Research Article

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On the range of \( n^{th} \) order derivations acting on commutative Banach positive squares \( \ell \)-algebras

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Abstract: In this paper we prove that the image of a \( n^{th} \) order derivation on real commutative Banach \( \ell \)-algebras with positive squares is contained in the set of nilpotent elements.

Keywords: \( f \)-algebra, almost \( f \)-algebra, Jacobson radical, \( n^{th} \) order derivation

MSC: 06F25, 13N15

1 Introduction

Let \( A \) be a commutative algebra. A linear mapping \( D : A \rightarrow A \) is called an \( n^{th} \) order derivation if \( D \) satisfies

\[
D(x_1x_2...x_{n+1}) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leq i_1 < ... < i_k \leq n+1} x_{i_1}...x_{i_k} D(x_{\tilde{i_1}}...\tilde{x_{i_k}}...x_{n+1})
\]

for all \( x_1, x_2, ..., x_{n+1} \in A \) such that the symbol \( \tilde{x} \) indicates that \( x \) is omitted. The first order derivation is exactly the ordinary derivation. The notion of high order derivation was introduced, in 1967, by Osborn [11]. Nakai [10] and Ebanks [5,6] focus their attention on higher order derivations. They give fundamental results in order to identify \( n^{th} \) order derivations.

Many mathematicians are interested on the range inclusion problems for the first order derivations. Singer and Wermer [13] proved that the range of every bounded derivation on a commutative Banach algebra is contained in the (Jacobson) radical. Simultaneously, they conjectured that the continuity assumption is superfluous. Johnson [8] showed that in order to confirm the conjecture of Singer-Wermer it was sufficient to suppose \( A^2 = A \oplus \mathbb{C} \); a commutative radical Banach algebra \( A \) with identity 1 adjoined and \( D : A^2 \rightarrow A^2 \) a derivation and work to prove the conjecture in this case. It took more than 30 years before the Singer and Wermer conjecture was finally confirmed by Thomas [14].

In the framework of (real) lattice-ordered algebras, Colville, Davis and Keimel [4] and Henriksen [7] showed that if \( A \) is an Archimedean \( f \)-algebra, then a positive operator \( D : A \rightarrow A \) is a derivation if and only if \( D(A) \subset N(A) \) and \( A^2 \subset \ker D \). Boulabiar [3] studied the positive derivations on Archimedean almost \( f \)-algebras. More precisely, he showed that if \( D \) is a positive derivation on an Archimedean almost \( f \)-algebra \( A \), then \( D(A) \subset N(A) \) and \( A^3 \subset \ker D \). Later, M. A. Toumi et al [15] and Ben Amor [1] generalized these results for the order bounded derivations. In 2012, Toumi [16] extended this results for the continuous derivations. Recently, Kouki and Toumi [9] focus their attention on the problem of range inclusion of derivation on non-Banach algebra. In fact, they proved that the image of any derivation on an universally complete \( f \)-algebra is contained in the cloradical (the intersection of all closed maximal modular ideals). Moreover, they proved...
that any derivation on a Banach $f$-algebra maps into the set of nilpotent elements.

To the best of our knowledge no attention at all has been paid in the literature to the range of $n^{th}$ order derivations problem. In this paper we focus our attention on $n^{th}$ order derivations acting on commutative Banach positive squares $\ell$-algebras. More precisely, we prove that every $n^{th}$ order derivation on real Banach $\ell$-algebras with positive squares maps into the set of nilpotent elements.

2 Preliminaries

An algebra $A$ which is simultaneously a vector lattice such that the partial ordering and the multiplication on $A$ are compatible, that is $a, b \in A^+$ implies $ab \in A$ is called a lattice-ordered algebra (briefly $\ell$-algebra).

In an $\ell$-algebra $A$ we denote the collection of all nilpotent elements of $A$ by $N(A)$. An $\ell$-algebra $A$ is said to be semiprime if $N(A) = \{0\}$. An $\ell$-algebra $A$ is called an $f$-algebra if $A$ verifies the property that $a \wedge b = 0$ and $c \geq 0$ imply $ac \wedge b = ca \wedge b = 0$. An $\ell$-algebra $A$ is called an almost $f$-algebra whenever it follows from $a \wedge b = 0$ that $ab = ba = 0$. An $\ell$-algebra $A$ is called an $d$-algebra if $A$ verifies the property that $a \wedge b = 0$ and $c \geq 0$ imply $ac \wedge bc = ca \wedge cb = 0$.

In the following lines, we recall some definitions on high order derivations. An $n^{th}$ order derivation on a commutative algebra $A$ is a linear mapping $D$ from $A$ into $A$ such that

$$D(x_1x_2\ldots x_{n+1}) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1, \ldots, i_k \leq n+1} x_{i_1}\ldots x_{i_k}D(x_{i_1}\ldots x_{i_k}\ldots x_{n+1})$$

for all $x_1, x_2, \ldots, x_{n+1} \in A$. The first order derivation $D : A \to A$ is a linear mapping such that

$$D(ab) = D(a) b + a D(b) \text{ for all } a, b \in A.$$

$\text{Der}^n(A)$ denoted the set of all $n^{th}$ order derivations on $A$. The set generated by the composites of $m$ ($m \leq n$) derivations denoted $\text{der}^n(A)$. Namely, $\text{der}^n(A) \subset \text{Der}^n(A)$ and $\text{der}^1(A) = \text{Der}^1(A) = \text{Der}(A)$.

A complex vector lattice is the complexification $A_C = A \oplus iA$ of a real vector lattice $A$ provided that each $z \in A_C$ has the absolute value $|z|$ defined by the formula

$$|z| = \sup_{0 \leq \theta \leq 2\pi} |x(cos \theta) + y(sin \theta)| \quad (z = x + iy \in A_C).$$

Define a complex Banach $\ell$-algebra $A_C$ to be the complexification of a real Banach $\ell$-algebra $A$. The multiplication in $A$ extends naturally to the multiplication in $A_C$ by the formula

$$(x + iy)(x' + iy') = \left( xx' - yy' \right) + i \left( xy' + x'y \right).$$

An ideal $I$ in $A_C$ is defined as the complexification $J \oplus iJ$ of an ideal $J \subset A$. An operator $T : E_C \to F_C$ is uniquely representable as $T = T_1 + iT_2$ where $T_1, T_2$ are real operators from $A$ to $A$. Let $D : A_C \to A_C$ be a linear mapping and $D = D_1 + iD_2$. Then $D$ is a complex derivation if and only if $D_1$ and $D_2$ are real derivations on $A$. For details on complex lattice algebras we refer to [2].

3 The range of a derivation acting on real commutative Banach positive squares $\ell$-algebras

Before proceeding with the main result, we need some prerequisites.

Let $A$ be a lattice-ordered algebra. A ring ideal $M$ is said to be modular if there exists $u$ in $A$ such that $x - ux \in M$ for each $x \in A$.

Lemma 1. Let $A$ be a real commutative Banach $\ell$-algebra, let $A_C$ be its complexification and let $I = J \oplus iJ$ be an ideal in $A_C$. Then the following properties are equivalent:

1. $D(I) \subset I$ for any $D$ in $\text{Der}^n(A_C)$.
2. $\text{Der}^n(A_C)$ is the set of all $n^{th}$ order derivations acting on $A_C$.
3. $\text{Der}^n(A_C)$ is the set of all $n^{th}$ order derivations acting on $A_C$. 

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1. \( I \) is a maximal modular ideal in \( AC \).
2. \( J \) is a maximal modular ideal in \( A \).

Proof. (1) \( \Rightarrow \) (2) Let \( K \) be a modular ideal in \( A \) such that \( J \subset K \). Then \( J \oplus iJ \subset K \oplus iK \). Since \( J \oplus iJ \) is a maximal modular ideal in \( AC \), it follow that

\[
K \oplus iK = J \oplus iJ \text{ or } K \oplus iK = AC.
\]

Hence

\[
K = J \text{ or } K = A.
\]

It remains to show that \( J \) is modular. Since \( I \) is modular, there exists \( u = u_1 + iu_2 \in AC \) such that

\[
x - ux = (x - u_1 x) + iux_2 \in J \oplus iJ, \text{ for all } x \in A.
\]

Then

\[
x - u_1 x \in J, \text{ for all } x \in A.
\]

Therefore, \( J \) is modular.

(2) \( \Rightarrow \) (1) Let \( K \oplus iK \) be a modular ideal of \( AC \) such that \( J \oplus iJ \subset K \oplus iK \). Then \( J \subset K \). Since \( J \) is a maximal modular ideal in \( A \), it follow that

\[
K = J \text{ or } K = A.
\]

Hence

\[
K \oplus iK = J \oplus iJ \text{ or } K \oplus iK = AC.
\]

Then \( J \oplus iJ \) is a maximal ideal in \( AC \). We have to show finally that \( J \oplus iJ \) is modular. Since \( J \) is modular, there exists \( u \in A \) such that

\[
x - ux \in J, \text{ for all } x \in A.
\]

Let \( x = a + ib \in AC \), then

\[
x - ux = (a + ib) - u (a + ib)
\]

\[
= a - ua + i (b - ub)
\]

\[
\in J \oplus iJ = I.
\]

\( \square \)

Let \( A \) be a real commutative \( \ell \)-algebra and let \( AC \) be its complexification. \( M(A) \) (respectively \( M(AC) \)) denotes the set of all maximal modular ring ideals of \( A \) (respectively, the set of all maximal modular ring ideals of \( AC \)). The Jacobson radical of \( A \) and the Jacobson radicals of \( AC \), which are denoted respectively by \( rad(A) \) and \( rad(AC) \), are defined as,

\[
rad(A) = \bigcap_{M \in M(A)} M
\]

and

\[
rad(AC) = \bigcap_{M \in M(AC)} M.
\]

**Proposition 1.** Let \( A \) be a real commutative Banach \( \ell \)-algebra with positive squares and let \( AC \) be its complexification. Then

\[
rad(AC) = N(AC)
\]

\[
= \{ a \in AC : a^3 = 0 \}
\]

\[
= \{ a \in AC : abc = 0 \text{ for all } b, c \in AC \}.
\]
Proof. By using the previous lemma, it follows that
\[
\text{rad}(A_C) = \bigcap_{I \in M(A_C)} I \\
= \bigcap_{J \in M(A)} (J \oplus iJ) \\
= \left( \bigcap_{J \in M(A)} J \right) \oplus i \left( \bigcap_{J \in M(A)} J \right) \\
= \text{rad}(A) \oplus i\text{rad}(A)
\]

According to Render [12],
\[
\text{rad}(A) = \{ a \in A : a^3 = 0 \} \\
= \{ a \in A : abc = 0 \text{ for all } b, c \in A \} \\
= N(A).
\]

Hence
\[
\text{rad}(A_C) = \text{rad}(A) \oplus i\text{rad}(A) \\
= N(A) \oplus iN(A) \\
= N(A_C)
\]

Consequently
\[
\text{rad}(A_C) = N(A_C) \\
= \{ a \in A_C : abc = 0 \text{ for all } b, c \in A_C \}.
\]

Theorem 1. Let A be a real commutative Banach ℓ-algebra with positive squares and let D : A → A be a derivation. Then D(A) ⊂ N(A)

Proof. Let D : A → A be a derivation. Let \( \tilde{D} : A_C \rightarrow A_C \) defined by \( \tilde{D}(x + iy) = D(x) + iD(y) \). Then \( \tilde{D} \) is a complex derivation on \( A_C \). Thomas in [14] showed that the image of a complex derivation is contained in the radical. According to the previous proposition \( \text{rad}(A_C) = N(A_C) \). Then \( \tilde{D}(A_C) \subset N(A_C) \). Since
\[
\tilde{D}(A_C) = D(A) \oplus iD(A) \text{ and } N(A_C) = N(A) \oplus iN(A).
\]

Then
\[
D(A) \subset N(A).
\]

We deduce also the following corollaries.

Corollary 1. Let A be a real Banach almost f-algebra and let D : A → A be a derivation. Then
\[
D(A) \subset N(A) \text{ and } D\left(A^3\right) = \{0\}
\]

Proof. If \( a \in N(A) \), then it follows from
\[
0 = D(abc) \\
= abD(c) + acD(b) + bcD(a) \\
= bcD(a).
\]
that \( bcD(a) \) for all \( b, c \in A \). Therefore \( D(a) \in N(A) \).

Let \( \bar{D} : A_{/N(A)} \to A_{/N(A)} \) defined by \( \bar{D}(a + N(A)) = D(a) + N(A) \). Hence it not hard to prove that \( \bar{D} \) is a derivation and that \( A_{/N(A)} \) is a semiprime Banach \( f \)-algebra. Then \( D \) is null and so \( D(a) \subset N(A) \) for all \( a \in A \). Hence \( D(A) \subset N(A) \) and \( D(A^2) = \{0\} \).

If in addition the nilpotency index does not exceed 2, the situation improves considerably.

**Corollary 2.** Let \( A \) be a real Banach almost \( f \)-algebra such that \( N(A) = \{ a \in A ; a^2 = 0 \} \) and let \( D : A \to A \) be a linear mapping. Then \( D \) is a derivation if and only if

\[
D(A) \subset N(A) \text{ and } D\left(A^2\right) = \{0\}.
\]

**Proof.** If \( D \) is a derivation. Then, by the above theorem, \( D(x) \subset N(A) \) for all \( x \in A \). It follows that \( D(x)y = 0 \) for all \( x, y \in A \) and so \( D(A^2) = \{0\} \). It is not hard to prove that \( D(A) \subset N(A) \) and \( D(A^2) = \{0\} \) implies that \( D \) is a derivation.

**Corollary 3.** Let \( A \) be a real Banach \( f \)-algebra and let \( D : A \to A \) be a linear mapping. Then \( D \) is a derivation if and only if

\[
D(A) \subset N(A) \text{ and } D\left(A^2\right) = \{0\}.
\]

**Corollary 4.** Let \( A \) be a real Banach \( d \)-algebra and let \( D : A \to A \) be a derivation. Then \( D(A) \subset N(A) \).

### 4 The Singer-Wermer conjecture of \( n \)th order derivation acting on real commutative Banach positive squares \( \ell \)-algebras

Nakai [10] proved that if \( D_1 \) and \( D_2 \) are derivations of orders \( m, n \) respectively acting on a commutative algebra \( A \), then \( D_1 \circ D_2 \) is an \( (m + n)^{th} \) order derivation. Then the product of two derivations is a second order derivation. If in addition \( A \) is an \( f \)-algebra, the situation improves considerably.

**Proposition 2.** Let \( A \) be a real Banach \( f \)-algebra and let \( D_1, D_2 : A \to A \) be two derivations. Then \( D_1 \circ D_2 \) is a derivation.

**Proof.** Let \( D_1, D_2 : A \to A \) be two derivations. It follows from [9, Theorem 8] that \( D_1(A) \subset N(A) \) and \( D_2(A) \subset N(A) \).

Therefore

\[
D_1 \circ D_2(ab) = D_1(aD_2(b) + D_2(a)b)
\]

\[
= aD_1D_2(b) + D_1(a)D_2(b) + D_2(a)D_1(b) + bD_1D_2(b)
\]

\[
= 0.
\]

and

\[
D_1 \circ D_2(A) \subset N(A).
\]

Then, \( D_1 \circ D_2 \) is a derivation.

According to the previous proposition, any second order derivation on a Banach \( f \)-algebra cannot be a product of two derivations. This is illustrated in the following example.

**Example 1.** Take \( A = \mathbb{R}^2 \) with the usual operations and order. We define the following multiplication:

\[
\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} aa' \ b b' \\ 0 \end{pmatrix}
\]
for all \((\alpha, \beta)\), \((\beta, \gamma)\) \(\in A\). A simple verification shows that \(A\) is a Banach \(f\)-algebra under the multiplication.

Let \(D : A \to A\) be the linear map defined by \(D(\alpha, \beta) = (\beta, \gamma)\) for all \((\alpha, \beta)\) \(\in A\). It is easy to show that \(D\) is a second order derivation that is not derivation and so \(D\) cannot be a product of two derivations.

**Proposition 3.** Let \(A\) be a commutative algebra and let \(D : A \to A\) be an \(n^{th}\) order derivation. Let \(x_1, x_2, \ldots, x_{n-1} \in A\) and let \(D_{x_1, x_2, \ldots, x_{n-1}} : A \to A\) defined by

\[
D_{x_1, x_2, \ldots, x_{n-1}}(x_n) = D(x_1 x_2 \ldots x_{n-1}) - \frac{n-1}{2}\sum_{k=1}^{n-2} (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq n-1} x_{i_1} \ldots x_{i_k} D(x_1 \ldots \hat{x}_{i_1} \ldots \hat{x}_{i_k} \ldots x_{n-1})
\]

Then \(D_{x_1, x_2, \ldots, x_{n-1}}\) is a derivation.

**Proof.** Let \(a, b \in A\). Since

\[
D_{x_1, x_2, \ldots, x_{n-1}}(a) = D(ax_1 x_2 \ldots x_{n-1}) - aD(x_1 x_2 \ldots x_{n-1})
\]

\[
- \sum_{k=1}^{n-2} (-1)^{k+2} \sum_{1 \leq i_1 < \cdots < i_k \leq n-1} ax_{i_1} \ldots x_{i_k} D(x_1 \ldots \hat{x}_{i_1} \ldots \hat{x}_{i_k} \ldots x_{n-1})
\]

\[
- \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq n-1} x_{i_1} \ldots x_{i_k} D(ax_1 \ldots \hat{x}_{i_1} \ldots \hat{x}_{i_k} \ldots x_{n-1})
\]

Then

\[
D_{x_1, x_2, \ldots, x_{n-1}}(ab) = D(abx_1 x_2 \ldots x_{n-1}) - abD(x_1 x_2 \ldots x_{n-1})
\]

\[
- \sum_{k=1}^{n-2} (-1)^{k+2} \sum_{1 \leq i_1 < \cdots < i_k \leq n-1} abx_{i_1} \ldots x_{i_k} D(x_1 \ldots \hat{x}_{i_1} \ldots \hat{x}_{i_k} \ldots x_{n-1})
\]

\[
- \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq n-1} x_{i_1} \ldots x_{i_k} D(abx_1 \ldots \hat{x}_{i_1} \ldots \hat{x}_{i_k} \ldots x_{n-1})
\]

Since

\[
D(abx_1 x_2 \ldots x_n) = abD(x_1 x_2 \ldots x_{n-1})
\]

\[
+ \sum_{k=1}^{n-2} (-1)^{k+3} \sum_{1 \leq i_1 < \cdots < i_k \leq n-1} abx_{i_1} \ldots x_{i_k} D(x_1 \ldots \hat{x}_{i_1} \ldots \hat{x}_{i_k} \ldots x_{n-1})
\]

\[
+ \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq n-1} x_{i_1} \ldots x_{i_k} D(abx_1 \ldots \hat{x}_{i_1} \ldots \hat{x}_{i_k} \ldots x_{n-1})
\]

\[
+ aD(bx_1 \ldots x_{n-1})
\]

\[
+ \sum_{k=1}^{n-1} (-1)^{k+2} \sum_{1 \leq i_1 < \cdots < i_k \leq n-1} ax_{i_1} \ldots x_{i_k} D(bx_1 \ldots \hat{x}_{i_1} \ldots \hat{x}_{i_k} \ldots x_{n-1})
\]

\[
+ bD(ax_1 \ldots x_{n-1})
\]

\[
+ \sum_{k=1}^{n-1} (-1)^{k+2} \sum_{1 \leq i_1 < \cdots < i_k \leq n-1} bx_{i_1} \ldots x_{i_k} D(ax_1 \ldots \hat{x}_{i_1} \ldots \hat{x}_{i_k} \ldots x_{n-1})
\]
Then
\[
D_{x_1, x_2, \ldots, x_{n-1}}(ab) = aD(bx_1 \ldots x_{n-1}) + \sum_{k=1}^{n-1} (-1)^{k+2} \sum_{1 \leq i_1 < \cdots < i_k \leq n-1} ax_{i_1} \ldots x_{i_k} D(bx_{i_{k+1}} \ldots x_{n-1}) \\
+ bD(ax_1 \ldots x_{n-1}) + \sum_{k=1}^{n-1} (-1)^{k+2} \sum_{1 \leq i_1 < \cdots < i_k \leq n-1} bx_{i_1} \ldots x_{i_k} D(ax_{i_{k+1}} \ldots x_{n-1}) \\
- 2 \sum_{k=1}^{n-2} (-1)^{k+2} \sum_{1 \leq i_1 < \cdots < i_k \leq n-1} abx_{i_1} \ldots x_{i_k} D(x_{i_{k+1}} \ldots x_{n-1}) \\
= aD(bx_1 \ldots x_{n-1}) + \sum_{k=1}^{n-1} (-1)^{k+2} \sum_{1 \leq i_1 < \cdots < i_k \leq n-1} x_{i_1} \ldots x_{i_k} D(bx_{i_{k+1}} \ldots x_{n-1}) \\
- \sum_{k=1}^{n-2} (-1)^{k+2} \sum_{1 \leq i_1 < \cdots < i_k \leq n-1} bx_{i_1} \ldots x_{i_k} D(ax_{i_{k+1}} \ldots x_{n-1}) \\
+ bD(ax_1 \ldots x_{n-1}) + \sum_{k=1}^{n-1} (-1)^{k+2} \sum_{1 \leq i_1 < \cdots < i_k \leq n-1} x_{i_1} \ldots x_{i_k} D(ax_{i_{k+1}} \ldots x_{n-1}) \\
- \sum_{k=1}^{n-2} (-1)^{k+2} \sum_{1 \leq i_1 < \cdots < i_k \leq n-1} ax_{i_1} \ldots x_{i_k} D(x_{i_{k+1}} \ldots x_{n-1}) \\
= aD_{x_1, x_2, \ldots, x_{n-1}}(b) + D_{x_1, x_2, \ldots, x_{n-1}}(a) b 
\]

\[\Box\]

**Proposition 4.** Any real \(n^{th}\) order derivation on a commutative semiprime Banach \(l\)-algebra with positive squares is null.

**Proof.** Let \(D : A \to A\) be a \(n^{th}\) order derivation. Let \(x_1, x_2, \ldots, x_{n-1} \in A\). The maps \(D_{x_1, x_2, \ldots, x_{n-1}} : A \to A\) defined by
\[
D_{x_1, x_2, \ldots, x_{n-1}}(x_n) = D(x_1 x_2 \ldots x_n) - \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \ldots x_{i_k} D(x_{i_{k+1}} \ldots x_{n})
\]
is a real derivation on a commutative Banach \(l\)-algebra with positive squares. Then \(D_{x_1, x_2, \ldots, x_{n-1}}(A) \subset N(A)\). Since \(N(A) = \{0\}\), then \(D_{x_1, x_2, \ldots, x_{n-1}}\) is null. It follows that \(D\) is a \((n-1)^{th}\) order derivation. By repeating this argument, we find that \(D\) is a first order derivation. Therefore \(D(A) \subset N(A)\). Since \(A\) is a semiprime, then \(D\) is null. \(\Box\)

**Theorem 2.** Let \(A\) be a real commutative Banach positive squares \(l\)-algebra and let \(D : A \to A\) be a derivation of order \(n\). Then
\[
D(A) \subset N(A) \text{ and } D \left( A^{n+2} \right) = \{ 0 \}.
\]

**Proof.** Let \(D : A \to A\) be a \(n^{th}\) order derivation. Let \(a \in N(A)\). Then \(abc = 0\) for all \(b, c \in A\). Let \(x_i \in A\), \(1 \leq i \leq n\). Hence
\[
0 = D \left( a \prod_{i=1}^{n} x_i^2 \right) \\
= aD \left( \prod_{i=1}^{n} x_i^2 \right) + (-1)^{n+1} \left( \prod_{i=1}^{n} x_i^2 \right) D(a).
\]
Proof. If \( N(A) = \{ a \in A : a^2 = 0 \} \), then for all \( x, y \in N(A) \) we deduce the following result.

\[
d ( D(x) ) \subseteq N(A) \quad \text{and} \quad D \left( A^{n+1} \right) = \{ 0 \}.
\]

Corollary 5. Let \( A \) be a real commutative Banach positive squares \( \ell \)-algebra such that \( N(A) = \{ a \in A : a^2 = 0 \} \) and let \( D : A \to A \) be a linear mapping. Then \( D \) is a \( n \)th order derivation if and only if

\[
D ( A ) \subseteq N ( A ) \quad \text{and} \quad D \left( A^{n+1} \right) = \{ 0 \}.
\]

Proof. If \( D \) is a \( n \)th order derivation. Then, by the above theorem, \( D(x) \subseteq N(A) \) for all \( x \in A \). It follows that \( D(xy) = 0 \) for all \( x, y \in A \) and so \( D \left( A^{n+1} \right) = \{ 0 \} \). It is not hard to prove that \( D(A) \subseteq N(A) \) and \( D \left( A^{n+1} \right) = \{ 0 \} \) implies \( D \) is a \( n \)th order derivation.

Corollary 6. Let \( A \) be a real Banach almost \( f \)-algebra and let \( D : A \to A \) be a \( n \)th order derivation. Then

\[
D ( A ) \subseteq N ( A ) \quad \text{and} \quad D \left( A^{n+2} \right) = \{ 0 \}.
\]

Since in an \( f \)-algebra, the nilpotency index does not exceed 2, we deduce the following result.

Corollary 7. Let \( A \) be a real Banach \( f \)-algebra and let \( D : A \to A \) be a linear mapping. Then \( D \) is a \( n \)th order derivation if and only if

\[
D ( A ) \subseteq N ( A ) \quad \text{and} \quad D \left( A^{n+1} \right) = \{ 0 \}.
\]

References