ASYMPTOTIC PROPERTIES OF TRINOMIAL DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. New criteria for asymptotic properties of the solutions of the third order delay differential equation

\[
\left(\frac{1}{r(t)} y''(t)\right)' - p(t)y'(t) + g(t)y(\tau(t)) = 0,
\]

by transforming this equation to its binomial canonical form are presented.

We consider the third order delay differential equation

\[
\left(\frac{1}{r(t)} y''(t)\right)' - p(t)y'(t) + g(t)y(\tau(t)) = 0
\]

and the corresponding second order differential equation

\[
\left(\frac{v'(t)}{r(t)}\right)' = p(t)v(t).
\]

We always assume that

- (H1) \( r(t), p(t) \) and \( g(t) \in C([t_0, \infty)) \), \( p(t) \geq 0, r(t) > 0, g(t) > 0 \);
- (H2) \( \tau(t) \in C([t_0, \infty)), \tau(t) \leq t \) and \( \tau(t) \to \infty \) as \( t \to \infty \);
- (H3) \( \int_{t_0}^{\infty} r(s) \, ds = \infty \).

The particular case of (1), namely differential equation without delay

\[
y'''(t) - p(t)y'(t) + g(t)y(t) = 0
\]

has been investigated in earlier papers [1, 3, 4, 7, 10, 12, 17] and [20]. In these papers the following structure of nonoscillatory solutions is presented:
**Lemma 1.** Let \( y(t) \) be a nonoscillatory solution of (3). Then there exists \( t_1 \geq t_0 \) such that either
\[
y(t)y'(t) < 0
\]
or
\[
y(t)y'(t) \geq 0
\]
for \( t \geq t_1 \) and moreover if \( y(t) \) satisfies (4), then also
\[
(-1)^i y(t)y^{(i)}(t) > 0, \quad 0 \leq i \leq 3, \quad t \geq t_1.
\]

It is known that equation (3) always has a solution satisfying (6). For the next considerations we say that (3) has property \((P_0)\) if every nonoscillatory solution of (3) satisfies (6).

Lazer presented a sufficient condition for property \((P_0)\) of (3) (see [17]). This result has been improved by several authors (see, e.g., [6, 10, 12, 19] and [20]). We denote
\[
D_0y = y, \quad D_1y = (D_0y)', \quad D_2y = \frac{1}{r}(D_1y)', \quad D_3y = (D_2y)'.
\]

We say that (1) has property \((P_0)\) if every nonoscillatory solution \( y(t) \) of (1) satisfies
\[
(-1)^i y(t)D_iy(t) \geq 0, \quad 0 \leq i \leq 3.
\]

**Lemma 2.** The operator \( Ly = \left( \frac{1}{r(t)}y''(t) \right)' - p(t)y'(t) \) can be written as
\[
Ly = \frac{1}{v} \left( \frac{v^2}{r} \left( \frac{1}{v} y' \right)' \right),
\]
where \( v(t) \) is a positive solution of (2).

**Proof.** Straightforward computation shows that
\[
Ly = \frac{1}{v} \left( \frac{v^2}{r} \left[ y'' \frac{1}{r} \frac{y'}{v} - \frac{v'}{rv^2} y' \right] \right)' = \left( \frac{y''}{r} \right)' - \left( \frac{v'}{r} \right)' \frac{y'}{v} = \left( \frac{1}{r(t)}y''(t) \right)' - p(t)y'(t).
\]

**Corollary 1.** If \( v(t) \) is a positive solution of (2), then equation (1) can be rewritten as
\[
\left( \frac{v^2}{r} \left( \frac{1}{v} y' \right)' \right)' + v(t)g(t)y(\tau(t)) = 0.
\]
We shall study properties of trinomial equation (1) with help of its binomial form (8). The properties of canonical equations are well known. For this reason it is useful for (8) to be in the canonical form. The equation (8) is in the canonical form if
\[
\int_0^\infty \frac{r(t)}{v(t)} \, dt = \infty \quad \text{and} \quad \int_0^\infty v(t) \, dt = \infty.
\]

\textbf{Lemma 3.} Equation (2) possesses the following couple of solutions
\[
v(t) > 0, \quad v'(t) \leq 0 \quad \text{and} \quad \left( \frac{v'(t)}{r(t)} \right)' \geq 0 \tag{9}
\]
and
\[
v(t) > 0, \quad v'(t) \geq 0 \quad \text{and} \quad \left( \frac{v'(t)}{r(t)} \right)' \geq 0, \tag{10}
\]
for all \( t \) large enough.

We say that a solution \( v(t) \) of (2) is of degree 0 if it satisfies (9), on the other hand \( v(t) \) is said to be of degree 2 if it satisfies (10).

\textbf{Lemma 4.} Let (H3) hold. If \( v(t) \) is a solution of degree 0 of (2), then
\[
\int_{t_0}^\infty \frac{r(t)}{v^2(t)} \, dt = \infty.
\]

\textbf{Proof.} It is easy to see that \( v(t) \) satisfies \( c > v(t) \), eventually which implies assertion of the lemma. \( \square \)

A solution \( v(t) \) of degree 0 is the key solution of (2). If it satisfies the condition
\[
\int_{t_0}^\infty v(t) \, dt = \infty, \tag{11}
\]
then equation (1) can be represented in the canonical form (8).

We present sufficient condition for every solution of degree 0 of (2) to satisfy (11). Let us denote \( \bar{P}_r(t) = r(t) \int_t^\infty p(s) \, ds \) (we suppose that \( \int_{t_0}^\infty p(s) \, ds < \infty \)).

\textbf{Lemma 5.} Assume that
\[
\int_{t_0}^\infty e^{-\int_{t_0}^t \bar{P}_r(s) \, ds} \, dt = \infty. \tag{12}
\]
Then every solution of degree 0 of (2) satisfies (11).

\textbf{Proof.} Let \( v(t) \) satisfy (9). Integrating (2) from \( t \) to \( \infty \), one gets
\[
\ell - \frac{v'(t)}{r(t)} = \int_t^\infty p(s) v(s) \, ds,
\]
where $\ell = \lim_{t \to \infty} \frac{v'(t)}{r(t)}$. We claim that $\lim_{t \to \infty} \frac{v'(t)}{r(t)} = 0$. If not, then $\lim_{t \to \infty} \frac{v'(t)}{r(t)} = \ell$, $\ell < 0$. Then $v'(t) \leq \ell r(t)$. Integrating from $t_1$ to $t$, we have $v(t) \leq v(t_1) + \ell \int_{t_1}^{t} r(t) \, ds \to -\infty$ as $t \to \infty$. This is a contradiction. Thus we conclude that

$$v'(t) = r(t) \int_{t}^{\infty} p(s)v(s) \, ds \leq v(t)r(t) \int_{t}^{\infty} p(s) \, ds = v(t)\tilde{P}_r(t).$$

Then integrating from $t_1$ to $t$, we have

$$v(t) = v(t_1) e^{-\int_{t_1}^{t} \tilde{P}_r(s) \, ds}.$$ Integrating from $t$ to $\infty$ we get the desired property. \hfill \Box

To simplify our notation, we set

$$L_0y = y, \quad L_1y = \frac{1}{v}(L_0y)', \quad L_2y = \frac{v^2}{r}(L_1y)', \quad L_3y = (L_2y)'.$$ We present structure of nonoscillatory solutions $y(t)$ of (8), where (8) is in canonical form (see, e.g., [9] or [14]).

**Lemma 6.** Let (8) be in canonical form. Every positive solution of (8) satisfies either

$$L_0y(t) > 0, \quad L_1y(t) < 0, \quad L_2y(t) > 0, \quad L_3y(t) < 0 \quad (13)$$

or

$$L_0y(t) > 0, \quad L_1y(t) > 0, \quad L_2y(t) > 0, \quad L_3y(t) < 0 \quad (14)$$

for large $t$.

**Remark 1.** It follows from Lemma 6 that if (8) is in canonical form, then the set of nonoscillatory solutions of (1) has the same structure as those of (2). That is

$$y(t)y'(t) < 0$$

or

$$y(t)y'(t) > 0,$$

where $y(t)$ is a nonoscillatory solutions of (1). Note that the corresponding inequality (5) is sharp for (1).
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So $D_2y$ is of constant sign. If we admit $D_2y < 0$, then $D_1y = y'$ is decreasing so $y'(t) \leq y'(t_1) = \ell < 0$. Integrating from $t_1$ to $t$ we get

$$y(t) \leq y(t_1) + \ell(t_1 - t) \to -\infty \quad \text{as} \quad t \to \infty.$$  

A contradiction with the positivity of $y(t)$. So we conclude $D_2y > 0$ and $y$ satisfies (7).

Now, $D_0y > 0$ and $D_1y < 0$ implies $L_0y > 0$ and $L_1y < 0$. It follows from (8) that $L_3y < 0$. Then $L_2y(t)$ is decreasing. If we admit $L_2y(t) < 0$ for $t \geq t_1$, then $L_1y(t) \leq \ell < 0$ and integrating from $t_1$ to $t$ one gets

$$y(t) < y(t_1) + \ell \int_{t_1}^{t} v(s) \, ds \to -\infty \quad \text{as} \quad t \to \infty.$$  

□

Following Kiguradze [13] we say that (8) has property (A) if every its nonoscillatory solution is of degree 0. Lemma 7 can now be formulated in the form:

**Theorem 1.** If $v(t)$ is a positive solution of degree 0 of (2) satisfying (11), then (1) has property $(P_0)$ if and only if (8) has property (A).

**Theorem 2.** Let $v(t)$ be a positive solution of degree 0 of (2) satisfying (11). Let

$$\tau'(t) > 0.$$  

(15)

If the differential inequality

$$\left( \frac{v^2(t)}{\tau(t)} z'(t) \right)' + \left[ v(\tau(t)) \tau'(t) \int_{t}^{\infty} v(s) g(s) \, ds \right] z(\tau(t)) \leq 0$$  

(16)

has no positive solution, then (1) has property $(P_0)$.

**Proof.** Assume that (1) has not property $(P_0)$. It implies from Theorem 1 that (8) has not property (A). Consequently a solution $y(t)$ exists that satisfies (14).

Integrating (8) from $t$ to $\infty$, we have

$$L_2y(t) = c + \int_{t}^{\infty} v(s) g(s) y(\tau(s)) \, ds,$$

(17)

where $c = \lim_{t \to \infty} L_2y(t)$. Since

$$y(t) = y(t_1) + \int_{t_1}^{t} v(x) L_1y(x) \, dx,$$

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then it implies from (17) that

\[ L_2 y(t) \geq \int_t^{\infty} v(s) g(s) \left( \int_{t_1}^{\tau(s)} v(x) L_1 y(x) \, dx \right) \, ds. \]

We may assume that \( \tau(t) > t_1 \), so

\[ L_2 y(t) \geq \int_t^{\tau(t)} v(s) g(s) \int_{\tau^{-1}(s)}^{\tau(s)} v(x) L_1 y(x) \, dx \, ds. \]

After changing the order of integration we get

\[ L_2 y(t) \geq \int_{\tau(t)}^{\infty} v(x) L_1 y(x) \int_{\tau^{-1}(x)}^{\infty} v(s) g(s) \, ds \, dx = \int_{\tau(t)}^{\infty} L_1 y(x) G(x) \, dx, \]

where

\[ G(x) = v(x) \int_{\tau^{-1}(x)}^{\infty} v(s) g(s) \, ds, \]

and \( \tau^{-1}(t) \) is the inverse function to \( \tau(t) \). Integration from \( t_1 \) to \( t \) leads to

\[ L_1 y(t) \geq \int_{t_1}^{t} \frac{r(s)}{v^2(s)} \int_{\tau(s)}^{\tau^{-1}(s)} L_1 y(x) G(x) \, dx. \] (18)

Let us denote the right hand side of (18) by \( z(t) \). Then \( z(t) > 0 \) and

\[ 0 = \left( \frac{v^2(t)}{r(t)} z'(t) \right)' + L_1 y(\tau(t)) G(\tau(t)) \tau'(t) \geq \left( \frac{v^2(t)}{r(t)} z'(t) \right)' + G(\tau(t)) \tau'(t) z(\tau(t)) \]

so \( z(t) \) is a positive solution of (16). A contradiction. \( \square \)

**Corollary 2.** Let \( v(t) \) be a positive solution of degree 0 of (2) satisfying (11). Let (15) hold. If the second order differential equation

\[ \left( \frac{v^2(t)}{r(t)} z'(t) \right)' + \left[ v(\tau(t)) \tau'(t) \int_t^{\infty} v(s) g(s) \, ds \right] z(\tau(t)) = 0 \] (19)

is oscillatory, then (11) has property \((P_0)\).

It is known that (19) is oscillatory if and only if (16) has no positive solution (see [9], [14]).
THEOREM 3. Let \( v(t) \) be a positive solution of degree 0 of (2) satisfying (11). Let (15) hold. If

\[
\liminf_{t \to \infty} \left( \int_{t_1}^{\tau(t)} \frac{r(s)}{v^2(s)} \, ds \right) \left( \int_{t}^{\infty} v(x) \tau'(x) \int_{x}^{\infty} v(s)g(s) \, ds \, dx \right) > \frac{1}{4}
\]  

(20)

then equation (1) has property (P). 

Proof. Condition (20) guaranties oscillation of (19) (see, e.g., [8, 9] or [14]). " 

EXAMPLE 1. Let us consider delay differential equation

\[
\left( \frac{1}{3t} y''(t) \right)' - \frac{1}{t^3} y'(t) + g(t)y(\lambda t) = 0, \quad 0 < \lambda < 1.
\]  

(21)

Now \( v(t) = \frac{1}{t} \) is a solution of degree 0 of equation \( \left( \frac{v'(t)}{3t} \right)' = \frac{1}{t^3} v(t) \). Moreover, it satisfies (11), then follows from Theorem 3 that (21) satisfies (P) if

\[
\liminf_{t \to \infty} \lambda^4 t^4 \int_{s}^{\infty} \frac{g(s)}{s} \ln \left| \frac{s}{t} \right| \, ds > \frac{1}{3}.
\]  

(22)

THEOREM 4. Let all assumptions of Theorem 3 hold. Moreover, if

\[
\int_{t_0}^{\infty} v(s_3) \int_{s_2}^{\infty} \frac{r(s_1)}{v^2(s_1)} \int_{s_1}^{\infty} v(s_0)g(s_0) \, ds_0 \, ds_1 \, ds_2 = \infty,
\]  

(23)

then every nonoscillatory solution of (11) tends to zero as \( t \to \infty \). 

Proof. Let \( y(t) > 0 \) be a solution of (11). From Theorem 3 it follows that \( y(t) \) satisfies (7), so \( y'(t) < 0 \). There exists \( \lim_{t \to \infty} y(t) = \ell \geq 0 \). If \( \ell > 0 \) then \( y(\tau(t)) > \ell \). Integrating (8) twice from \( t \to \infty \) and then from \( t_1 \) to \( t \), we have

\[
y(t) = y(t_1) - \int_{t_1}^{t} v(s_2) \int_{s_2}^{\infty} \frac{r(s_1)}{v^2(s_1)} \int_{s_1}^{\infty} v(s_0)g(s_0)y(\tau(s_0)) \, ds_0 \, ds_1 \, ds_2
\]

(24)

\[
y(t) \leq y(t_1) - \ell \int_{t_1}^{t} v(s_2) \int_{s_2}^{\infty} \frac{r(s_1)}{v^2(s_1)} \int_{s_1}^{\infty} v(s_0)g(s_0) \, ds_0 \, ds_1 \, ds_2 \to -\infty
\]

as \( t \to \infty \). Therefore, \( \lim_{t \to \infty} y(t) = 0 \). 

EXAMPLE 2. We consider a differential equation

\[
\left( \frac{1}{3t} y''(t) \right)' - \frac{1}{t^3} y'(t) + \frac{a}{t^3} y(\lambda t) = 0, \quad 0 < \lambda < 1, \quad a > 0.
\]  

(25)
It follows from Theorem 3 that \((E_3)\) has a property \((P_0)\) and from Theorem 4 that every nonoscillatory solution of \((25)\) tends to zero as \(t \to \infty\).

Now we recall the following result (see, e.g., [9] or [14]), which enables us to deduce property \((A)\) of delay equation from that of equation without delay:

**Theorem 5.** Assume that \(v(t)\) is a solution of degree 0 of \((2)\) satisfying \((11)\). Let \((15)\) hold. If
\[
\left(\frac{v^2(t)}{r(t)} \left(\frac{1}{v(t)} y'(t)\right)\right)' + \frac{v(\tau^{-1}(t))g(\tau^{-1}(t))}{r'(\tau^{-1}(t))} y(t) = 0
\]
has property \((A)\), then so does \((8)\).

When we combine Theorem 1 and Theorem 5, we get:

**Theorem 6.** Assume that \(v(t)\) is a solution of degree 0 of \((2)\) satisfying \((11)\). Let \((15)\) hold. If \((26)\) has property \((A)\) then \((1)\) has property \((P_0)\).

Applying Theorem 1 to \((26)\), we get

**Theorem 7.** Assume that \(v(t)\) is a solution of degree 0 of \((2)\) satisfying \((11)\). Let \((15)\) hold. Then \((??)\) has property \((A)\) if and only if
\[
\left(\frac{1}{v(t)} y''(t)\right)' - p(t)y'(t) + \frac{v(\tau^{-1}(t))g(\tau^{-1}(t))}{v(t)r'(\tau^{-1}(t))} y(t) = 0
\]
has property \((P_0)\).

Finally Theorems 6 and 7 provide:

**Theorem 8.** Assume that \(v(t)\) is a solution of degree 0 of \((2)\) satisfying \((11)\). Let \((15)\) hold. If \((27)\) has property \((P_0)\), then so does \((1)\).

**Remark 2.** Our results here present easily verifiable criteria for studied properties of \((1)\). We have reduced properties of general delay trinomial equation to oscillation of the corresponding second order differential equation. Moreover, using Mahfoud \([18]\) types of comparison theorems, we can examine properties of delay trinomial equation with help of that without deviating argument.

REFERENCES


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