Inclusion theorems of double Deferred Cesàro means II

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Abstract

In 1932 R. P. Agnew present a definition for Deferred Cesàro mean. Using this definition R. P. Agnew present inclusion theorems for the deferred and none Deferred Cesàro means. This paper is part 2 of a series of papers that present extensions to the notion of double Deferred Cesàro means. Similar to part 1 this paper uses this definition and the notion of regularity for four dimensional matrices, to present extensions and variations of the inclusion theorems presented by R. P. Agnew in [2].

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1 Introduction

This paper is part 2 of a series of papers characterization the inclusion between Cesàro means and double Deferred Cesàro means. In part 1[11] we presented the notion of double Deferred Cesàro means which is a multi-dimensional analog and Agnew’s Deferred Cesàro means in [2]. Using this notions and as series of basic results in [11], this paper present a series of inclusion theorems similar to the following: The double Cesàro mean includes $D_{m-1,q_m,n-1,p_n}$ be a Deferred Cesàro mean with $q_m = m$, $p_n = n; m \neq \alpha_1, \alpha_2, \ldots$ and $n \neq \beta_1, \beta_2, \ldots$ with

\[ q_{\alpha_i} = \alpha_{i+1} - 1; i = 1, 2, 3, \ldots, \alpha_m \]

and

\[ p_{\beta_j} = \beta_{j+1} - 1; j + 1, 2, 3, \ldots, \beta_n \]

where \( \{q_{\alpha_i}\} \) and \( \{p_{\beta_j}\} \) are increasing single dimensional sequences of integers such that $\alpha_m > m$ and $\beta_n > n$.

2 Definitions, notations and preliminary results

The definitions, notations, and preliminary results are similar to those in Part 1 [11] which are restated here for the purpose of completeness.

Definition 2.1 (Pringsheim, 1900). A double sequence $x = \{x_{k,l}\}$ has a Pringsheim limit $L$ (denoted by $\text{P-lim } x = L$) provided that, given an $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > N$. Such an $\{x\}$ is described more briefly as “P-convergent”.

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Definition 2.2 (Patterson, 2000). A double sequence \( \{y\} \) is a double subsequence of \( \{x\} \) provided that there exist increasing index sequences \( \{n_j\} \) and \( \{k_j\} \) such that, if \( \{x_j\} = \{x_{n_j,k_j}\} \), then \( \{y\} \) is formed by

\[
\begin{align*}
x_1 & \quad x_2 & \quad x_5 & \quad x_{10} \\
x_4 & \quad x_3 & \quad x_6 & \quad - \\
x_9 & \quad x_8 & \quad x_7 & \quad - \\
- & \quad - & \quad - & \quad -
\end{align*}
\]

In [13] Robison presented the following notion of conservative four-dimensional matrix transformation and a Silverman-Toeplitz type characterization of such notion.

Definition 2.3. The four-dimensional matrix \( A \) is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

The assumption of bounded was added because a double sequence which is P-convergent is not necessarily bounded. Along these same lines, Robison and Hamilton presented a Silverman-Toeplitz type multidimensional characterization of regularity in [3] and [13].

Theorem 2.4. (Hamilton [3], Robison [13]) The four-dimensional matrix \( A \) is RH-regular if and only if

\[
\begin{align*}
R\mathcal{H}_1: & \quad \text{P-lim}_{m,n} a_{m,n,k,l} = 0 \text{ for each } k \text{ and } l; \\
R\mathcal{H}_2: & \quad \text{P-lim}_{m,n} \sum_{k,l=0,0}^{\infty} a_{m,n,k,l} = 1; \\
R\mathcal{H}_3: & \quad \text{P-lim}_{m,n} \sum_{k=0}^{\infty} |a_{m,n,k,l}| = 0 \text{ for each } l; \\
R\mathcal{H}_4: & \quad \text{P-lim}_{m,n} \sum_{l=0}^{\infty} |a_{m,n,k,l}| = 0 \text{ for each } k; \\
R\mathcal{H}_5: & \quad \sum_{k,l=0,0}^{\infty,\infty} |a_{m,n,k,l}| \text{ is P-convergent}; \\
R\mathcal{H}_6: & \quad \text{there exist finite positive integers } \Delta \text{ and } \Gamma \text{ such that } \sum_{k,l>\Gamma} |a_{m,n,k,l}| < \Delta.
\end{align*}
\]

The main goals of this paper includes the comparison of double Cesàro mean transformation

\[
(C, 1, 1)_{m,n,k,l} := \begin{cases} \frac{1}{mn}, & \text{if } k \leq m \text{ and } l \leq n \\ 0, & \text{if otherwise} \end{cases}
\]

with the double Deferred Cesàro mean

\[
D_{m,n,k,l} := \begin{cases} \frac{1}{(\alpha_m - \beta_m)(q_n - p_n)}, & \text{if } \beta_m < k \leq \alpha_m \text{ and } p_n < l \leq q_n; \\ 0, & \text{if otherwise} \end{cases}
\]

where \([p_n]\) \([q_n]\) \([\alpha_m]\), and \([\beta_m]\) are sequences of nonnegative integers satisfying

\[
\alpha_m < \beta_m, \text{ and } p_n < q_n \text{ for } m,n = 1,2\ldots; \quad (1.1)
\]

and

\[
\lim_{m} \beta_m = +\infty, \text{ and } \lim_{n} q_n = +\infty. \quad (1.2)
\]

Using these four dimensional transformations we shall present a catalog of inclusion theorems such as the following. The four dimensional summability method \( M \) include \( D_{p_n,\alpha_n,\alpha_n,\beta_n} \) where \( p_n \) and \( q_n \) for almost all \( n \) is a give non-negative integer \( p \) if and only if \( \alpha_n \) and \( \beta_n \) are almost all positive integers.
3 Main results

Theorem 3.1. The Double Cesàro transformation includes every Double Deferred Cesàro mean of the form \(D_{p_n, \alpha_n, q_n, \beta_n}\) for which \(\alpha_n\) and \(\beta_n\) contains almost all positive integers.

Proof. Let \([x_{k,l}]\) be summable by \(D_{p_n, \alpha_n, q_n, \beta_n}\) (say to \(L\)) such that \(P - \lim_{m,n} D_{m,n} = L\) and choose two integers \(K\) and \(L\) large such that \([p_m]\) and \([q_n]\) contains all integers greater than \(K\) and \(L\), respectively. Thus let \(i_1 = i_2 = i_3 = \cdots = i_K = 1\) and \(j_1 = j_2 = j_3 = \cdots = j_L = 1\) and determine for \(m > K\) and \(n > L\) index \(i_m\) and \(j_n\) is such that \(p_{i_m} = m\) and \(q_{j_n} = n\). Since \(\lim_{m} i_m = +\infty\) and \(\lim_{n} j_n = +\infty\), it follows

\[
P - \lim_{m,n} D_{m,n} = L \text{ and } P - \lim_{m,n} D_{i_m,j_n} = L.
\]

Therefore \([x]\) is summable by \(D_{p_m, q_n, \alpha, \beta}\) to \(L\). The result follows from Lemma 3.3 of [11]. Q.E.D.

Theorem 3.2. The Double Cesàro transformation fails to contain includes \(D_{p_n, \alpha_n, q_n, \beta_n}\) if there exists an Pringsheim increasing sequence double sequence \([\alpha_k, i]\) of integers whose elements belong to neither \([p_m]\) nor \([q_n]\).

Proof. Let us consider the following

\[
M_{m,n} = \begin{cases} 
0, & \text{if } (m,n) \neq (\alpha_m, \beta_n); \quad m,n = 1,2,3,\ldots \\
x_{m,n}, & \text{if } (m,n) = (\alpha_m, \beta_n); \quad m,n = 1,2,3,\ldots 
\end{cases}
\]

where \([x]\) is a \(P\)-divergent double sequence. Let \([s_{m,n}]\) be double sequence that is mapped by \(M\) into \(\alpha_m, \beta_n\), and determine that \(D_{p_m, \alpha_m, q_n, \beta_n}\) sum \([x]\) to zero. Since \(M\) fails to sum \([x]\).

The following theorem follows from Theorem 3.1 and 3.2.

Theorem 3.3. The four dimensional summability method \(M\) include \(D_{p_n, \alpha_n, q_n, \beta_n}\) where \(p_n\) and \(q_n\) for almost all \(n\) is a give non-negative integer \(p\) if and only if \(\alpha_n\) and \(\beta_n\) are almost all positive integers.

Theorem 3.4. The four dimensional summability method \(M\) include \(D_{m-1, q_m, n-1, \beta_n}\) where \(q_m - m\) and \(p_n - n\) both increases monotonically with \(m\) and \(n\), respectively if and only if \(q_m - m\) and \(p_n - n\) both are both bounded.

Proof. To establish to sufficiency part not that \(q_m - m\) and \(p_n - n\) must have a limit, say \(\alpha\) and \(\beta\), respectively and that \(q_m - m = \alpha\) and \(p_n - n = \beta\) for almost all \(m\) and \(n\). Thus \([q_m]\) and \([p_n]\) contains almost all positive integers and Theorem 3.1 grants us the results.

To established the necessary part, suppose \(q_m - m\) and \(p_n - n\) increases monotonically with \(m\) and \(n\) are both unbounded. The goal now is to show that the set of double sequences that are double Cesàro summable are not summable by the double Deferred Cesàro mean. Let \(m_1 = n_1 = 1\) and \(m_2\) and \(n_2\) are the smallest integers such that

\[
q_m - m > q_{m_1} - m_1 \text{ and } p_n - n > p_{n_1} - n_1
\]

Then choose \(m_3\) and \(n_3\) to be the smallest integers \(m\) and \(n\) such that

\[
q_m - m > q_{m_2} - m_2 \text{ and } p_n - n > p_{n_2} - n_2.
\]
Thus having chosen

\[ m_1 < m_2 < \cdots < m_α \] \[ n_1 < n_2 < \cdots < n_β. \]

We then choose \( m_{α+1} \) and \( n_{β+1} \) to be the smallest integers such that

\[ q_m - m > q_{m_α} - m_α \] \[ p_n - n > p_{n_β} - n_β. \]

We then define a double sequence \( \{s_{k,l}\} \) as follows:

\[
s_{k,l} = \begin{cases} 
q_m p_n, & \text{if } k = q_m \text{ and } l = p_n; i, j = 1, 2, 3, \ldots \\
k l, & \text{if } k \neq q_m \text{ and/or } l \neq p_n; i, j = 1, 2, 3, \ldots 
\end{cases}
\]

Note \( D_{m, n} \) maps \( \{s_{k,l}\} \) into 1. for all \((m, n)\). Thus \( \{s_{k,l}\} \) is D-summable to 1. Also \( \{s_{k,l}\} \) is not M-summable, since \( P^-\lim_{k,l} s_{k,l} \neq 0 \). Thus the double Cesàro mean is contained in the double Deferred Cesàro mean. Q.E.D.

**Theorem 3.5.** Let \( D_{m-1,q_m,n-1,p_n} \) be a Deferred Cesàro mean with \( q_m = m, p_n = n, m \neq α_1, α_2, \ldots \) and \( n \neq β_1, β_2, \ldots \) with

\[ q_{α_i} = α_{i+1} - 1; i = 1, 2, 3, \ldots, α_m \]

and

\[ p_{β_j} = β_{j+1} - 1; j + 1, 2, 3, \ldots, β_n \]

where \( \{q_{α_i}\} \) and \( \{p_{β_j}\} \) are increasing single dimensional sequences of integers such that \( α_m > m \) and \( β_n > n \). Then \( D \) is included in \( M \) if and only if \( \frac{q_m}{m} \) and \( \frac{p_n}{n} \) are bounded for all \( m \) and \( n \).

**Proof.** Note \( D_{m-1,m,n-1,n} \) is the identity transformation. Let us consider the ordered pair \((m, n)\) and observe that for each pair \((m, n)\), let

\[ i = i_m \text{ and } j = j_n \]

be such that \( α_i < m < α_{i+1} \) and \( β_j < n < β_j \). Let \( \{s_{m,n}\} \) be a given double sequence and consider the transformation

\[
M_{m,n} = \frac{1}{mn} \begin{bmatrix}
s_{1,1} + s_{1,2} + s_{1,3} + \cdots + s_{1,n} \\
2,1 + s_{2,2} + s_{2,3} + \cdots + s_{2,n} \\
\vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots 
\mbox{ } \\
s_{m,1} + s_{m,2} + s_{m,3} + \cdots + s_{m,n}
\end{bmatrix}
\]
Using the definition of double Deferred Cesàro mean we obtain the following

\[
\begin{bmatrix}
  s_{1,1} + \cdots + s_{1,\beta_1-1} \\
  s_{2,1} + \cdots + s_{2,\beta_1-1} \\
  \vdots & \ddots & \vdots \\
  s_{\alpha_1-1,1} + \cdots + s_{\alpha_1-1,\beta_1-1} \\
  s_{\alpha_1,1} + \cdots + s_{\alpha_1,\beta_1-1} \\
  s_{\alpha_1+1,1} + \cdots + s_{\alpha_1+1,\beta_1-1} \\
  \vdots & \ddots & \vdots \\
  s_{\alpha_2-1,1} + \cdots + s_{\alpha_2-1,\beta_1-1} \\
  s_{\alpha_2,1} + \cdots + s_{\alpha_2,\beta_1-1} \\
  s_{\alpha_2+1,1} + \cdots + s_{\alpha_2+1,\beta_1-1} \\
  \vdots & \ddots & \vdots \\
  s_{\alpha_3-1,1} + \cdots + s_{\alpha_3-1,\beta_1-1} \\
\end{bmatrix}
+ \cdots +
\begin{bmatrix}
  s_{1,\beta_j} + \cdots + s_{1,\beta_{j+1}-1} \\
  s_{2,1} + \cdots + s_{2,\beta_1-1} \\
  \vdots & \ddots & \vdots \\
  s_{\alpha_1-1,\beta_j} + \cdots + s_{\alpha_1-1,\beta_{j+1}-1} \\
  s_{\alpha_1,\beta_j} + \cdots + s_{\alpha_1,\beta_{j+1}-1} \\
  s_{\alpha_1+1,\beta_j} + \cdots + s_{\alpha_1+1,\beta_{j+1}-1} \\
  \vdots & \ddots & \vdots \\
  s_{\alpha_2-1,\beta_j} + \cdots + s_{\alpha_2-1,\beta_{j+1}-1} \\
  s_{\alpha_2,\beta_j} + \cdots + s_{\alpha_2,\beta_{j+1}-1} \\
  s_{\alpha_2+1,\beta_j} + \cdots + s_{\alpha_2+1,\beta_{j+1}-1} \\
  \vdots & \ddots & \vdots \\
  s_{\alpha_3-1,\beta_j} + \cdots + s_{\alpha_3-1,\beta_{j+1}-1} \\
\end{bmatrix}
\]

Let us denote the above sum by \( \Omega_{m,n} \) and the sum below by \( \Lambda_{m,n} \)

\[
\begin{bmatrix}
  s_{1,n+1} + \cdots + s_{1,\beta_{j+1}-1} \\
  s_{2,n+1} + \cdots + s_{2,\beta_{j+1}-1} \\
  \vdots & \ddots & \vdots \\
  s_{m,n+1} + \cdots + s_{m,\beta_{j+1}-1} \\
  s_{m+1,1} + \cdots + s_{m+1,n+1} + s_{m+1,\beta_{j+1}-1} \\
  s_{m+2,1} + \cdots + s_{m+2,n+1} + s_{m+2,\beta_{j+1}-1} \\
  \vdots & \ddots & \vdots \\
  s_{\alpha_{i+1},1} + \cdots + s_{\alpha_{i+1},\beta_{j+1}-1} \\
\end{bmatrix}
+ \cdots +
\begin{bmatrix}
  s_{1,\beta_{j+1}} + \cdots + s_{1,\beta_{j+1}-1} \\
  s_{2,\beta_{j+1}} + \cdots + s_{2,\beta_{j+1}-1} \\
  \vdots & \ddots & \vdots \\
  s_{\alpha_{i+1},\beta_{j+1}} + \cdots + s_{\alpha_{i+1},\beta_{j+1}-1} \\
\end{bmatrix}
\]

Therefore \( M_{m,n} = \frac{1}{mn} (\Omega_{m,n} - \Lambda_{m,n}) \). It is important to observe that if \( m = \alpha_{i+1} - 1 \) and/or \( n = \beta_{j+1} - 1 \) then the terms in \( \Lambda_{m,n} \) will not exist that is if \( m = \alpha_{i+1} - 1 \) and/or \( n = \beta_{j+1} - 1 \) then the terms in the rows and/or columns will not exists. Let us also denote the following sum by
The double Cesàro mean includes

\[ \frac{1}{mn} m,n \sum_{k,l=1}^{\alpha_1+\beta_1} s_{k,l} + \frac{1}{mn} m,n \frac{\alpha_1(\beta_2-\beta_1)}{D_{0,1}} D_{0,1} + \cdots + \frac{1}{mn} m,n \frac{\alpha_1(\beta_j+1-\beta_i)}{D_{0,j}} D_{0,j} \]

\[ \frac{1}{mn} m,n \sum_{k,l=1}^{\alpha_2+\beta_2} s_{k,l} + \frac{1}{mn} m,n \frac{(\alpha_2-\alpha_1)(\beta_2-\beta_1)}{D_{1,1}} D_{1,1} + \cdots + \frac{(\alpha_2-\alpha_1)(\beta_j+1-\beta_i)}{D_{1,j}} D_{1,j} \]

\[ \frac{1}{mn} m,n \sum_{k,l=1}^{\alpha_3+\beta_3} s_{k,l} + \frac{1}{mn} m,n \frac{(\alpha_3-\alpha_2)(\beta_2-\beta_1)}{D_{2,1}} D_{2,1} + \cdots + \frac{(\alpha_3-\alpha_2)(\beta_j+1-\beta_i)}{D_{2,j}} D_{2,j} \]

\[ + \cdots + + + + \]

and also denote the following sum by \( \bar{\Lambda}_{m,n} \)

\[ D_{1,n+1} + \cdots + D_{1,\beta_{j+1}+1} \]

\[ D_{2,n+1} + \cdots + D_{2,\beta_{j+1}+1} \]

\[ \vdots + \cdots + \]

\[ D_{m,n+1} + \cdots + D_{m,\beta_{j+1}+1} \]

\[ D_{m+1,1} + \cdots + D_{m+1,n} + D_{m+1,n+1} + \cdots + D_{m+1,\beta_{j+1}+1} \]

\[ D_{m+2,1} + \cdots + D_{m+2,n} + D_{m+2,n+1} + \cdots + D_{m+2,\beta_{j+1}+1} \]

\[ \vdots + \cdots + \]

\[ D_{\alpha_{i+1},1} + \cdots + D_{\alpha_{i+1},n} + D_{\alpha_{i+1},n+1} + \cdots + D_{\alpha_{i+1},\beta_{j+1}+1} \]

Then we can now rewrite \( M_{m,n} \) in the following manner \( \bar{\Omega}_{m,n} = \frac{1}{mn} \bar{\Lambda}_{m,n} \). The relation \( \bar{\Omega}_{m,n} = \frac{1}{mn} \bar{\Lambda}_{m,n} \) hold for each \( (m,n) \) and defines a four-dimensional transformation of the form

\[ \sigma_{m,n} = \sum_{k,l=1}^{\infty,\infty} a_{m,n,k,l} s_{k,l} \]

which carries \( D_{m,n} \) into \( M_{m,n} \). This transformation clearly satisfies RH1 and RH2. This transformation satisfies RH3 and RH4 only if \( 2\alpha_{i+1} - m - 2 \) and \( 2\beta_{j+1} - n - 2 \) are bounded respectively for each \( (m,n) \), which is equivalent to \( \frac{\alpha_{i+1}+1}{m} \) and \( \frac{\alpha_{i+1}+1}{m} \) are bounded, which is also equivalent to the boundedness of \( \frac{q_m}{m} \) and \( \frac{p_n}{n} \) for each \( (m,n) \). Condition RH5 and RH6 hold only when both \( \frac{2\alpha_{i+1} - m - 2}{m} \) and \( \frac{2\beta_{j+1} - n - 2}{n} \) are bounded, and as above the is equivalent to boundedness of \( \frac{q_m}{m} \) and \( \frac{p_n}{n} \) for each \( (m,n) \). Since \( D \) is a factorable four-dimensional summability matrix the main theorem in [1] assure us that it has an inverse. Thus the result follows for the Robison-Hamilton characterization of regularity.

Q.E.D.

Theorem 3.6. The double Cesàro mean includes \( D_{m-1,q_m,n-1,p_n} \) be a Deferred Cesàro mean with \( q_m = m, p_n = n; m \neq \alpha_1, \alpha_2, \ldots \) and \( n \neq \beta_1, \beta_2, \ldots \) with

\[ q_{\alpha_i} = \alpha_{i+1} - 1; i = 1, 2, 3, \ldots, \alpha_m \]

and

\[ p_{\beta_j} = \beta_{j+1} - 1; j = 1, 2, 3, \ldots, \beta_n \]

where \( \{q_{\alpha_i}\} \) and \( \{p_{\beta_j}\} \) are increasing single dimensional sequences of integers such that \( \alpha_m > m \) and \( \beta_n > n \).
Proof. Observe that for each pair \((m, n)\), let

\[ i = i_m \text{ and } j = j_n \]

be such that \(h_i \leq m < h_{i+1}\) and \(t_j \leq n < t_{j+1}\). Let \(\{s_{m,n}\}\) be a given double sequence and consider the following four dimensional Cesàro transformation

\[
M_{m,n} = \frac{1}{mn} \begin{bmatrix}
    s_{1,1} + s_{1,2} + \cdots + s_{1,n} \\
    s_{2,1} + s_{2,2} + \cdots + s_{2,n} \\
    \vdots \\
    s_{m,1} + s_{m,2} + \cdots + s_{m,n}
\end{bmatrix}.
\]

Using the definition of double Deferred Cesàro mean we can rewrite \(mnM_{m,n}\) using the following, respectively, \(A^i_{m,n}, A^{i-1}_{m,n}, A^{i-2}_{m,n} \cdots, A^\alpha_{m,n}\) and \(K_{m,n}\) where \(K_{m,n}\) is

\[
s_{1,1} + s_{1,2} + \cdots + s_{1,\beta_3-1} \\
    s_{2,1} + s_{2,2} + \cdots + s_{2,\beta_3-1} \\
    \vdots \\
    s_{\beta_\Delta-1,1} + s_{\beta_\Delta-1,2} + \cdots + s_{\beta_\Delta-1,\beta_\Delta-1}
\]

with \(\Delta\) and \(\delta\) are 2 or 1 depending on weather \(\alpha\) and/ or \(\beta\) are odd or even, and the \(A\)'s are define below, respectively

\[
s_{m,n} + s_{m,n-1} + \cdots + s_{m,t_j+1} \\
    s_{m-1,n} + s_{m-1,n-1} + \cdots + s_{m-1,t_j+1} \\
    \vdots \\
    s_{h_i+1,n} + s_{h_i+1,n-1} + \cdots + s_{h_i+1,t_j+1}
\]

\[
s_{h_i,n} + s_{h_i,n-1} + \cdots + s_{h_i,t_j+1} + s_{\alpha_i,t_j} + \cdots + s_{h_i,t_j-1} \\
    \vdots \\
    s_{h_i-1,n} + s_{h_i-1,n-1} + \cdots + s_{h_i-1,t_j+1} + s_{h_i-1,t_j} + \cdots + s_{h_i-1,t_j-1}
\]

\[
s_{h_i-1,n} + s_{h_i-1,n-1} + \cdots + s_{h_i-1,t_j+1} + s_{h_i-1,t_j} + \cdots + s_{h_i-1,t_j-1} \\
    \vdots \\
    s_{h_i,n} + s_{h_i,n-1} + \cdots + s_{h_i,t_j+1} + s_{\alpha_i,t_j} + \cdots + s_{h_i,t_j-1} \\
    \vdots \\
    s_{m,n} + s_{m,n-1} + \cdots + s_{m,t_j+1} \\
    s_{m-1,n} + s_{m-1,n-1} + \cdots + s_{m-1,t_j+1} \\
    \vdots \\
    s_{m,n} + s_{m,n-1} + \cdots + s_{m,t_j+1}
\]
\[ s_{m,t_j-2} + \cdots + s_{m,t_j-3} + s_{m,t_j-1} \]

\[ : + \cdots + : \]

\[ s_{h_{i-2},n} + s_{h_{i-2},n-1} + \cdots + s_{h_{i-2},t_j+1} + s_{h_{i-2},t_j-2} + \cdots + s_{h_{i-2},t_j-3} + s_{h_{i-2},t_j-1} , \]

\[ \vdots + \vdots + \cdots + \vdots + \vdots + \cdots + \vdots \]

\[ s_{h_{i-3}+1,n} + s_{h_{i-3}+1,n-1} + \cdots + s_{h_{i-3}+1,t_j+1} + s_{h_{i-3}+1,t_j-2} + \cdots + s_{h_{i-3}+1,t_j-3} + s_{h_{i-3}+1,t_j-1} \]

\[ s_{m,t_{\beta+1}-2} + \cdots + s_{m,t_{\beta}} \]

\[ \vdots + \cdots + \vdots \]

\[ s_{h_{\alpha+1},n} + s_{h_{\alpha+1},n-1} + \cdots + s_{h_{\alpha+1},t_{\beta+1}-1} + s_{h_{\alpha+1},t_{\beta+1}-2} + \cdots + s_{h_{\alpha+1},t_{\beta}} \]

\[ \vdots + \vdots + \cdots + \vdots + \vdots + \cdots + \vdots \]

\[ s_{h_{\alpha+1},n} + s_{h_{\alpha+1},n-1} + \cdots + s_{h_{\alpha+1},t_{\beta+1}-1} + s_{h_{\alpha+1},t_{\beta+1}-2} + \cdots + s_{h_{\alpha+1},t_{\beta}} \]

It is clear that

\[ M_{m,n} = \frac{1}{mn} \left[ A_{m,n}^i + A_{m,n}^{i-1} + A_{m,n}^{i-2} + \cdots + A_{m,n}^0 + K_{m,n} \right] . \]

Now using the above identities we can rewrite our equation as follow

\[ T_{m,n} = M_{m,n} - \frac{K_{m,n}}{mn} \]

and the D’s grants us the following:

\[
\begin{bmatrix}
D_{m,n} & D_{m,n-1} & \cdots & D_{m,\beta_j+1} \\
D_{m-1,n} & D_{m-1,n-1} & \cdots & D_{m-1,\beta_j+1} \\
\vdots & \vdots & \cdots & \vdots \\
D_{\alpha_i+1,n} & D_{\alpha_i+1,n-1} & \cdots & D_{\alpha_i+1,\beta_j+1}
\end{bmatrix}
\]

\[
\left( \frac{\alpha_i-\alpha_{i-1}+1}{mn} \right) (\beta_j+1-n) D_{\alpha_i,n} + \left( \frac{\alpha_i-\alpha_{i-1}+1}{mn} \right) (\beta_j-\beta_{j-1}+1-n) D_{\alpha_{i-1},\beta_{j-1}}
\]

\[
+ \left( \frac{\alpha_i-m+1}{mn} \right) (\beta_j-n) D_{m,\beta_j}
\]

\[
+ \left( \frac{\alpha_{i+1}-m+1}{mn} \right) (\beta_{\Delta+1}-\beta_{\Delta}+1) D_{m,\beta_{\Delta}}
\]

\[
+ \left( \frac{\alpha_{i+1}-\alpha_{i+1}}{mn} \right) (\beta_{\Delta+1}-\beta_{\Delta}+1) D_{\alpha_{i+1},\beta_{\Delta}}
\]

Since the above equalities define four dimensional RH-regular transformation from \{D_{m,n}\} to \{T_{m,n}\} we are granted the if the double sequence \{D_{m,n}\} convergence to \(L\) in the Pringsheim sense then \{T_{m,n}\} convergence to \(L\) in the Pringsheim sense and since the double sequence \{K_{m,n}\} is bounded then \{M_{m,n}\} convergence to \(L\) in the Pringsheim sense. Thus double Cesàro means includes double Deferred Cesàro means. The completes the proof.

Q.E.D.
References


