A model of quotient spaces

Hawete Hattab*

DOI 10.1515/taa-2017-0003
Received November 3, 2015; accepted November 1, 2016

Abstract: Let $R$ be an open equivalence relation on a topological space $E$. We define on $E$ a new equivalence relation $\tilde{R}$ by $x \tilde{R} y$ if the closure of the $R$-trajectory of $x$ is equal to the closure of the $R$-trajectory of $y$. The quotient space $E/\tilde{R}$ is called the trajectory class space. In this paper, we show that the space $E/\tilde{R}$ is a simple model of the quotient space $E/R$. This model can provide a finite model. Some applications to orbit spaces of groups of homeomorphisms and leaf spaces are given.

Keywords: Homotopy, orbit space, leaf space, leaf class space, orbit class space.

MSC: 55P10, 55P15, 18B35, 54F65, 54B15

1 Introduction

The idea of modeling spaces with ones which are easier to describe and to work with is standard in Algebraic Topology. We say that a topological space $Y$ is a model of a topological space $X$ if it is weak homotopy equivalent to $X$. If moreover $Y$ is finite, we say that $Y$ is a finite model. For example, the singleton is the unique finite model of every contractible space. McCord found in [22] a finite model of the $n$-sphere $S^n$ with only $2n + 2$ points. In [3], the authors showed that this model is a minimal finite model of $S^n$. For example, a minimal finite model of the unit circle $S^1$ is the 4-point space $K_{22} = \{a, b, c, d\}$ such that $a < c$, $a < d$, $b < c$ and $b < d$.

Let $E$ be a topological space and let $R$ be an open equivalence relation. We define the class of a trajectory $O$ as being the union of all trajectories $O'$ having the same closure as $O$. In other words, we define on $E$ a new equivalence relation $\tilde{R}$ by:

$x \tilde{R} y$ if $\tilde{R}(x) = \tilde{R}(y)$.

Since, for each open subset $U$ of $E$, the saturated sets of $U$ by $R$ and by $\tilde{R}$ are equal, it follows that this new equivalence relation is open.

We denote by $E/R$ the trajectory space and $E/\tilde{R}$ the trajectory class space. We denote by $q : E \rightarrow E/R$ and $p : E \rightarrow E/\tilde{R}$ the canonical projections. In this paper, we show that the trajectory class space is homotopy equivalent to the trajectory space.

A codimension-$q$ foliation on a smooth $m$-manifold $M$ is an open equivalence relation $\mathcal{F}$ on $M$ such that each trajectory (called a leaf) is a weakly embedded submanifold of dimension $p = m - q$ and such that the canonical projection of $M$ on the leaf space $M/\mathcal{F}$ is a locally submersion. The leaf space is a very uninformative and complicated quotient space, and the problem is to define a more refined quotient which captures aspects of the geometric structure of the foliation. One of the interesting approach to this problem is to construct, up to homotopy, a quotient, this construction takes the form of a classifying space. This approach goes back...
to Haefliger, who constructed a classifying $BΓ_q$ for foliations of codimension $q$, called universal leaf space [8, 17]. On the other hand, if one considers a Hausdorff quotient of the leaf space, one looses most of the dynamical information of our initial foliation. For this reason, [24] considered an intermediary quotient called the leaf class space which is a $T_0$-space but keep more information on the initial foliation. Recently, many authors studied the relations between dynamical properties of the foliation and the leaf class space [6, 9, 19]. In this paper, using [6], we give finite models of some complicated leaf spaces of a codimension-one foliation on a three manifold.

Let $E$ be a topological space and let $\text{Homeo}(E)$ be its group of homeomorphisms. Consider a subgroup $G \subset \text{Homeo}(E)$, the family of orbits $G(x) = \{g(x) : g \in G\}$ by $G$ determines an open equivalence relation on $E$. In general, the orbit space $E/G$ is very complicated. For instance, if $E = S^1$ and $G = \langle R_α \rangle$ is the group generated by an irrational rotation $R_α$, then the orbit space can hardly be explicitly described but the orbit class space $E/\tilde{G}$ is reduced to a singleton. Thus the singleton is a finite model of the orbit space $S^1/R_α$.

Let $f : (X, x_0) \to (Y, y_0)$ be a quotient map of topological spaces, where $X$ is locally path-connected and $Y$ is semilocally simply-connected³. If each fiber $f^{-1}(y)$ is connected, then the induced homomorphism $f_* : π_1(X, x_0) \to π_1(Y, y_0)$ is surjective.

Calcut et al. noted that Theorem 1.1 may be applied to orbit spaces of vector fields and p-dimensional foliations of a manifold and their associated leaf spaces. We show that this theorem is also true for some orbit class spaces of groups of homeomorphisms of CW-complex.

### 2 Finite spaces, foliations and groups of homeomorphisms

The most important results concerning finite spaces can be summarized by the following four items:

1. The connection between finite topological spaces and finite partially ordered sets, first considered by Alexandroff in [1].
2. The combinatorial description of homotopy types of finite spaces, discovered by Stong in his beautiful article [25].
3. The correspondence between finite spaces and polyhedra, found by McCord [22].
4. The connection between finite spaces on one side, and foliations, groups of homeomorphisms on the other side (Bonatti et al. in [6, 7]).

Items (1), (2) and (3) are developed in [2, Chapter 1].

#### 2.1 Finite spaces and foliations

Let $\mathcal{F}$ be a transversally oriented codimension-one foliation on a closed manifold $M$. For notions: proper leaves, stable leaves, locally dense leaves, minimal sets and local minimal sets one can see [16]. The fact that the leaf class space $X = M/\mathcal{F}$ is a $T_0$-space, allows us to define an order on $X$ as follows:

$$a = p(x) \leq p(y) = b$$

if $\mathcal{F}_x \subset \mathcal{F}_y$

³ A space $Z$ is semilocally simply-connected if each $z \in Z$ has a neighborhood $U$ such that the induced homomorphism $π_1(U, z) \to π_1(Z, z)$ is trivial.
where \( F_x \) and \( F_y \) are the leaves containing \( x \) and \( y \) respectively. According to [9], the quotient topology of the leaf class space \( X \) is compatible with the inverse order \( \succeq \) that is if \( a \in X \), then \([a]\succeq] \leftrightarrow a\). On the other hand, according to [16], the subset \([a, \to] = \{x \in X : x \succeq a\}\) is always open and if the leaf \( F_x \) is proper and attracting (the pseudo-group of holonomy is without fixed point) or non proper and contained in a local minimal set then the subset \([a, \to] = \{x \in X : x \succeq a\}\) is open.

Let \((Y, \phi)\) be a partially ordered set. A chain of \((Y, \phi)\) is a totally ordered subset of \( Y \). Given an integer \( n \geq 0 \), we say that a point \( x \in Y \) has height \( ht(x) = n \) if \( n + 1 \) is the upper bound of the cardinality of the chains of \((Y, \phi)\) bounded by \( x \). So a point \( x \) is minimal for \( \phi \) if it has height \( ht(x) = 0 \). We say that \( Y \) has a finite height \( ht(Y) \in \mathbb{N} \) if the height \( ht(y) \) are defined for every \( y \in Y \) and if \( ht(Y) = \sup(ht(y), y \in Y) \). A finite partially ordered set has a finite height.

Let \( X \) be a \( T_0 \)-space, partially ordered by \( a \preceq b \) if \([a] \subset [b] \). A point \( a \in X \) is a separated point if for every \( b \not\in [a] \) there exists an open neighborhood \( U_a \) (resp. \( U_b \)) of \( a \) (resp. of \( b \)) such that \( U_a \cap U_b = \emptyset \) [19]. The regular part \( X_0 \) of \( X \) is the interior of the set of separated points. The singular part of \( X \) is the complementary \( X - X_0 \). One deduces that, if the singular part of \( X \) is finite, then the height of \( X \) is finite. For \( 0 \leq k \leq ht(X) \), the set of points of height less or equal to \( k \) is:

\[
X^k = \{x \in X, \, ht(x) \leq k\}.
\]

By convention we denote \( X^{-1} = \emptyset \). Notice that each set \( X^k \) is a closed subset of \( X \).

According to [9], if \( X \) is the leaf class space of a transversally oriented codimension-one foliation on a closed manifold, then \( X_0 \) is the union of open subsets of \( X \) homeomorphic to the line \( \mathbb{R} \) or to the circle \( S^1 \). In this case, according to [18], \( p^{-1}(X_0) \) is the union of all stable proper leaves.

One of the main results of [6] is the following theorem:

**Theorem 2.1.** Every connected finite \( T_0 \)-space \( Y \) is homeomorphic to the singular part \( X - X_0 \) of the leaf class space \( X \) of a transversally oriented codimension one \( C^1 \)-foliation \( \mathcal{F} \) on a closed oriented three manifold \( M \).

### 2.2 Finite spaces and groups of homeomorphisms

According to [7], we have:

**Theorem 2.2.** Any finite poset \( X \) can be realized as the orbit class space built from a dynamical system \((E, G)\) where \( E \) is a CW-complex of dimension \( ht(X) + 1 \) and \( G \) is a finitely generated group of homeomorphisms of \( E \).

### 3 The trajectory space and the class trajectory space

Let \( E \) be a topological space and let \( \mathcal{R} \) be an open equivalence relation on \( E \). Recall that a continuous map \( f : X \to Y \) between two topological spaces is called a quasi-homeomorphism if the map which assigns to each open set \( V \subset Y \) the open set \( f^{-1}(V) \) is a bijective map [15]. The goal of this section is to show the following theorem:

**Theorem 3.1.** The trajectory space and the class trajectory space are homotopy equivalent.

**Proof.** According to [19], the map \( \varphi : E/\mathcal{R} \to E/\mathcal{R} \) which associates to each trajectory its class is a surjective quasi-homeomorphism.

In the following, we will prove a stronger statement: if \( f : X \to Y \) is a surjective quasi-homeomorphism, then \( f \) is a homotopy equivalence.

Let \( X \) be the universal \( T_0 \)-space associated to the space \( X \) by identifying pairs of points \( x \) and \( y \) such that \( \{x\} = \{y\} \). If \( h \) is the quotient map, then there exist a continuous map \( g \) such the following diagram is
commutative
\[
\begin{array}{c}
X \xrightarrow{f} Y \\
\downarrow h^g \\
\tilde{X}
\end{array}
\]
i.e \(g \circ h = f\). According to [10, Proposition 2.7], \(g\) is a homeomorphism and so it is a homotopy equivalence. It remains to show that \(h\) is a homotopy equivalence. Let \(i : \tilde{X} \to X\) be a map such that \(h \circ i = 1\) (i.e., \(i\) picks a representative from each equivalence class); we pretend that \(i\) is a homotopy inverse to \(h\). It suffices to show that \(i \circ h\) is homotopic to the identity on \(X\). Define \(H : X \times I \to X\) by \(H(x, t) = i \circ h(x)\) for \(t = 0\) and \(H(x, t) = x\) for \(t \in ]0, 1]\). By definition of \(h\), \(i \circ h(x)\) is contained in exactly the same open sets of \(X\). Thus for any \(U \subset X\) open, \(H^{-1}(U) = U \times I\), so \(H\) is continuous. Hence \(H\) is a homotopy from \(i \circ h\) to 1. By transitivity of the homotopy equivalence, we deduce that \(f\) is a homotopy equivalence.

\section{Finite model of leaf spaces}

\textbf{Proposition 4.1.} The leaf space of a Kronecker foliation is contractible.

\textit{Proof.} A Kronecker foliation is obtained as the suspension of an irrational rotation on the circle \(S^1\) and so all of the leaves are dense which implies that the leaf class space is a singleton. By applying Theorem 3.1, the leaf space of a Kronecker foliation is contractible. \hfill \Box

\textbf{Example 4.2.} We define a foliation \(\mathcal{F}_k\) on a 3-manifold \(N = S \times S^1\) where \(S\) is a closed connected oriented surface of genus 2. Recall that there is an onto (surjective) morphism \(\rho\) from the fundamental group \(\pi_1(S)\) onto the free group \(F^2\). The foliation \(\mathcal{F}_k\) is obtained as a suspension of a morphism of \(\pi_1(S)\) in the set of diffeomorphisms of the circle \(S^1\) (which factorizes by \(\rho\)), whose image is the free group generated by:

\begin{itemize}
  \item a diffeomorphism \(f : S^1 \to S^1\) having a Denjoy minimal set \(m_f\) and precisely \(k\) orbits of wandering intervals (connected components of \(S^1 - m_f\)); consider \(I_1, \ldots, I_k\) \(k\) wandering intervals, which are connected components of \(S^1 - m_f\), not pairwise equivalent; hence every orbit of \(S^1 - m_f\) meets \(\bigcup_{i=1}^k I_i\) in precisely one point;
  \item \(g\) is a diffeomorphism of \(S^1\) which is the identity map out of \(\bigcup_{i=1}^k I_i\) and satisfies \(g(x) > x\) for \(x \in \bigcup_{i=1}^k I_i\).
\end{itemize}

We remark that for any connected component \(c\) of \(X_0\) there is a closed transversal curve \(\gamma_c\) embedded in \(N\) such that \(\gamma_c\) cuts each leaf \(L \subset p^{-1}(c)\) in exactly one point, and is disjoint from the minimal set \(m_k\) and from the leaves corresponding to the other components \(c' \neq c\) of \(X_0\).

\textbf{Proposition 4.3.} The leaf space \(N/\mathcal{F}_k\) is contractible.

\textit{Proof.} First \(X - X_0\) is a singleton and \(X_0\) is composed by \(k\) circles. \(K_{22}\) is a minimal model of each circle with the weak homotopy equivalence \(\varphi\) defined in [29]. If we replace each circle by a copy of \(K_{22}\), then we obtain a finite space \(Y\) having a unique minimal element. We define the map \(\psi : X \to Y\) as follows: the restriction on each circle is equal to \(\varphi\) and the identity elsewhere. \(\psi\) is an increasing map and so it is continuous. By applying [22, Theorem 6], we obtain that \(\psi\) is a weak homotopy equivalence. Since \(Y\) has a unique minimal element, according to [29, Proposition 2.4], \(Y\) is contractible and so \(X\) is contractible and so, by applying Theorem 3.1, the leaf space \(N/\mathcal{F}_k\) is contractible. \hfill \Box

\textbf{Example 4.4.} Let \(N_1\) and \(N_2\) be two copies of the 3-manifold \(N\) endowed with the foliation \(\mathcal{F}_2\) given by Example 4.2. In \(N_1\) and \(N_2\) we remove a solid torus which are small tubular neighborhoods of closed transversal curves parallel to the curve \(\gamma_{c_2}\). Then we get manifolds \(N_1\) and \(N_2\) with one boundary component diffeomorphic to \(T^2\) transverse to the foliation. Now we glue the torus \(\partial N_2\) to the torus \(\partial N_1\), in order to glue the induced foliation.
One gets a closed connected orientable 3-manifold $M_{2,1}$ endowed with an orientable codimension one foliation $\mathcal{F}_{2,1}$.

Proposition 4.5. A finite model of the leaf space $M_{2,1}/\mathcal{F}_{2,1}$ is the space $Y$, whose Hasse diagram is the one of Figure 1.

![Figure 1: The finite model $Y$ of the leaf space $M_{2,1}/\mathcal{F}_{2,1}$.](image)

Proof. The leaf class space $M_{2,1}/\widetilde{\mathcal{F}}_{2,1}$ is the space of Figure 2:

![Figure 2: The leaf class space $M_{2,1}/\widetilde{\mathcal{F}}_{2,1}$.](image)

If we replace each circle by a copy of $K_{22}$, then we obtain a finite space $Y$ (see Figure 1). We define the map $\psi : X \to Y$ as follows: the restriction of $\psi$ on each circle is equal to $\varphi$ and the identity elsewhere. $\psi$ is an increasing map and so it is continuous. By applying [22, Theorem 6], we obtain that $\psi$ is a weak homotopy equivalence. Therefore $Y$ is a finite model of the leaf space $M_{2,1}/\mathcal{F}_{2,1}$.

5 The fundamental group of finite spaces

Our main result is:

Theorem 5.1. Let $X$ be a connected finite space of height $n$. There exists a CW-complex of dimension $n + 1$ $E$ and a subgroup $G$ of homeomorphisms of $E$ such that $\pi_1(X)$ is isomorphic to $\pi_1(E)/\text{Ker}(p_\#)$ where $\text{Ker}(p_\#)$ is the kernel of the induced homomorphism of the quotient map $p$.

Proof. Let $X$ be a connected finite space of height $n$. According to Theorem 2.2, there exists a CW-complex of dimension $n + 1$ $E$ and a subgroup $G$ of homeomorphisms of $E$ such that $X$ is homeomorphic to the orbit class space $E/\widetilde{G}$. Therefore $X$ and $E/\widetilde{G}$ are homotopy equivalent. A CW-complex $E$ is always locally path-connected and from the fact that finite spaces are locally contractible, we deduce that $E/\widetilde{G}$ is a semilocally
simply-connected space. It only remains to prove that for any \( a \in E/\tilde{G} \), \( p^{-1}(a) \) is connected. According to the proof of Theorem 2.2 (as in [7, pages 739–740]), there exist two exclusive cases:

- If \( a \) has even height \( 2j \), then \( p^{-1}(a) \) is the interior of a \((2j + 1)\)-dimensional sphere.
- If \( a \) has odd height \( 2j + 1 \), then \( p^{-1}(a) \) is the interior of a \((2j + 2)\)-dimensional sphere with holes \( Y_{2j+2,k(a)} \), where \( k(a) \) is the number of outgoing edges of \( a \). Note that for all integers \( n \geq 2 \) and \( k \geq 1 \),

\[
Y_{n,k} = S^n \setminus \left( \bigcup_{i=1}^{k} \text{Int}(A_i) \right)
\]

where all \( A_i \) are pairwise disjoint and homeomorphic to \( D^n \).

In both cases \( p^{-1}(a) \) is connected.

Then, according to Theorem 1.1, the induced homomorphism \( p_* \) is subjective and so by applying the first group isomorphism theorem, we have \( \pi_1(X) \) is isomorphic to \( \pi_1(E)/\text{Ker}(p_*) \). \( \square \)

References