \leq r(V, K, \psi).$$

(3)

Taking suprema w.r.t. $V \in \mathcal{U}$ gives the desired equality $h'_U(K, \psi) = h_U(K, \psi)$. 

□
3 The interplay between uniform entropies and topological entropies

Here we discuss the possibility to define new entropy functions in the category of Tychonov spaces, using the uniform entropy. To this end we need to connect the category $\text{Tych}$ of Tychonov spaces and continuous maps with the category $\text{Unif}$ of uniform spaces and uniformly continuous maps.

A functorial uniformity on $\text{Tych}$ is nothing else but a functor $F : \text{Tych} \to \text{Unif}$ such that the uniform space $FX$ has the same underlying set as the topological space $X$ and the uniformly continuous map $FF$, for a continuous map $f : X \to Y$ in $\text{Tych}$, is the same set-map as $f$ (using a more rigorous categorial language, if $T : \text{Unif} \to \text{Tych}$ is the forgetful functor, then the composition $T \cdot F$ coincides with the identity functor of $\text{Tych}$).

For every functor $F$ as above one can define a (topological) entropy $h_F$ in $\text{Tych}$ by letting $h_F(f) = h_U(Ff)$ for every selfmap $f : X \to X$ in $\text{Tych}$. In this way $h_F$ inherits for free the nice properties of $h_U$ (e.g., Logarithmic Law, etc.).

**Example 3.1.** [15, Definition 2.21] For $X \in \text{Tych}$ let $\mathcal{F}X$ denote the universal (fine) uniform structure on $X$. Since every continuous map from $X$ to a uniform space is uniformly continuous for the universal uniform structure $\mathcal{F}$ (see, for instance, [17, 15G (5)]), the assignment $X \mapsto \mathcal{F}X$ is a functorial uniformity.

For a Tychonov space $X$ one can also consider the finest totally bounded uniformity $\mathcal{C}$ usually termed Čech uniformity of $X$ (generated by all continuous bounded real-valued functions of $X$). Since this is also a functorial uniformity (actually, the coarsest functorial uniformity), one may ask whether the entropy $h_{\mathcal{C}}$ defined on Tychonov spaces coincides with some of the previously defined entropies $h_{\mathfrak{fl}}, h_{\beta}$ and $h_{\mathcal{F}}$. In [10, Example 4.2.4 (b)] it was stated (without proof) that $h_{\mathcal{C}}$ coincides with $h_{\mathfrak{fl}}$. We shall see below that this equality fails. More precisely, we shall see in Theorem 3.7 that always $h_{\mathcal{C}} = h_{\mathcal{F}} \leq h_{\beta} \leq h_{\mathfrak{fl}}$, but these entropies do not coincide in non-normal spaces (as the example from [16] shows). In particular, this negatively answers the natural question on whether the entropies $h_{\beta}$ and $h_{\mathfrak{fl}}$ can be obtained by an appropriate (functorial) uniformity (as $h_{\mathcal{F}}$ is the largest such entropy).

The following example from [15] shows that $h_{\mathcal{F}}(-)$ differs from both $h_{\beta}(-)$ and $h_{\mathfrak{fl}}(-)$.

**Example 3.2.** [15, Example 2.23] Let $X = \mathbb{Z}$ carrying the discrete topology and let $f : X \to X$ be the map defined by $f(x) = kx$, where $k > 1$ is an integer. As $X$ has no infinite compact subsets, $h_{\mathcal{F}}(f) = 0$. We show that $h_{\beta}(f) = h_{\mathfrak{fl}}(f) = \infty$. According to Corollary 1.4, it suffices to check that $h_{\beta}(f) = \infty$. Let $X^\#$ denote the group $X$ equipped with its Bohr topology. Then $f : X^\# \to X^\#$ is continuous, and its extension $bf$ to the Bohr compactification $bX$ of $X^\#$ (i.e., the completion of $X^\#$) has $h_{\mathfrak{fl}}(bf) = \infty$, by [3, Theorem 3.3, Case 1]. Since the continuous extension $\beta f : \beta X \to bX$ of the inclusion map $j : X \to bX$ satisfies $\beta f = bf \circ \beta j$, one concludes that $\infty = h_{\mathfrak{fl}}(bf) \leq h_{\beta}(f)$.

(b) Analogously, for the locally compact abelian group $\mathbb{R}$ that has an endomorphism $f : x \mapsto 2x$ with $h_{\mathcal{F}}(f) = \log 2$ and $h_{\mathfrak{fl}}(f) = h_{\beta}(f) = \infty$ (see Proposition 1.10).

(c) The map $T : x \mapsto x + 1$ in $\mathbb{R}$ has $h_{\mathcal{F}}(T) = 0$ (being a non-expanding map) and $h_{\mathfrak{fl}}(T) = h_{\beta}(T) = \infty$ as the restriction of $T$ on the closed $T$-invariant subspace $\mathbb{Z}$ has infinite entropy (see Example 1.2).

This example leaves open the following question raised already in [15]:

**Question 3.3.** [15, Question 2.24] Does the inequality $h_{\mathcal{F}}(\phi) \leq h_{\beta}(\phi)$ hold for every continuous selfmap $\phi : X \to X$ of a Tychonov space $X$?

We give an affirmative answer of this question below.

Now we use the new Theorem 2.3 to show that the use of functorial uniformities cannot provide many distinct entropies. More precisely, $h_{\mathcal{F}}$ appears to be the only entropy function that can be obtained in this...
Corollary 3.6. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be two uniformities on a set $X$ giving rise to the same topology $T(\mathcal{U}_1) = T(\mathcal{U}_2)$ on $X$. Then for every selfmap $f : X \to X$ that is uniformly continuous in both uniformities $\mathcal{U}_1$ the entropies computed in $(X, \mathcal{U}_1)$ and $(X, \mathcal{U}_2)$ coincide.

Proof. For the sake of precision, let us denote by $f_i$ the uniformly continuous map $(X, \mathcal{U}_i) \to (X, \mathcal{U}_i)$, for $i = 1, 2$. Then we have to prove that $h_{\mathcal{U}_1}(f_1) = h_{\mathcal{U}_2}(f_2)$. Since both $h_{\mathcal{U}_1}(f_1)$ and $h_{\mathcal{U}_2}(f_2)$ are obtained as suprema over the family of compact sets $K$ of $(X, \tau)$, it suffices to prove that $h_{\mathcal{U}_1}(K, f_1) = h_{\mathcal{U}_2}(K, f_2)$. For a fixed compact subset $K$ both uniformities coincide on $K$. In particular, for every $U_1 \in \mathcal{U}_1$ there exists $U_2 \in \mathcal{U}_2$ such that $(K \times K) \cap U_2 \subseteq U_1$. Therefore, $r_n(U_1, K, f_1) \leq r_n(U_2, K, f_2)$. After taking logarithms and dividing by $n$ one gets

$$\frac{\log r_n(U_1, K, f_1)}{n} \leq \frac{\log r_n(U_2, K, f_2)}{n}.$$ 

This inequality is available for each $n$, from the definitions of $r(U_1, K, f_1)$ and $r(U_2, K, f_2)$ we deduce that for every $U_1 \in \mathcal{U}_1$ there exists $U_2 \in \mathcal{U}_2$ such that $r(U_1, K, f_1) \leq r(U_2, K, f_2)$. Hence,

$$r(U_1, K, f_1) \leq h_{\mathcal{U}_2}(K, f_2) = \sup\{r(U_2, K, f_2) : U_2 \in \mathcal{U}_2\}$$

for every $U_1 \in \mathcal{U}_1$. This proves that $h_{\mathcal{U}_1}(K, f_1) \leq h_{\mathcal{U}_2}(K, f_2)$.

Analogously, one checks that $h_{\mathcal{U}_2}(K, f_2) \leq h_{\mathcal{U}_1}(K, f_1)$. \hfill $\square$

Corollary 3.5. Let $X$ be a Tychonov space. Then for every continuous selfmap $f : X \to X$ the entropies $h_{A}(f)$ and $h_{U}(f)$ coincide.

For a Tychonov space $X$ consider the family $\mathcal{A}(X)$ of all neighborhoods of the diagonal in $X \times X$. In general $\mathcal{A}(X)$ is not a uniformity, although it is a filter. However, one can take the largest uniformity coarser than $\mathcal{A}(X)$. Note that that uniformity induces the given topology, since it is finer than $\mathcal{C}^t(X)$, but coarser than $\mathcal{A}(X)$. That uniformity induces the given topology, since it is finer than $\mathcal{C}^t(X)$, but coarser than $\mathcal{A}(X)$. Hence it coincides with the fine uniformity $\mathcal{T}(X)$ of $X$. If $X$ is metric, then $\mathcal{A}(X) = \mathcal{T}(X)$. Hence, from Theorem 3.4 we obtain:

Corollary 3.6. Let $X$ be a metric space. Then for every uniformly continuous selfmap $f : X \to X$ the entropies $h_{A}(f)$ and $h_{U}(f)$ coincide.

This corollary corrects an erroneous assertion from [18]. Indeed, in Remark on page 240 one can find a claim (without proof) that for $X = \mathbb{R}$ and the map $T : x \mapsto 2x$ one has $h_{A}(T) = \infty$. Since $h_{U}(T) = \log 2$, this contradicts the above corollary.

Now we can finally answer Question 3.3:

Theorem 3.7. For every continuous selfmap $f : X \to X$ of a Tychonov space $X$ one has $h_{C}(f) \leq h_{\beta}(f) \leq h_{\text{fin}}(f)$. Moreover, there exist maps $f$ and $g$ such that $h_{C}(f) < h_{\beta}(f)$ and $h_{\beta}(g) < h_{\text{fin}}(g)$. In particular, neither $h_{\beta}$, nor $h_{\text{fin}}(g)$ can be obtained by the use of appropriate factorial uniformities.

Proof. The uniformity $\mathcal{C}^t$ on $X$ is the one induced by the unique uniformity of the compact space $\beta X$. Therefore, the monotonicity of $h_{U}$ with respect to taking invariant subspaces implies

$$h_{C}(f) \leq h_{U}(\beta f) = h(\beta f) = h_{\beta}(f) \leq h_{\text{fin}}(f).$$

A map $f$ with $h_{C}(f) < h_{\beta}(f)$ was given in Example 3.2. A map with $h_{\beta}(g) < h_{\text{fin}}(g)$ was produced in [16] (see Example 1.5). \hfill $\square$

This theorem leaves open the following natural question:

Question 3.8. Does there exist a continuous selfmap $\phi : X \to X$ of a Tychonov space $X$ such that

$$h_{C}(f) < h_{\beta}(f) < h_{\text{fin}}(f)?$$
Let us note that such a Tychonov space cannot be normal. On the other hand, both examples, distinguishing \( h_{C^*}(f) \) from \( h_\beta \) and \( h_\beta \) from \( h_{\text{fin}} \), are extremal (i.e., the example from Example 1.5 has \( 0 = h_\beta(f) < h_{\text{fin}}(f) = \infty \), while \( 0 = h_{C^*}(f) < h_\beta(f) = \infty \) in Example 3.2), so none of them works.

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