ON PERFECT CONES AND ABSOLUTE BAIRE-ONE RETRACTS

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ABSTRACT. We introduce perfect cones over topological spaces and study their connection with absolute $B_1$-retracts.

1. Introduction

A subset $E$ of a topological space $X$ is a retract of $X$ if there exists a continuous mapping $r: X \to E$ such that $r(x) = x$ for all $x \in E$. Different modifications of this notion in which $r$ is allowed to be discontinuous (in particular, almost continuous or a Darboux function) were considered in [3], [10]–[12]. The author introduced in [5] the notion of $B_1$-retract, i.e., a subspace $E$ of $X$ for which there exists a Baire-one mapping $r: X \to E$ with $r(x) = x$ on $E$. Moreover, the following two results were obtained in [5].

Theorem 1.1. Let $X$ be a normal space and $E$ be an arcwise connected and locally arcwise connected metrizable $F_{\sigma}$- and $G_\delta$-subspace of $X$. If

(i) $E$ is separable, or
(ii) $X$ is collectionwise normal,
then $E$ is a $B_1$-retract of $X$.

Theorem 1.2. Let $X$ be a completely metrizable space and let $E$ be an arcwise connected and locally arcwise connected $G_\delta$-subspace of $X$. Then, $E$ is a $B_1$-retract of $X$.

Note that, in the above mentioned results, $E$ is a locally arcwise connected space. Therefore, it is natural to ask

Question 1.3. Is any arcwise connected $G_\delta$-subspace $E$ of a completely metrizable space $X$ a $B_1$-retract of this space?

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In this work we introduce the notions of the perfect cone over a topological space and an absolute $B_1$-retract (see definitions in Section 2). We show that the perfect cone over a $\sigma$-compact metrizable zero-dimensional space is an absolute $B_1$-retract. Moreover, we give a negative answer to Question 1.3.

2. Preliminaries

Throughout the paper, all topological spaces have no separation axioms if it is not specified.

A mapping $f : X \to Y$ is a Baire-one mapping if there exists a sequence of continuous mappings $f_n : X \to Y$ which converges to $f$ pointwise on $X$.

A subset $E$ of a topological space $X$ is called:

- a $B_1$-retract of $X$ if there exists a sequence of continuous mappings $r_n : X \to E$ such that $r_n(x) \to r(x)$ for all $x \in X$ and $r(x) = x$ for all $x \in E$; the mapping $r : X \to E$ is called a $B_1$-retraction of $X$ onto $E$;
- a $\sigma$-retract of $X$ if $E = \bigcup_{n=1}^{\infty} E_n$, where $(E_n)_{n=1}^{\infty}$ is an increasing sequence of retracts of $X$;
- ambiguous if it is simultaneously $F_{\sigma}$ and $G_\delta$ in $X$.

A topological space $X$ is:

- perfectly normal if it is normal and every closed subset of $X$ is $G_\delta$;
- an absolute $B_1$-retract (in the class $C$ of topological spaces) if $X \in C$ and for any homeomorphism $h$, which maps $X$ onto a $G_\delta$-subset $h(X)$ of a space $Y \in C$, the set $h(X)$ is a $B_1$-retract of $Y$; in this paper we will consider only the case $C$ is the class of all perfectly normal spaces;
- a space with the regular $G_\delta$-diagonal if there exists a sequence $(G_n)_{n=1}^{\infty}$ of open neighbourhoods of the diagonal $\Delta = \{(x,x) : x \in X\}$ in $X^2$ such that $\Delta = \bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty} \overline{G_n}$;
- contractible if there exists a continuous mapping $\gamma : X \times [0,1] \to X$ and $x^* \in X$ such that $\gamma(x,0) = x$ and $\gamma(x,1) = x^*$ for all $x \in X$; the mapping $\gamma$ is called a contraction.

A family $(A_s : s \in S)$ of subsets of $X$ is said to be a partition of $X$ if $X = \bigcup_{s \in S} A_s$ and $A_s \cap A_t = \emptyset$ for all $s \neq t$.

If a space $X$ is homeomorphic to a space $Y$, then we denote this fact by $X \simeq Y$.

For a mapping $f : X \times Y \to Z$ and a point $(x,y) \in X \times Y$, let $f^x(y) = f_y(x) = f(x,y)$. 
3. Perfect cones and their properties

The cone \( \Delta(X) \) over a topological space \( X \) is the quotient space \( (X \times [0,1]) / (X \times \{0\}) \) with the quotient mapping \( \lambda : X \times [0,1] \to \Delta(X) \). By \( v \) we denote the vertex of the cone, i.e., \( v = \lambda(X \times \{0\}) \). We call the set \( \lambda(X \times \{1\}) \) the base of the cone.

Let \( X_1 = (0,1) \) and \( X_2 = [0,1] \). Then, \( \Delta(X_2) \) is homeomorphic to a triangle \( T \subseteq [0,1]^2 \) while \( \Delta(X_1) \) is not even metrizable, since there is no countable base of neighbourhoods at the cone vertex. Consequently, the naturally embedding of \( \Delta(X_1) \) into \( \Delta(X_2) \) is not a homeomorphism. Therefore, on the cone \( \Delta(X) \) over a space \( X \), one more topology \( \mathcal{T}_p \) which coincides with the quotient topology \( \mathcal{T} \) on \( X \times (0,1] \) naturally appears and the base of neighbourhoods of the vertex \( v \) forms the system \( \{ \lambda(X \times [0,\varepsilon]) : \varepsilon > 0 \} \). The cone \( \Delta(X) \) equipped with the topology \( \mathcal{T}_p \) is said to be perfect and is denoted by \( \Delta_p(X) \).

For all \( x \in X \), we write
\[
v x = \lambda^x([0,1]).
\]

Obviously,
\[
\Delta(X) = \bigcup_{x \in X} v x.
\]

It is easily seen that \( v x \cap vy = \{v\} \) for all distinct \( x, y \in X \) and \( v x \simeq [0,1] \) for all \( x \in X \).

For every \( y \in \Delta(X) \setminus \{v\} \), we set
\[
\alpha(y) = \text{pr}_X(\lambda^{-1}(y)). \tag{3.1}
\]

Obviously, \( \alpha : \Delta(X) \setminus \{v\} \to X \) is continuous in both topologies \( \mathcal{T} \) and \( \mathcal{T}_p \).

Let
\[
\beta(y) = \begin{cases} 
\text{pr}_{[0,1]}(\lambda^{-1}(y)), & y \neq v, \\
0, & y = v.
\end{cases} \tag{3.2}
\]

Then, \( \beta : \Delta_p(X) \to [0,1] \) is a continuous function. Indeed, it is evident that \( \beta \) is continuous on \( \Delta_p(X) \setminus \{v\} \). Since
\[
\beta^{-1}([0,\varepsilon]) = \lambda(X \times [0,\varepsilon]) \tag{3.3}
\]
for any \( \varepsilon > 0 \), the set \( \beta^{-1}([0,\varepsilon]) \) is a neighbourhood of \( v \). Consequently, \( \beta \) is continuous at \( v \).

We observe that
\[
\lambda(\alpha(y), \beta(y)) = y
\]
for all \( y \in \Delta(X) \setminus \{v\} \).
Remark 3.1.

1. The concept of the perfect cone over a separable metrizable space was also defined in [8, p. 55].
2. We observe that \( x \mapsto \lambda(x, 1) \) is a homeomorphism of \( X \) onto \( \lambda(X, \times \{1\}) \subseteq \Delta_p(X) \). Therefore, we can identify \( X \) with its image and consider \( X \) as a subspace of \( \Delta_p(X) \).
3. In the light of the previous observation, we may assume that the mapping \( \alpha \) defined by formula (3.1) is a retraction.
4. The system \( \{ \beta^{-1}((0, \varepsilon)) : \varepsilon > 0 \} \) is the base of neighbourhoods of the vertex of the cone according to (3.3).

Proposition 3.2 ([8, p. 55]). The cone \( \Delta(X) \) over a compact space \( X \) is perfect.

Proof. Let \( W \) be an open neighbourhood of \( v \) in \( \Delta(X) \). Then, for every \( x \in X \), there exist a neighbourhood \( U_x \) of \( x \) and \( \delta_x > 0 \) such that \( \lambda(U_x \times [0, \delta_x]) \subseteq W \). Choose a finite subcover \( (U_1, \ldots, U_n) \) of \( (U_x : x \in X) \) and put \( \varepsilon = \min\{\delta_1, \ldots, \delta_n\} \). Then, \( \lambda(X \times [0, \varepsilon]) \subseteq W \). Hence, \( \Delta(X) \) is the perfect cone. \( \square \)

Theorem 3.3. Let \( X \) be a topological space.

1. If \( X \) is Hausdorff, then \( \Delta_p(X) \) is Hausdorff.
2. If \( X \) is regular, then \( \Delta_p(X) \) is regular.
3. If \( X \) is a countable regular space, then \( \Delta_p(X) \) is perfectly normal.
4. \( \Delta_p(X) \) is contractible.
5. If \( X \) is locally (arcwise) connected, then:
   a) \( \Delta(X) \) is locally (arcwise) connected;
   b) \( \Delta_p(X) \) is locally (arcwise) connected.
6. If \( X \) is metrizable, then \( \Delta_p(X) \) is metrizable.

Proof.

1) Let \( x, y \in \Delta_p(X) \) and \( x \neq y \). Since \( \Delta_p(X) \setminus \{v\} \) is homeomorphic to the Hausdorff space \( X \times (0, 1] \), it is sufficient to consider the case \( x = v \) or \( y = v \). Assume that \( x = v \) and \( y \neq v \). Then, \( 0 = \beta(x) < \beta(y) \leq 1 \), where \( \beta \) is defined by formula (3.2). Set \( O_x = \lambda(X \times [0, \beta(y)/2]) \) and \( O_y = \lambda(X \times (\beta(y)/2, 1]) \). Then, \( O_x \) and \( O_y \) are disjoint neighbourhoods of \( x \) and \( y \) in \( \Delta_p(X) \), respectively.

2) Fix \( y \in Y \) and a closed set \( F \subseteq \Delta_p(X) \) such that \( y \notin F \). Since \( X \times (0, 1] \) is regular, the case \( y \neq v \) and \( v \notin F \) is obvious.

Let \( y = v \). Choose \( \varepsilon > 0 \) such that \( F \cap \lambda(X \times [0, \varepsilon)) = \emptyset \). Then, \( U = \lambda(X \times [0, \varepsilon/2]) \) and \( V = \lambda(X \times (\varepsilon/2, 1]) \) are disjoint open neighbourhoods of \( v \) and \( F \) in \( \Delta_p(X) \), respectively.
Now, let \( y \neq v \) and \( v \in F \). We take \( \varepsilon > 0 \) such that \( O_v = \beta^{-1}([0, \varepsilon)) \) is an open neighbourhood of \( v \) with \( y \notin O_v \). Moreover, since \( \Delta_p(X) \setminus \{v\} \) is regular, there exists an open neighbourhood \( O_y \) of \( y \) in \( \Delta_p(X) \) with \( \overline{O_y} \cap G \subseteq \Delta_p \setminus F \). Then, \( U = O_y \setminus O_v \) is an open neighbourhood of \( y \) in \( \Delta_p(X) \) such that \( U \cap F = \emptyset \).

3) We observe that \( \Delta_p(X) \) is regular by the previous proposition. Moreover, \( Y \) is hereditarily Lindelöf (and, consequently, normal) as the union of countably many homeomorphic copies of \([0, 1]\). Hence, \( \Delta_p(X) \) is perfectly normal.

4) For all \( y \in \Delta_p(X) \) and \( t \in [0, 1] \) define
\[
\gamma(y, t) = \begin{cases} \lambda(\alpha(y), t \cdot \beta(y)), & y \neq v, \\ v, & y = v, \end{cases}
\]
where \( \alpha \) and \( \beta \) are defined by (3.1) and (3.2), respectively. Then, \( \gamma(y, 0) = v \) and \( \gamma(y, 1) = y \) for all \( y \in Y \). Clearly, \( \gamma \) is continuous on \( (Y \setminus \{v\}) \times [0, 1] \). Let \( \varepsilon > 0 \) and \( W = \lambda(X \times [0, \varepsilon)) = \beta^{-1}([0, \varepsilon)) \). Since \( \beta(\gamma(y, t)) = t \cdot \beta(y), \gamma(W \times [0, 1]) \subseteq W \). Hence, \( \gamma \) is continuous at each point of the set \( \{v\} \times [0, 1] = \beta^{-1}(0) \).

5a) Notice that the space \( \Delta(X) \setminus \{v\} \) is locally (arcwise) connected, since it is homeomorphic to the locally (arcwise) connected space \( X \times [0, 1] \). Let us check that \( \Delta(X) \) is locally (arcwise) connected at \( v \). Fix an open neighbourhood \( W \) of \( v \) in \( \Delta(X) \). Then, \( V = \lambda^{-1}(W) \) is open in \( X \times [0, 1] \) and \( X \times \{0\} \subseteq V \). For every \( x \in X \), we denote by \( G_x \) the (arcwise) component of \( V \) with \((x, 0) \in G_x \). Then, \( G_x \) is open in \( \Delta(X) \) and, consequently, in \( \Delta(X) \). Let \( G = \bigcup_{x \in X} G_x \). Then, the set \( \lambda(G) \) is an open neighbourhood of \( v \) such that \( \lambda(G) \subseteq W \). It remains to observe that \( \lambda(G) \) is (arcwise) connected, since \( v \in \lambda(G_x) \) and \( \lambda(G_x) \) is (arcwise) connected for all \( x \in X \).

5b) Let \( \varepsilon > 0 \) and \( W = \beta^{-1}([0, \varepsilon)) \). Since each element of \( W \) can be joined by a segment with \( v \), \( W \) is an arcwise connected neighbourhood of \( v \).

6) Let \( \varrho \) be a metric generating the topology of \( X \) with \( \varrho \leq 1 \). For all \( x, y \in X \) and \( s, t \in [0, 1] \) we set
\[
d(\lambda(x, s), \lambda(y, t)) = |t - s| + \min\{s, t\} \varrho(x, y).
\]
Then, \( d \) is a correctly defined, symmetric, nonnegative and nondegenerate mapping of \( \Delta_p(X) \times \Delta_p(X) \). Moreover, the triangle inequality is satisfied, i.e.,
\[
d(\lambda(x, s), \lambda(z, u)) \leq d(\lambda(x, s), \lambda(y, t)) + d(\lambda(y, t), \lambda(z, u))
\]
for all \((x, s), (y, t), (z, u) \in X \times [0, 1]\). If \( t \geq \min\{s, u\} \), the inequality is obvious. Let \( t < \min\{s, u\} \). Without loss of generality, we may assume that \( t < u \leq s \).

Then, the above inequality is equivalent to
\[
s - u + u \varrho(x, z) \leq s - t + t \varrho(x, y) + u - t + t \varrho(y, z),
\]
i.e.,
\[
\varrho(x, z) \leq 2 \left(1 - \frac{t}{u}\right) + \frac{t}{u}\left(\varrho(x, y) + \varrho(y, z)\right),
\]
where
which does hold since \( p(x, z) \leq 2 \) and \( p(x, z) \leq p(x, y) + p(y, z) \). Moreover, \( d \) generates the topology of the perfect cone. It is obvious that the \( d \)-neighbourhoods of \( v \) are the correct ones and the metric \( d \) on \( \lambda(X \times (0, 1]) \) is equivalent to the summing metric inherited from \( X \times (0, 1] \). □

A subset \( E \) of a topological vector space \( X \) is bounded if for any neighbourhood of zero \( U \) there is such \( \gamma > 0 \) that \( E \subseteq \delta U \) for all \( |\delta| \geq \gamma \).

**Proposition 3.4.** Let \( Z \) be a topological vector space and \( X \subseteq Z \) be a bounded set. Then \( \Delta_p(X) \) is embedded to \( Z \times \mathbb{R} \).

**Proof.** Consider the set \( C = \{(xt, t) : x \in X, t \in [0, 1]\} \). Let \( \varphi(x, t) = (xt, t) \) for all \( (x, t) \in X \times [0, 1] \) and \( v^* = (0, 0) \in X \times [0, 1] \). Then, the restriction \( \varphi|_{X \times \{0, 1\}} \) is a homeomorphism onto \( C \setminus \{v^*\} \). Moreover, the mapping \( \beta : C \to [0, 1], \beta(z, t) = t \), is continuous. Therefore, \( \beta^{-1}\left([0, \varepsilon]\right) \) is an open neighbourhood of \( v^* \) in \( C \) for any \( \varepsilon > 0 \). Now, we show that the system \( \{\beta^{-1}(\{0, \varepsilon\}) : \varepsilon > 0\} \) is a base of \( v^* \). Take an open neighbourhood of zero in \( Z \), \( \delta > 0 \) and let \( W = U \times (-\delta, \delta) \). Choose \( \varepsilon \in (0, \delta) \) such that \( tX \subseteq U \) for all \( t \) with \( |t| < \varepsilon \). Then, for each \( y = (z, t) \in \beta^{-1}\left([0, \varepsilon]\right) \) we have \( |t| < \delta \) and \( z \in tx \subseteq U \). Consequently, \( \beta^{-1}\left([0, \varepsilon]\right) \subseteq W \). Hence, \( C \) is homeomorphic to the perfect cone \( \Delta_p(X) \). □

The following result easily follows from [8, Theorem 1.5.9].

**Corollary 3.5.** The cone \( \Delta_p(X) \) over a finite Hausdorff space \( X \) is an absolute retract.

### 4. Weak \( B_1 \)-retracts

A subset \( E \) of a topological space \( X \) is called a weak \( B_1 \)-retract of \( X \) if there exists a sequence of continuous mappings \( r_n : X \to E \) such that \( \lim_{n \to \infty} r_n(x) = x \) for all \( x \in E \). Clearly, every \( B_1 \)-retract is a weak \( B_1 \)-retract. The converse proposition is not true (see Example [77]).

A space \( X \) is called an absolute weak \( B_1 \)-retract if, for any space \( Y \) and for any homeomorphic embedding \( h : X \to Y \), the set \( h(X) \) is a weak \( B_1 \)-retract of \( Y \).

Let \( E = \bigcup_{n=1}^{\infty} E_n \) and let \( (r_n)_{n=1}^{\infty} \) be a sequence of retractions \( r_n : X \to E_n \). If the sequence \( (E_n)_{n=1}^{\infty} \) is increasing, then \( \lim_{n \to \infty} r_n(x) = x \) for every \( x \in E \). Thus, we have proved the following fact.

**Proposition 4.1.** Every \( \sigma \)-retract of a topological space \( X \) is a weak \( B_1 \)-retract of \( X \).

**Proposition 4.2.** Let \( X \) be a countable Hausdorff space. Then, \( \Delta_p(X) \) is an absolute weak \( B_1 \)-retract.
ON PERFECT CONES AND ABSOLUTE BAIRE-ONE RETRACTS

**Proof.** Assume that $\Delta_p(X)$ is a subspace of a topological space $Z$. Let $X = \{x_n : n \in \mathbb{N}\}$ and $X_n = \{x_1, \ldots, x_n\}$. Then, $\Delta_p(X) = \bigcup_{n=1}^{\infty} \Delta_p(X_n)$ and every $\Delta_p(X_n)$ is a retract of $Z$ by Corollary 3.5. Then, $\Delta_p(X)$ is a weak $B_1$-retract of $Z$ by Proposition 1.1.

It was proved in [5] that a $B_1$-retract of a connected space is connected. It turns out that this is still valid for weak $B_1$-retracts.

**Theorem 4.3.** Let $X$ be a connected space. Then, any weak $B_1$-retract $E$ of $X$ is connected.

**Proof.** Let $(r_n)_{n=1}^{\infty}$ be a sequence of continuous mappings $r_n : X \to E$ such that $\lim_{n \to \infty} r_n(x) = x$ for all $x \in E$. Denote $H = \bigcup_{n=1}^{\infty} r_n(X)$. We show that $H$ is connected. Conversely, suppose that $H = H_1 \cup H_2$, where $H_1$ and $H_2$ are disjoint sets which are closed in $H$. Observe that each set $B_n = r_n(X)$ is connected. Then, $B_n \subseteq H_1$ or $B_n \subseteq H_2$. Choose an arbitrary $x \in H_1$. Then, there exists a number $n_1$ such that $r_n(x) \in H_1$ for all $n \geq n_1$. Hence, $B_n \subseteq H_1$ for all $n \geq n_1$. Similarly, there exists a number $n_2$ such that $B_n \subseteq H_2$ for all $n \geq n_2$. Therefore, $B_n \subseteq H_1 \cap H_2$ for all $n \geq \max\{n_1, n_2\}$, which is impossible.

It is easy to see that $H \subseteq E \subseteq \overline{H}$. Since $H$ and $\overline{H}$ are connected, $E$ is connected, too.

**Lemma 4.4.** Let $X$ be a normal space, $Y$ be a contractible space, $(F_i)_{i=1}^{n}$ be a sequence of disjoint closed subsets of $X$ and let $g_i : X \to Y$ be a continuous mapping for every $1 \leq i \leq n$. Then, there exists a continuous mapping $g : X \to Y$ such that $g(x) = g_i(x)$ on $F_i$ for every $1 \leq i \leq n$.

**Proof.** Let $y^* \in Y$ and $\gamma : Y \times [0,1] \to Y$ be a continuous mapping such that $\gamma(y, 0) = y$ and $\gamma(y, 1) = y^*$ for all $y \in Y$. For all $x, y \in X$ and $t \in [0,1]$, define

$$h(x, y, t) = \begin{cases} 
\gamma(x, 2t), & 0 \leq t \leq 1/2, \\
\gamma(y, -2t + 2), & 1/2 < t \leq 1.
\end{cases}$$

Then, the mapping $h : Y \times Y \times [0,1] \to Y$ is continuous, $h(x, y, 0) = x$ and $h(x, y, 1) = y$.

Let $n = 2$. By Urysohn’s lemma, there is a continuous function $\varphi : X \to [0,1]$ such that $\varphi(x) = 0$ on $F_1$ and $\varphi(x) = 1$ on $F_2$. For all $x \in X$, let

$$g(x) = h(g_1(x), g_2(x), \varphi(x)).$$

Clearly, $g : X \to Y$ is continuous and $g(x) = g_1(x)$ if $x \in F_1$, and $g(x) = g_2(x)$ if $x \in F_2$. 

95
Assume the assertion of the lemma is true for \( k \) sets, where \( k = 1, \ldots, n - 1 \), and prove it for \( n \) sets. According to our assumption, there exists a continuous mapping \( \tilde{g}: X \rightarrow Y \) such that \( \tilde{g}|_{F_i} = g_i \) for every \( i = 1, \ldots, n - 1 \). Since the sets \( F = \bigcup_{i=1}^{n-1} F_i \) and \( F_n \) are closed and disjoint, there exists a continuous mapping \( g: X \rightarrow Y \) such that \( g|_{F} = \tilde{g} \) and \( g|_{F_n} = g_n \). Then, \( g|_{F_i} = g_i \) for every \( 1 \leq i \leq n \).

**Theorem 4.5.** Let \( E \) be a contractible ambiguous weak \( B_1 \)-retract of a normal space \( X \). Then, \( E \) is a \( B_1 \)-retract of \( X \).

**Proof.** Let \( (r_n)_{n=1}^{\infty} \) be a sequence of continuous mappings \( r_n: X \rightarrow E \) such that \( \lim_{n \rightarrow \infty} r_n(x) = x \) for all \( x \in E \). Choose increasing sequences \( (E_n)_{n=1}^{\infty} \) and \( (F_n)_{n=1}^{\infty} \) of closed subsets of \( X \) such that \( E = \bigcup_{n=1}^{\infty} E_n \) and \( X \setminus E = \bigcup_{n=1}^{\infty} F_n \). Fix \( x^* \in E \). Then, for every \( n \in \mathbb{N} \), by Lemma 4.4 there exists a continuous mapping \( f_n: X \rightarrow E \) such that \( f_n(x) = r_n(x) \) if \( x \in E_n \), and \( f_n(x) = x^* \) if \( x \in F_n \). It is easy to verify that the sequence \( (f_n)_{n=1}^{\infty} \) is pointwise convergent on \( X \) and \( \lim_{n \rightarrow \infty} f_n(X) \subseteq E \). Let \( r(x) = \lim_{n \rightarrow \infty} f_n(x) \) for all \( x \in X \). Then, \( r(x) = \lim_{n \rightarrow \infty} r_n(x) = x \) for all \( x \in E \).

**Proposition 4.6.** The perfect cone \( \Delta_p(X) \) over a countable regular space \( X \) is an absolute \( B_1 \)-retract.

**Proof.** We first note that \( \Delta_p(X) \) is perfectly normal by Theorem 3.3. Assume that \( \Delta_p(X) \) is a \( G_\delta \)-subset of a perfectly normal space \( Z \). Then, \( \Delta_p(X) \) is a weak \( B_1 \)-retract of \( Z \) by Proposition 4.2. Moreover, \( \Delta_p(X) \) is a contractible \( F_\sigma \)-subspace of \( Z \). Hence, Theorem 4.3 implies that \( \Delta_p(X) \) is a \( B_1 \)-retract of \( Z \).

Let us observe that any \( B_1 \)-retract of a space with a regular \( G_\delta \)-diagonal is a \( G_\delta \)-subset of this space [5 Proposition 2.2]. But it is not valid for weak \( B_1 \)-retracts as the following example shows.

**Example 4.7.** Let \( \mathbb{Q} \) be a set of all rational numbers and \( X = \mathbb{Q} \cap [0, 1] \). Then, \( \Delta_p(X) \) is a weak \( B_1 \)-retract of \( \mathbb{R}^2 \) but not a \( B_1 \)-retract of \( \mathbb{R}^2 \).

**Proof.** Indeed, \( \Delta_p(X) \) is a weak \( B_1 \)-retract of \( \mathbb{R}^2 \) by Proposition 4.2. Since \( \Delta_p(X) \) is not a \( G_\delta \)-set in \( \mathbb{R}^2 \), \( \Delta_p(X) \) is not a \( B_1 \)-retract.

**Theorem 4.8.** Let \( X \) be a perfectly normal space, \( E \) be a contractible \( G_\delta \)-subspace of \( X \), \( x^* \in E \) and let \( (E_n: n \in \mathbb{N}) \) be a cover of \( E \) such that

1. \( E_n \cap E_m = \{x^*\} \) for all \( n \neq m \);
2. \( E_n \) is a relatively ambiguous set in \( E \) for every \( n \);
3. \( E_n \) is a (weak) \( B_1 \)-retract of \( X \) for every \( n \).

Then \( E \) is a (weak) \( B_1 \)-retract of \( X \).
ON PERFECT CONES AND ABSOLUTE BAIRE-ONE RETRACTS

Proof. From [6, p. 359], it follows that for every \( n \) there exists an ambiguous set \( C_n \) in \( X \) such that \( C_n \cap E = E_n \setminus \{ x^* \} \). Moreover, there exists a sequence \( (F_n)_{n=1}^\infty \) of closed subsets of \( X \) such that \( X \setminus E = \bigcup_{n=1}^\infty F_n \). Let \( D_n = C_n \cup F_n \), \( n \geq 1 \). Now, define \( X_1 = D_1 \) and \( X_n = D_n \setminus (\bigcup_{k<n} D_k) \) if \( n \geq 2 \). Then, \( (X_n : n \in \mathbb{N}) \) is a partition of \( X \setminus \{ x^* \} \) by ambiguous sets \( X_n \) and \( X_n \cap E = E_n \setminus \{ x^* \} \) for every \( n \geq 1 \).

Suppose that \( E_n \) is a weak \( B_1 \)-retract of \( X \) for every \( n \). Choose a sequence \( (r_{n,m})_{m=1}^\infty \) of continuous mappings \( r_{n,m} : X \to E_n \) such that \( \lim_{m \to \infty} r_{n,m}(x) = x \) for all \( x \in E_n \). Since \( X_n \) is \( F_\alpha \) in \( X \), for every \( n \), there is an increasing sequence \( (B_{n,m})_{m=1}^\infty \) of closed subsets \( B_{n,m} \) of \( X \) such that \( X_n = \bigcup_{m=1}^\infty B_{n,m} \). Let \( A_{n,m} = \emptyset \) if \( n > m \), and \( A_{n,m} = B_{n,m} \) if \( n \leq m \). Then, Lemma 4.4 implies that for every \( m \in \mathbb{N} \) there is a continuous mapping \( r_m : X \to E \) such that \( r_m|_{A_{n,m}} = r_{n,m} \) and \( r_m(x^*) = x^* \).

We will show that \( \lim_{m \to \infty} r_m(x) = x \) on \( E \). Fix \( x \in E \). If \( x = x^* \), then \( r_m(x) = x \) for all \( m \). If \( x \neq x^* \), then there is a unique \( n \) such that \( x \in E_n \). Since \( (A_{n,m})_{m=1}^\infty \) increases, there exists a number \( m_0 \) such that \( x \in A_{n,m} \) for all \( m \geq m_0 \). Hence, \( \lim_{m \to \infty} r_m(x) = \lim_{m \to \infty} r_{n,m}(x) = x \). Therefore, \( E \) is a weak \( B_1 \)-retract of \( X \).

If \( E_n \) is a \( B_1 \)-retract of \( X \) for every \( n \), we apply similar arguments. \( \square \)

5. Cones over ambiguous sets

**Theorem 5.1.** Let \( \Delta_p(X) \) be the perfect cone over a metrizable locally arcwise connected space \( X \), \( Z \) be a normal space, and let \( h : \Delta_p(X) \to Z \) be an embedding such that \( h(\Delta_p(X)) \) is an ambiguous set in \( Z \). If

a) \( X \) is separable, or
b) \( \Delta_p(X) \) is collectionwise normal,

then \( h(\Delta_p(X)) \) is a \( B_1 \)-retract of \( Z \).

**Proof.** We notice that \( h(\Delta_p(X)) \) is metrizable, arcwise connected and locally arcwise connected according to Theorem 3.3. Then, the set \( h(\Delta_p(X)) \) is a \( B_1 \)-retract of \( Z \) by Theorem 1.1. \( \square \)

By \( B_{c}(x_0) \), we denote an open ball in a metric space \( X \) with center at \( x_0 \in X \) and with radius \( \varepsilon \).

**Theorem 5.2.** Let \( \Delta_p(X) \) be the perfect cone over a zero-dimensional metrizable separable space \( X \), \( Z \) be a normal space and let \( h : \Delta_p(X) \to Z \) be such a homeomorphic embedding that \( h(\Delta_p(X)) \) is a closed set in \( Z \). Then \( h(\Delta_p(X)) \) is a weak \( B_1 \)-retract of \( Z \).
Proof. Without loss of generality, we may assume that $\Delta_p(X)$ is a closed subspace of a normal space $Z$. Consider a metric $d$ on $X$ which generates its topological structure and $(X,d)$ is a completely bounded space. For every $n \in \mathbb{N}$, there exists a finite set $A_n \subseteq X$ such that the family $B_n = (B_n(a) : a \in A_n)$ is a cover of $X$. Since $X$ is strongly zero-dimensional [2 Theorem 6.2.7], for every $n$ there exists a finite cover $U_n = (U_{i,n} : i \in I_n)$ of $X$ by disjoint clopen sets $U_{i,n}$ which refines $B_n$. Take an arbitrary $x_{i,n} \in U_{i,n}$ for every $n \in \mathbb{N}$ and $i \in I_n$. For all $x \in X$ and $n \in \mathbb{N}$, define

$$f_n(x) = x_{i,n},$$

if $x \in U_{i,n}$ for some $i \in I_n$. Then, every mapping $f_n : X \to X$ is continuous and $\lim_{n \to \infty} f_n(x) = x$ for all $x \in X$.

Fix $n \in \mathbb{N}$. For all $y \in \Delta_p(X)$, we set

$$g_n(y) = \begin{cases} \lambda(f_n(\alpha(y)), \beta(y)) & \text{if } y \neq v, \\
\ v & \text{if } y = v. \end{cases}$$

We prove that $g_n : \Delta_p(X) \to \Delta_p(X)$ is continuous at $y = v$. Indeed, let $(y_m)_{m=1}^\infty$ be a sequence of points $y_m \in Y$ such that $y_m \to v$. Assume that $y_m \neq v$ for all $m$. Show that $g_n(y_m) \to v$. Fix $\varepsilon > 0$. Since $\beta(y_m) \to 0$, there is a number $m_0$ such that $\beta(y_m) < \varepsilon$ for all $m \geq m_0$. Then, $g_n(y_m) = \lambda(f_n(\alpha(y_m)), \beta(y_m)) \in \lambda(X \times [0,\varepsilon))$ for all $m \geq m_0$. Hence, $g_n$ is continuous at $v$.

Note that $g_n(\Delta_p(X)) \subseteq K_n$, where $K_n = \bigcup_{i \in I_n} vx_{i,n}$. Since $K_n$ is a compact absolute retract by Corollary 3.1, $K_n$ is an absolute extensor. Taking into account that $\Delta_p(X)$ is closed in $Z$, we get that there exists a continuous extension $r_n : Z \to K_n$ of $g_n$.

It remains to show that $\lim_{n \to \infty} r_n(y) = y$ for all $y \in \Delta_p(X)$. Fix $y \in \Delta_p(X)$. If $y = v$, then $r_n(y) = g_n(y) = v$ for all $n \geq 1$. Let $y \neq v$. Since $\lim_{n \to \infty} f_n(\alpha(y)) = \alpha(y)$ and $\lambda$ is continuous,

$$\lim_{n \to \infty} r_n(y) = \lim_{n \to \infty} \lambda(f_n(\alpha(y)), \beta(y)) = \lambda(\alpha(y), \beta(y)) = y.$$ 

Hence, $\Delta_p(X)$ is a weak $B_1$-retract of $Z$.

Theorem 5.3. The perfect cone $\Delta_p(X)$ over a $\sigma$-compact zero-dimensional metrizable space $X$ is an absolute $B_1$-retract.

Proof. Assume that $\Delta_p(X)$ is a $G_δ$-subspace of a perfectly normal space $Z$. Since $X$ is $\sigma$-compact, there exists an increasing sequence $(F_n)_{n=1}^{\infty}$ of compact subsets of $Z$ such that $X = \bigcup_{n=1}^{\infty} F_n$. Since for every $n \in \mathbb{N}$ the set $F_{n+1} \setminus F_n$ is open in the zero-dimensional metrizable separable space $F_{n+1}$, there exists a partition $(B_{n,m} : m \in \mathbb{N})$ of $F_{n+1} \setminus F_n$ by relatively clopen sets $B_{n,m}$ in $F_{n+1}$. Let $\mathbb{N}^2 = \{(n_k, m_k : k \in \mathbb{N}), H_0 = F_1$ and let $H_k = B_{n_k,m_k}$ for every $k \in \mathbb{N}$. Then, the family $(H_k : k = 0, 1, \ldots)$ is a partition of $X$ by compact sets $H_k$.
ON PERFECT CONES AND ABSOLUTE BAIRE-ONE RETRACTS

Fix \( k \in \mathbb{N} \). Let \( E_k = \Delta_p (H_k) \) be the perfect cone over zero-dimensional metrizable separable space \( H_k \). Then, \( E_k \) is a closed subset of \( Z \). Therefore, \( E_k \) is a weak \( B_1 \)-retract of \( Z \) by Theorem 5.2.

Since \( \Delta_p (X) = \bigcup_{k=1}^{\infty} E_k \), Theorem 4.8 implies that \( \Delta_p (X) \) is a weak \( B_1 \)-retract of \( Z \). It remains to apply Theorem 4.5. □

The perfect cone \( \Delta_p (X) \) over a \( \sigma \)-compact space \( X \subseteq \mathbb{R} \) is an absolute \( B_1 \)-retract.

Proof. Suppose that \( \Delta_p (X) \) is a \( G_\delta \)-subspace of a perfectly normal space \( Z \).

Since \( \Delta_p (X) \) is \( \sigma \)-compact, \( \Delta_p (X) \) is \( F_\sigma \) in \( Z \).

Let \( G = \text{int}_Z X \), \( F = X \setminus G \), \( A = \Delta_p (G) \) and \( B = \Delta_p (F) \). Since \( G \) and \( F \) are \( \sigma \)-compact sets, \( A \) and \( B \) are \( \sigma \)-compact sets, too. Hence, \( A \) and \( B \) are ambiguous subsets of \( \Delta_p (X) \). Consequently, \( A \) and \( B \) are ambiguous in \( Z \). Since \( G \) is metrizable locally arcwise connected separable space, \( A \) is a \( B_1 \)-retract of \( Z \) by Theorem 5.1. Since \( F \) is zero-dimensional metrizable \( \sigma \)-compact space, \( B \) is a \( B_1 \)-retract of \( Z \) according to Theorem 5.3. Theorem 4.8 implies that the set \( \Delta_p (X) = A \cup B \) is a \( B_1 \)-retract of \( Z \). □

Note that the condition of \( \sigma \)-compactness of \( X \) in Theorems 5.2 and 5.3 is essential (see Example 6.4).

6. The weak local connectedness point set of \( B_1 \)-retracts

Let \( (Y,d) \) be a metric space. A sequence \((f_n)_{n=1}^{\infty}\) of mappings \( f_n : X \to Y \) is uniformly convergent to a mapping \( f \) at a point \( x_0 \) of \( X \) if for any \( \varepsilon > 0 \) there exists a neighbourhood \( U \) of \( x_0 \) and \( N \in \mathbb{N} \) such that

\[
d(f_n(x), f(x)) < \varepsilon
\]

for all \( x \in U \) and \( n \geq N \). We observe that if every \( f_n \) is continuous at \( x_0 \) and the sequence \((f_n)_{n=1}^{\infty}\) converges uniformly to \( f \) at \( x_0 \), then \( f \) is continuous at \( x_0 \).

By \( R((f_n)_{n=1}^{\infty}, f, X) \) we denote the set of all points of uniform convergence of the sequence \((f_n)_{n=1}^{\infty}\) to the mapping \( f \).

The closure of a set \( A \) in a subspace \( E \) of a topological space \( X \) is denoted by \( A^E \).

A space \( X \) is weakly locally connected at \( x_0 \in X \) if every open neighbourhood of \( x_0 \) contains a connected (not necessarily open) neighbourhood of \( x_0 \). The set of all points of weak local connectedness of \( X \) will be denoted by \( WLC(X) \).
Theorem 6.1. Let $X$ be a locally connected space, $(E,d)$ be a metric subspace of $X$ and let $r: X \to E$ be a $B_1$-retraction which is a pointwise limit of a sequence of continuous mappings $r_n: X \to E$. Then,

$$R((r_n)_{n=1}^\infty, r, X) \cap E \subseteq WLC(E).$$

Proof. Fix $x_0 \in R((r_n)_{n=1}^\infty, r, X) \cap E$ and $\varepsilon > 0$. Set $W = B_\varepsilon(x_0)$. Choose a neighbourhood $U_1$ of $x_0$ in $X$ and a number $n_0$ such that

$$d(r_n(x), r(x)) < \frac{\varepsilon}{4}$$

for all $x \in U_1$ and $n \geq n_0$. Since $r$ is continuous at $x_0$, there exists a neighbourhood $U_2 \subseteq X$ of $x_0$ such that

$$d(r(x), r(x_0)) < \frac{\varepsilon}{4}$$

for all $x \in U_2$. The locally connectedness of $X$ implies that there is a connected neighbourhood $U$ of $x_0$ such that $U \subseteq U_1 \cap U_2$. Since $\lim_{n \to \infty} r_n(x_0) = x_0$, there exists a number $n_1$ such that $r_n(x_0) \in U \cap E$ for all $n \geq n_1$. Let $N = \max\{n_0, n_1\}$ and

$$F = \bigcup_{n \geq N} r_n(U).$$

We show that $F \subseteq W$. Let $x \in U$ and $n \geq N$. Then,

$$d(r_n(x), x_0) = d(r_n(x), r(x_0)) \leq d(r_n(x), r(x)) + d(r(x), r(x_0)) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$ 

Thus, $r_n(x) \in B_{\varepsilon/2}(x_0)$. Then, $\bigcup_{n \geq N} r_n(U) \subseteq B_{\varepsilon/2}(x_0)$. Hence,

$$F \subseteq B_{\varepsilon/2}(x_0) \subseteq W.$$

Moreover, $r(U) \subseteq F$, provided $\lim_{n \to \infty} r_{N+n}(x) = r(x)$ for every $x \in U$. Observe that $U \cap E = r(U \cap E) \subseteq r(U)$. Therefore,

$$x_0 \in U \cap E \subseteq F \subseteq W,$$

which implies that $F$ is a closed neighbourhood of $x_0$ in $E$.

It remains to prove that $F$ is a connected set. To obtain a contradiction, assume that $F = F_1 \cup F_2$, where $F_1$ and $F_2$ are nonempty disjoint closed subsets of $F$. Clearly, $F \cap U \neq \emptyset$.

Consider the case $F_i \cap U \neq \emptyset$ for $i = 1, 2$. The continuity of $r_n$ implies that $r_n(U)$ is a connected set for every $n \geq 1$. Since, $r_n(U) \subseteq F$, $r_n(U) \subseteq F_1$ or $r_n(U) \subseteq F_2$ for every $n \geq N$. Choose $x_i \in F_i \cap U$ for $i = 1, 2$. Taking into account that $\lim_{n \to \infty} r_n(x_i) = x_i$ for $i = 1, 2$, we choose a number $k \geq N$ such that $r_n(x_i) \in F_i$ for all $n \geq k$ and for $i = 1, 2$. Then, $r_k(U) \subseteq F_1 \cap F_2$, which implies a contradiction.
Now, let \( F_1 \cap U \neq \emptyset \) and \( F_2 \cap U = \emptyset \). Then, \( U \cap E \subseteq F_1 \). Since \( r_n(x_0) \in U \cap E \), \( r_n(x_0) \in F_1 \), consequently, \( r_n(U) \subseteq F_1 \) for all \( n \geq N \). Then, \( F \subseteq \overline{F_1} = F_1 \). Therefore, \( F_2 = \emptyset \), a contradiction. One can similarly prove that the case when \( F_1 \cap U = \emptyset \) and \( F_2 \cap U \neq \emptyset \) is impossible.

Hence, the set \( F \) is connected and \( x_0 \in WLC(E) \). □

Note that we cannot replace the set \( R((r_n)_{n=1}^{\infty}, r, X) \) with a wider set \( C(r) \) of all points of continuity of the mapping \( r \) in Theorem 6.1 as the following example shows.

**Example 6.2.** There exists an arcwise connected closed subspace \( E \) of \( \mathbb{R}^2 \) and a \( B_1 \)-retraction \( r: \mathbb{R}^2 \to E \) such that \( C(r) \cap E \not\subseteq WLC(E) \).

**Proof.** Let \( a_0 = (0; 0) \), \( a_n = (1; n) \) for \( n \geq 1 \) and \( X = \{a_n : n = 0, 1, 2, \ldots \} \). Denote by \( v a_n \) the segment which connects the points \( v = (1; 0) \) and \( a_n \) for every \( n = 0, 1, \ldots \) Define \( E = \bigcup_{n=0}^{\infty} v a_n \). Then, \( E \) is an arcwise connected compact subspace of \( \mathbb{R}^2 \) and \( WLC(E) = (E \setminus v a_0) \cup \{v\} \). For all \( x \in \mathbb{R}^2 \), write

\[
r(x) = \begin{cases} x & \text{if } x \in E, \\
a_0 & \text{if } x \notin E. \end{cases}
\]

It is easy to see that \( r: \mathbb{R}^2 \to E \) is continuous at the point \( x = a_0 \). We show that \( r \in B_1(\mathbb{R}^2, E) \). Since \( X \setminus E = F_\sigma \), choose an increasing sequence of closed subsets \( X_n \subseteq \mathbb{R}^2 \) such that \( \mathbb{R}^2 \setminus E = \bigcup_{n=1}^{\infty} X_n \). Let \( E_n = \bigcup_{k=0}^{n} v a_k \), \( n \geq 1 \). For every \( n \in \mathbb{N} \) define \( A_n = X_n \cup E_n \). Then, for every \( n \), the set \( A_n \) is closed in \( \mathbb{R}^2 \), \( A_n \subseteq A_{n+1} \) and \( \bigcup_{n=1}^{\infty} A_n = \mathbb{R}^2 \). Clearly, the restriction \( r|_{A_n}: A_n \to E_n \) is continuous for every \( n \). By the Tietze Extension Theorem there is a continuous extension \( f_n: \mathbb{R}^2 \to \mathbb{R}^2 \) of \( r|_{A_n} \) for every \( n \). Notice that for every \( n \) there exists a retraction \( \alpha_n: \mathbb{R}^2 \to E_n \). Let \( r_n = \alpha_n \circ f_n \). Then, \( r_n: \mathbb{R}^2 \to E_n \) is a continuous mapping such that \( r_n|_{A_n} = r|_{A_n} \) for every \( n \).

It remains to show that \( \lim_{n \to \infty} r_n(x) = r(x) \) for all \( x \in \mathbb{R}^2 \). Indeed, fix \( x \in \mathbb{R}^2 \). Then there is a number \( N \) such that \( x \in A_n \) for all \( n \geq N \). Then \( r_n(x) = r(x) \) for all \( n \geq N \). Hence, \( r \in B_1(\mathbb{R}^2, E) \). □

**Theorem 6.3.** Let \( X \) be a locally connected Baire space and \( E \) be a metrizable \( B_1 \)-retract of \( X \). Then, the set \( E \setminus WLC(E) \) is of the first category in \( X \).

If, moreover, \( X \) has a regular \( G_\delta \)-diagonal and \( E \) is dense in \( X \), then \( WLC(E) \) is a dense \( G_\delta \)-subset of \( X \).

**Proof.** Let \( d \) be a metric on the set \( E \) which generates its topological structure. Consider a \( B_1 \)-retraction \( r: X \to E \) and choose a sequence \( (r_n)_{n=1}^{\infty} \) of continuous mappings \( r_n: X \to E \) such that \( \lim_{n \to \infty} r_n(x) = r(x) \) for all \( x \in E \).

Denote \( R = R((r_n)_{n=1}^{\infty}, r, X) \). Then, \( R \cap E \subseteq WLC(E) \) by Theorem 6.1. According to Osgood’s theorem [9], \( X \setminus R \) is an \( F_\sigma \)-set of the first category in \( X \). Hence, \( E \setminus WLC(E) \) is a set of the first category in \( X \).
OLENA KARLOVA

Now, assume that $X$ has a regular $G_\delta$-diagonal and $\overline{E} = X$. It follows from [5, Proposition 2.2] that $E$ is $G_\delta$ in $X$. Moreover, the set $R$ is dense in $X$, since $X$ is Baire. Then, $R \cap E$ is dense in $X$. Hence, $WLC(E)$ is dense in $X$. Observe that $WLC(E)$ is a $G_\delta$-subset of $E$ by [7, p. 233]. Then, $WLC(E)$ is $G_\delta$ in $X$. □

The following example gives a negative answer to Question 1.3.

Example 6.4. There exists an arcwise connected $G_\delta$-set $E \subseteq \mathbb{R}^2$ such that $E$ is the perfect cone over zero-dimensional metrizable separable space $X \subseteq \mathbb{R}$ and $E$ is not a $B_1$-retract of $\mathbb{R}^2$.

Proof. Let $I$ be the set of irrational numbers and $X = I \cap [0, 1]$. Define

$$E = \{(x, t) : x \in X, t \in [0, 1]\}.$$ 

Then, $E \simeq \Delta_p(X)$. Moreover, $E$ is an arcwise connected $G_\delta$-subset of $\mathbb{R}^2$. Clearly, $\overline{E} = [0, 1]^2$ and $WLC(E) = \{v\}$. Therefore, Theorem 6.3 implies that $E$ is not a $B_1$-retract of $[0, 1]^2$. Consequently, $E$ is not a $B_1$-retract of $\mathbb{R}^2$. □

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