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EXTENSION OF THE EXAMPLE BY MOORE–NEHARI

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ABSTRACT. R. Moore and Z. Nehari developed the variational theory for superlinear boundary value problems of the form $x'' = -p(t)|x|^{2\varepsilon}x$, $x(a) = 0 = x(b)$, where $\varepsilon > 0$ and $p(t)$ is a positive continuous function. They constructed simple example of the equation considered in the interval $[0, b]$ so that the problem had three positive solutions. We show that this example can be extended so that the respective BVP has infinitely many groups of solutions with a presribed number of zeros.

1. INTRODUCTION

R. Moore and Z. Nehari in the paper [5] considered equations of the form

$$x'' = -q(t)|x|^{2\varepsilon}x \tag{1}$$

in connection with Nehari's theory of *characteristic values* [6]. The main question was whether the equation may have multiple solutions of the boundary value problem (1),

$$x(a) = 0, \quad x(b) = 0. \tag{2}$$

The example was fairly simple: the coefficient $q(t)$ was chosen as a step-wise function

$$q(t) = \begin{cases} 1, & t \in [0,1], \\ 0, & t \in [1,\alpha], \qquad \alpha = 2.02585, \\ 1, & t \in [\alpha, \alpha+1]. \end{cases} \tag{3}$$

Two triples of symmetric solutions given on the intervals $[0,1]$ and $[\alpha, \alpha+1]$ can be connected by the straight line segments (solutions of $x'' = 0$) if the length of the middle interval $[1, \alpha]$ is chosen appropriately (Fig. 1, 2). This is the main idea of the example.

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We consider boundary value problem (BVP) of the form

$$x'' = -q(t)x^3, \tag{4}$$

$$x(-1) = 0, \quad x(1) = 0, \tag{5}$$

where the coefficient $q(t)$ is specially chosen.

Equations of the type (1) were considered in connection with the Nehari theory [6] of functionals

$$H(x) = \int_a^b \left[x'^2(t) - (1+\varepsilon)^{-1} q(t) x^{2+2\varepsilon}(t)\right] dt \tag{6}$$

that have minima in variational problems with some restrictions.

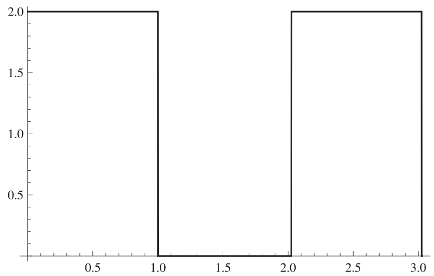

FIGURE 1. The coefficient $q(t)$ in Moore's example.

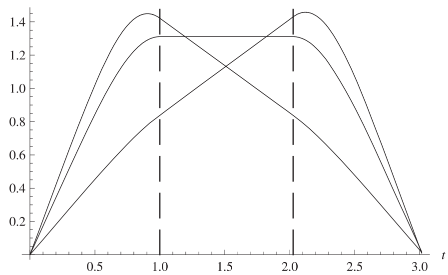

FIGURE 2. Three positive solutions in Moore's example.

Z. Nehari introduced numbers $\lambda_n(a,b)$, which are, by definition, the minimal values of the functional over the set Γ_n of all functions $x(t)$, which are:

1) continuous and piece-wise continuously differentiable in $[a,b]$;
2) there exist numbers a_ν such that
$$a = a_1 < \cdots < a_{n-1} = b \quad \text{and} \quad x(a_\nu) = 0 \text{ in any } a_\nu;$$
3) in any $[a_{\nu-1}, a_\nu]$ $x(t) \not\equiv 0$ and
$$\int_{a_{\nu-1}}^{a_\nu} x'^2(t)\, dt = \int_{a_{\nu-1}}^{a_\nu} q(t) x^2 |x|^{2\varepsilon}\, dt. \tag{7}$$

It was proved in [6] that minimizers in the above variational problem are C^2-solutions of the boundary value problem

$$x'' = -q(t)|x|^{2\varepsilon} x, \ \ x(a) = x(b) = 0, \ \ x(t) \text{ has exactly } n-1 \text{ zeros in } (a,b). \tag{8}$$

Putting (7) into (6) one gets that

$$\lambda_n(a,b) = \min_{x\in\Gamma_n} H(x) = \frac{\varepsilon}{1+\varepsilon}\int_a^b q(t)x^{2+2\varepsilon}(t)\,dt = \frac{\varepsilon}{1+\varepsilon}\int_a^b x'^2(t)\,dt, \tag{9}$$

where $x(t)$ is an appropriate solution of the BVP (8). The Nehari solution therefore can be defined as a solution of the BVP (8) which minimizes the functional

$$\int_a^b x'^2(t)\,dt \tag{10}$$

over all solutions of the problem (8).

The example constructed in the paper [5] confirmed that there may exist multiple solutions of the BVP for a given n.

In this note we show that the Moore–Nehari example can be changed slightly in order to the respective BVP has more than three solutions. Namely, the problem (4), (5) is considered with the coefficient

$$q(t) = \begin{cases} 2, & t\in[-1,-1+\varepsilon,], \\ 0, & t\in[-1+\varepsilon, 1-\varepsilon], \\ 2, & t\in[1-\varepsilon, 1]. \end{cases} \tag{11}$$

(ε has now different meaning than in (1)). If $\varepsilon > 0$ is small enough, then the problem (4), (5) has multiple solutions.

Moreover, if $\varepsilon \to 0$, then the number of solutions of the problem tends to infinity.

2. Lemniscatic functions

The lemniscatic functions, namely, the lemniscatic sine $\operatorname{sl} t$ and cosine $\operatorname{cl} t$ appear in our constructions. These functions can be introduced as solutions of the equation $x'' = -2x^3$ subject to the initial conditions $x(0) = 0,\ x'(0) = 1$ or $x(0) = 1,\ x'(0) = 0$, respectively.

Alternatively, the lemniscatic sine function $\operatorname{sl} t$ can be defined in implicit way as $t = \int_0^{\operatorname{sl} t} \frac{ds}{\sqrt{1-s^4}}$ in the interval $\big[0, \int_0^1 \frac{ds}{\sqrt{1-s^4}}\big]$ and extended to infinity by periodicity. The period is $4A$, where $A = \int_0^1 \frac{ds}{\sqrt{1-s^4}}$. The number $A \approx 1.311$ for the lemniscatic functions is an analogue of the number $\frac{\pi}{2}$ for usual trigonometric functions.

The lemniscatic functions $\operatorname{sl} t$ and $\operatorname{cl} t$ are much like the usual sine and cosine functions (Fig. 3), but they are not the derivatives of each other, but the following holds:

$$\operatorname{sl}' t = \operatorname{cl} t(1+\operatorname{sl}^2 t), \quad \operatorname{cl}' t = -\operatorname{sl} t(1+\operatorname{cl}^2 t). \tag{12}$$

Both a complete list of formula for the lemniscatic functions as well as classical references can be found in [2], [3]. The lemniscatic functions can be handled symbolically by the Wolfram Mathematica program using the representation in terms of the built-in Jacobi elliptic functions.

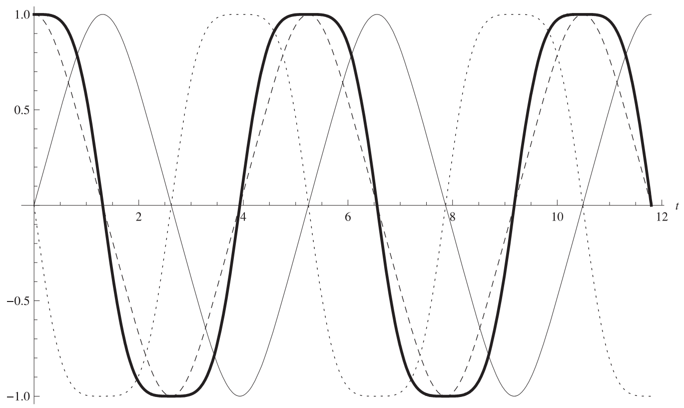

FIGURE 3. $\mathrm{sl}\,t$-thin, $\mathrm{cl}\,t$-dashed, $\mathrm{sl}'t$-thick line, $\mathrm{cl}'t$-dotted line.

We consider two auxiliary the initial value problems

$$x'' = -2x^3, \qquad x(-1) = 0, \qquad x'(-1) = \beta, \tag{13}$$

$$x'' = -2x^3, \qquad x(1) = 0, \qquad x'(1) = \gamma. \tag{14}$$

A solution of the Cauchy problem

$$x'' = -2x^3, \quad x(0) = 0, \quad x'(0) = \alpha \qquad \text{is} \quad \alpha^{1/2}\mathrm{sl}\,(\alpha^{1/2}t).$$

Then, using the change of the independent variable, solutions of the problems (13), (14) are

$$x_1(t;\beta) = \beta^{1/2}\mathrm{sl}\,\big(\beta^{1/2}(t+1)\big)$$

and

$$x_3(t;\gamma) = \pm\gamma^{1/2}\mathrm{sl}\,\big(\gamma^{1/2}(t-1)\big).$$

The sign is "–" if solutions with even number of zeros in $(-1, 1)$ are considered, and, respectively, "+" for solutions with odd number of zeros. Denote a solution in the middle interval $x_2(t)$. This solution is a linear function that connects smoothly solutions $x_1(t)$ and $x_3(t)$.

3. Multiple solutions of BVP

Consider equation (4) with the coefficient $q(t)$ given in (11). A solution $x(t)$ of this equation consists of three parts x_1, x_2 and x_3, each given in the intervals $[-1,-1+\varepsilon]$, $[-1+\varepsilon,1-\varepsilon]$, $[1-\varepsilon,1]$ respectively ($\varepsilon\in(0,1)$).

To define $x_1(t)$ consider the Cauchy problem

$$x''=-2x^3,\quad x(-1)=0,\quad x'(-1)=\beta>0. \tag{15}$$

This problem has an explicit solution

$$x_1(t)=\beta^{\frac{1}{2}}\operatorname{sl}\left[\beta^{\frac{1}{2}}(t+1)\right]. \tag{16}$$

A solution $x_3(t)\colon[1-\varepsilon,1]\to R$ of the Cauchy problem

$$x''=-2x^3,\quad x(1)=0,\quad x'(1)=\pm\gamma,\quad \gamma>0 \tag{17}$$

is

$$x_3(t)=\pm\gamma^{\frac{1}{2}}\operatorname{sl}\left[\gamma^{\frac{1}{2}}(t-1)\right]. \tag{18}$$

Suppose we are looking for solutions of (4), (2) which have even number of zeros (the most interesting case). Then

$$x_3(t)=-\gamma^{\frac{1}{2}}\operatorname{sl}[\gamma^{\frac{1}{2}}(t-1)] \tag{19}$$

and

$$x_3'(1)=-\gamma.$$

In order $x(t)$ to be C^2-function both solutions x_1 and x_3 are to be smoothly connected by a middle function $x_2(t)$ which is linear since $q(t)\equiv 0$ in the middle interval $[-1+\varepsilon,1-\varepsilon]$.

The following relations are to be satisfied:

$$\begin{cases} x_1(-1+\varepsilon)=x_2(-1+\varepsilon),\\ x_2'(t)=x_1'(-1+\varepsilon)=x_3'(1-\varepsilon), \qquad \text{for all}\quad t\in[-1+\varepsilon,1-\varepsilon],\\ x_3(1-\varepsilon)=x_2(1-\varepsilon).\end{cases} \tag{20}$$

The middle relation is written as

$$\beta\operatorname{sl}'\left(\sqrt{\beta}\varepsilon\right)=-\gamma\operatorname{sl}'\left(\sqrt{\gamma}\varepsilon\right). \tag{21}$$

A linear function

$$x_2(t)=kt+c\qquad\text{with}\qquad k=\beta\operatorname{sl}'\left(\sqrt{\beta}\varepsilon\right)=-\gamma\operatorname{sl}'\left(\sqrt{\gamma}\varepsilon\right)$$

satisfies

$$x_2(-1+\varepsilon)=x_1(-1+\varepsilon)\quad\text{if}\quad c=\sqrt{\beta}\operatorname{sl}\left(\sqrt{\beta}\varepsilon\right)+k(1-\varepsilon).$$

In order to satisfy also

$$x_2(1-\varepsilon)=x_3(1-\varepsilon),$$

the relation

$$k(1-\varepsilon)+c=x_3(1-\varepsilon)=\sqrt{\gamma}\operatorname{sl}\left(\sqrt{\gamma}\varepsilon\right)$$

should hold or

$$k(1-\varepsilon)+\sqrt{\beta}\,\mathrm{sl}\left(\sqrt{\beta}\varepsilon\right)+k(1-\varepsilon)=\sqrt{\gamma}\,\mathrm{sl}\left(\sqrt{\gamma}\varepsilon\right).$$

This line can be rewritten as

$$k(1-\varepsilon)+\sqrt{\beta}\,\mathrm{sl}\left(\sqrt{\beta}\varepsilon\right)=-k(1-\varepsilon)+\sqrt{\gamma}\,\mathrm{sl}\left(\sqrt{\gamma}\varepsilon\right)$$

or

$$\beta\,\mathrm{sl}'\big(\sqrt{\beta}\varepsilon\big)(1-\varepsilon)+\sqrt{\beta}\,\mathrm{sl}\left(\sqrt{\beta}\varepsilon\right)=\gamma\,\mathrm{sl}'\big(\sqrt{\gamma}\varepsilon\big)(1-\varepsilon)+\sqrt{\gamma}\,\mathrm{sl}\left(\sqrt{\gamma}\varepsilon\right).$$

Thus in order to define a solution of the problem (4), (2) which have even number of zeros the values

$$\beta=x_1'(-1)\quad\text{and}\quad-\gamma=x_3'(1)$$

have to satisfy the system

$$\begin{cases}\beta\,\mathrm{sl}'\big(\sqrt{\beta}\varepsilon\big)(1-\varepsilon)+\sqrt{\beta}\,\mathrm{sl}\left(\sqrt{\beta}\varepsilon\right)=\gamma\,\mathrm{sl}'\big(\sqrt{\gamma}\varepsilon\big)(1-\varepsilon)+\sqrt{\gamma}\,\mathrm{sl}\left(\sqrt{\gamma}\varepsilon\right),\\ \beta\,\mathrm{sl}'\big(\sqrt{\beta}\varepsilon\big)=-\gamma\,\mathrm{sl}'\big(\sqrt{\gamma}\varepsilon\big).\end{cases}\tag{22}$$

Introducing the new variables

$$u=\beta^{\frac{1}{2}}\varepsilon,\quad v=\gamma^{\frac{1}{2}}\varepsilon\tag{23}$$

the system (22) turns to

$$\begin{cases}\frac{1}{\varepsilon}u\,\mathrm{sl}\,u+\frac{1-\varepsilon}{\varepsilon^2}u^2\mathrm{sl}'u=\frac{1}{\varepsilon}v\,\mathrm{sl}\,v+\frac{1-\varepsilon}{\varepsilon^2}v^2\mathrm{sl}'v,\\ \frac{1}{\varepsilon^2}u^2\mathrm{sl}'u=-\frac{1}{\varepsilon^2}v^2\mathrm{sl}'v\end{cases}\tag{24}$$

or

$$\begin{cases}u\,\mathrm{sl}\,u+\frac{1-\varepsilon}{\varepsilon}u^2\mathrm{sl}'u=v\,\mathrm{sl}\,v+\frac{1-\varepsilon}{\varepsilon}v^2\mathrm{sl}'v,\\ u^2\mathrm{sl}'u=-v^2\mathrm{sl}'v\end{cases}\tag{25}$$

or finally,

$$\begin{cases}\delta\,u\,\mathrm{sl}\,u+u^2\mathrm{sl}'u=\delta\,v\,\mathrm{sl}\,v+v^2\mathrm{sl}'v,\\ u^2\mathrm{sl}'u=-v^2\mathrm{sl}'v,\end{cases}\tag{26}$$

where

$$\delta=\frac{\varepsilon}{1-\varepsilon}.$$

Denote

$$\begin{aligned}\Phi(u,v,\delta)&=\delta\,u\,\mathrm{sl}\,u+u^2\mathrm{sl}'u-\delta\,v\,\mathrm{sl}\,v-v^2\mathrm{sl}'v,\\ \Psi(u,v)&=u^2\mathrm{sl}'u+v^2\mathrm{sl}'v.\end{aligned}\tag{27}$$

For graphical description of zeros of the above functions see Fig. 4 and Fig. 5.

We will investigate system (26). Any positive solution (u,v) of this system generates a solution of the BVP (4), (11), (5). Our aim is to show that there exist infinitely many solutions of BVP and there are solutions with any prescribed number of zeros in $(-1,1)$.

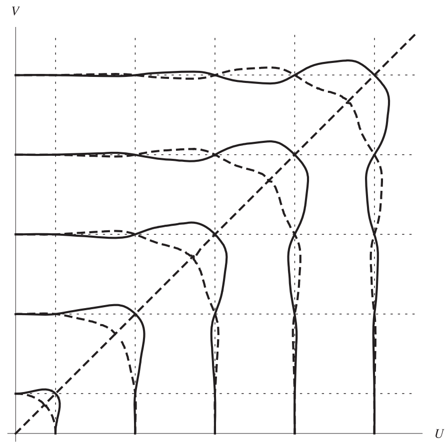

FIGURE 4. Zeros of Φ and $\Psi : \Psi(u,v) = 0$ – solid line, $\Phi(u,v,0) = 0$ – dashed line.

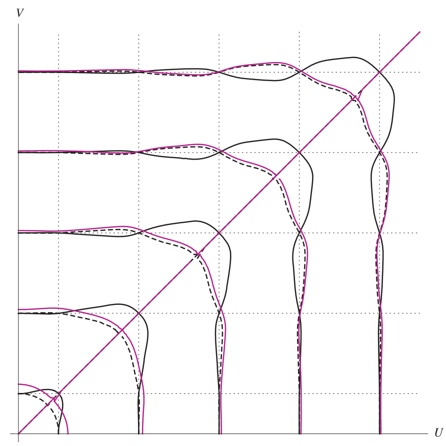

FIGURE 5. $\Phi(u,v,1) = 0$ – colored line.

First, consider the auxiliary system

$$\begin{cases} u^2\mathrm{sl}'u = v^2\mathrm{sl}'v, \\ u^2\mathrm{sl}'u = -v^2\mathrm{sl}'v \end{cases} \tag{28}$$

which for u, v positive reduces to a very simple system

$$\begin{cases} \mathrm{sl}'u = 0, \\ \mathrm{sl}'v = 0. \end{cases} \tag{29}$$

The latter system has infinitely many solutions $\big((2i-1)A, (2j-1)A\big)$, where $i, j = 1, \dots,$ $A = \int_0^1 \frac{ds}{\sqrt{1-s^4}}$. These solutions are cross points of gridlines depicted in Fig. 4. Any positive pair (u, v), which satisfies the system (26), corresponds to a solution of the boundary value problem (4), (5), where $q(t)$ is as in (11). For any such solution $x(t)$ one has that

$$x'(-1) = \beta > 0 \quad \text{and} \quad x'(1) = -\gamma < 0.$$

The number of zeros of such solutions is even, including zero (for positive solutions in $(-1, 1)$).

Our main result is that the system (26) has a solution in a small vicinity of a mesh point $\big((2i-1)A, (2j-1)A\big)$ for any positive i and j provided that δ is sufficiently small. Accordingly the existence of a solution to the BVP (4), (5) follows.

3.1. Auxiliary system (28)

The auxiliary system has solutions at the points $p_{ij} = \big((2i-1)A, (2j-1)A\big)$, where A is the above defined constant and i, j are positive integers. We would like to show more.

Consider the vector field

$$\Upsilon(u,v) = \big\{\Phi(u,v,0), \Psi(u,v)\big\} = \big\{u^2\mathrm{sl}'u - v^2\mathrm{sl}'v, u^2\mathrm{sl}'u + v^2\mathrm{sl}'v\big\} \tag{30}$$

associated with the system (28). We will show that the winding number $\gamma\big(C_{ij},\ \Upsilon(u,v)\big)$ of this vector field on a circle of radius 1 centered at a point p_{ij} is equal to either +1 or –1. All the needed information about planar vector fields, winding numbers etc. can be found in [1, Ch. 1].

Let us prove this for the point $p_{11} = (A, A)$. Make the change of variables $U = u - A$, $V = v - A$ and consider the vector field $\Upsilon(u,v)$ which in new coordinates is

$$\big\{(U+A)^2\mathrm{sl}'(U+A) - (V+A)^2\mathrm{sl}'(V+A),\\ (U+A)^2\mathrm{sl}'(U+A) + (V+A)^2\mathrm{sl}'(V+A)\big\}.$$

Using the formula

$$\mathrm{sl}'(U+A) = \mathrm{cl}\,(U+A)\big(1+\mathrm{sl}^2(U+A)\big) = -\mathrm{sl}\,U(1+\mathrm{cl}^2U) = -\mathrm{cl}'U$$

one obtains the vector field

$$\big\{-(U+A)^2\mathrm{cl}'U + (V+A)^2\mathrm{cl}'(V+A), -(U+A)^2\mathrm{cl}'U - (V+A)^2\mathrm{cl}'V\big\},$$

which is to be considered on a circle centered at the origin $(0,0)$ and with radius 1. Therefore $U = \cos\Theta$, $V = \sin\Theta$, $\Theta \in [0, 2\pi)$. The analysis of sign changing of the above functions confirms that $\gamma\big(C_{11}, \Upsilon(u,v)\big) = 1$.

This is illustrated by Fig. 6. The end points of vectors of the latter vector field given on a unity circle centered at the origin go along the curve shown in the picture.

3.2. Solvability of system (26) at a separate point p_{ij}

We wish to show that for any fixed point p_{ij} there exists a small $\delta > 0$ such that a solution of system (26) exists located in a vicinity of a point p_{ij}.

Proposition 3.1. *Suppose that $\big|\gamma\big(C_{ij}, \Upsilon(u,v)\big)\big| = 1$, where C_{ij} is a circle of radius 1 centered at the point p_{ij}. Then $\delta > 0$ exists such that*

$$\gamma\big(C_{ij}, \Xi(u,v)\big) = \gamma\big(C_{ij}, \Upsilon(u,v)\big),$$

where

$$\Xi(u,v) = \big\{\Phi(u,v,\delta), \Psi(u,v)\big\}.$$

Consequently, there exists a solution of system (26) *and a corresponding solution of the BVP* (4), (5).

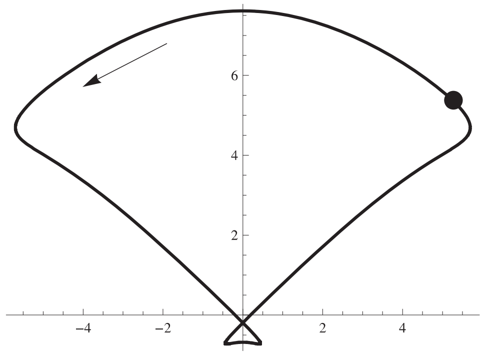

FIGURE 6. The small disk corresponds to $\Theta = 0$. The arrow shows the direction of rotation as Θ increases, $\gamma = 1$.

P r o o f. We use the notion of the principal part of the vector field as introduced in [1, Ch. 1, § 4]. The vector field $\Psi(u, v)$ is called the principal part of $\Phi(u, v)$ if

$$\Phi = \Psi + \omega$$

and $\|\omega\| < \|\Psi\|$, where $\|\cdot\|$ is the length of the vector.

Let us show that $\Upsilon(u, v)$ is the principal part for $\Xi(u, v)$ if $\delta > 0$ is sufficiently small. Comparing both vector fields we conclude that

$$\omega(u, v) = \Xi(u, v) - \Upsilon(u, v) = \big\{\delta(u \operatorname{sl} u - v \operatorname{sl} v), 0\big\}.$$

One has that

$$\|\omega(u, v)\| = |\delta(u \operatorname{sl} u - v \operatorname{sl} v)|$$

and

$$\|\Upsilon(u, v)\| = \sqrt{(u^2 \operatorname{sl}' u - v^2 \operatorname{sl}' v)^2 + (u^2 \operatorname{sl}' u + v^2 \operatorname{sl}' v)^2}.$$

The right side reduces to

$$\sqrt{2(u^4 \operatorname{sl}'^2 u + v^4 \operatorname{sl}'^2 v)}.$$

The function $u^4 \operatorname{sl}'^2 u + v^4 \operatorname{sl}'^2 v$ has a positive minimum m_{ij} on any circle of radius 1 centered at a point p_{ij}. Choosing δ sufficiently small, one has that

$$|\delta(u \operatorname{sl} u - v \operatorname{sl} v)| < \sqrt{2m_{ij}}$$

for any (u, v) on a circle.

Then the assertion follows from Theorem 4.6 in [1] since the vector fields Ξ and Υ are homotopic on a circle.

3.3. Solvability of system (26)

We wish to show that for any fixed $\delta > 0$ there exist infinitely many solutions of the system (26). First consider the system (28) which has solutions at the points

$$p_{ij} = \big((2i-1)A, (2j-1)A\big), \qquad i, j = 1, \dots \tag{31}$$

The vector field

$$\Upsilon(u,v) = \big\{\Phi(u,v,0), \Psi(u,v)\big\} = \big\{u^2 \mathrm{sl}'u - v^2 \mathrm{sl}'v, u^2 \mathrm{sl}'u + v^2 \mathrm{sl}'v\big\} \tag{32}$$

is defined for positive (u,v). The winding number $\gamma\big(C_{ij}, \Upsilon(u,v)\big) \neq 0$ for any point of the mesh (31). Consider the normalized vector field

$$\Upsilon_{norm}(u,v) = \frac{1}{u^2+v^2}\Upsilon(u,v).$$

Since vectors of the vector fields $\Upsilon(u,v)$ and $\Upsilon_{norm}(u,v)$ are collinear, the following holds.

Proposition 3.2. *Let C_{ij} be a unity circle centered at a point p_{ij} of the mesh* (31)*. Then the winding number $\gamma\big(C_{ij}, \Upsilon_{norm}(u,v)\big)$ is not zero.*

Consider the vector field $\Xi(u,v) = \big\{\Phi(u,v,\delta), \Psi(u,v)\big\}$, where the components are defined in (27).

Theorem 3.1. *For any $\delta > 0$ there exists $R > 0$ such that*

$$\gamma\big(C_{ij}, \Xi(u,v)\big) = \gamma\big(C_{ij}, \Upsilon(u,v)\big)$$

for any mesh point p_{ij} located in the region

$$K_R = \big\{u^2+v^2 > R^2\big\} \bigcap \{u>0, v>0\}.$$

P r o o f. If we rewrite the system (26) in the form

$$\begin{cases} \dfrac{\delta\, u\, \mathrm{sl}\, u}{u^2+v^2} + \dfrac{u^2 \mathrm{sl}'u}{u^2+v^2} = \dfrac{\delta\, v\, \mathrm{sl}\, v}{u^2+v^2} + \dfrac{v^2 \mathrm{sl}'v}{u^2+v^2}, \\ \dfrac{u^2 \mathrm{sl}'u}{u^2+v^2} = -\dfrac{v^2 \mathrm{sl}'v}{u^2+v^2} \end{cases} \tag{33}$$

and compare it with the "unperturbed" system

$$\begin{cases} \dfrac{u^2 \mathrm{sl}'u}{u^2+v^2} = \dfrac{v^2 \mathrm{sl}'v}{u^2+v^2}, \\ \dfrac{u^2 \mathrm{sl}'u}{u^2+v^2} = -\dfrac{v^2 \mathrm{sl}'v}{u^2+v^2}, \end{cases} \tag{34}$$

then the conclusion is that for $u^2 + v^2$ large enough the vector field $\Upsilon_{norm}(u, v)$ associated with the system (34) is a principal part of the vector field associated with the system (33) on the set

$$\Big\{\{u^2 + v^2 > R^2\} \bigcap \{u > 0, v > 0\}\Big\} \setminus \bigcup_{ij} C_{ij},$$

that is, on the set K_R minus neighbourhoods of points p_{ij}.

Therefore the system (33) has a solution in a neighborhood of any mesh point p_{ij} with large enough i and j.

Corollary 3.1. *For any $\varepsilon \in (0, 1)$ the boundary value problem* (4), (5), (11) *has infinitely many solutions.*

3.4. Samples of solutions

Evidently the system (26) has symmetric solutions $u = v$. These points give rise to even solutions of the problem (one of them is depicted in Fig. 11).

Below a number of solutions of the system (26) is indicated (Fig. 7) and seven solutions with four internal zeros are computed (Fig. 8 to Fig. 11).

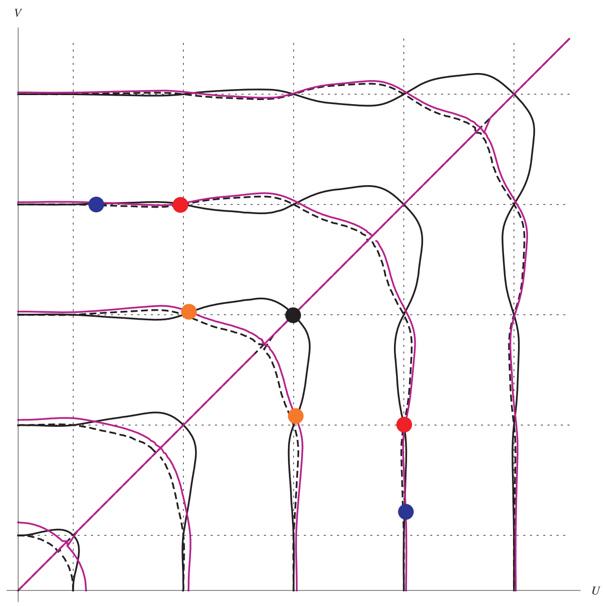

Figure 7. Some solutions of system (26), $\delta = 1$.

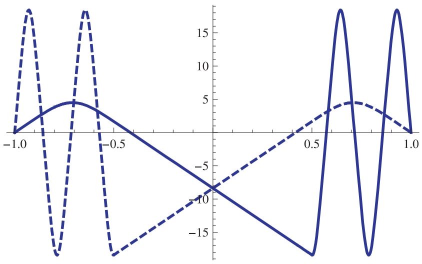

FIGURE 8. Solutions of the problem corresponding to two external (with respect to the diagonal) points in Fig. 7.

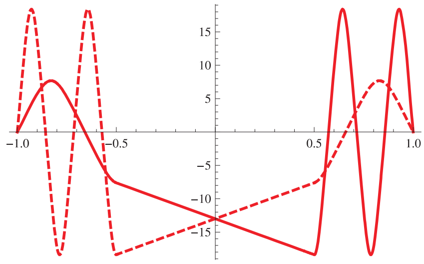

FIGURE 9. Solutions of the problem corresponding to two intermediate points.

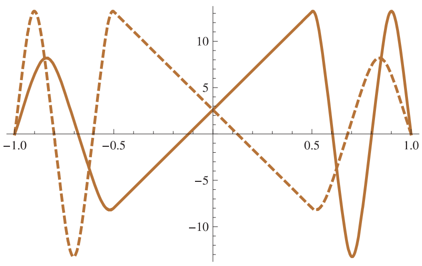

FIGURE 10. Solutions of the problem corresponding to two points adjacent to the diagonal in Fig. 7.

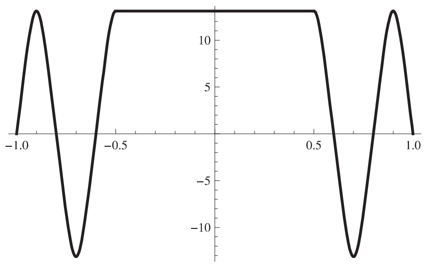

FIGURE 11. Solution of the problem corresponding to the point on the diagonal.

REFERENCES

[1] KRASNOSELSKII, M. A.—PEROV, A. I.—POVOLOTSKII, A. I.—ZABREJKO, P. P.: *Plane vector fields*. Nauka, Moscow, 1963. (In Russian)

[2] GRITSANS, A.—SADYRBAEV, F.: *Remarks on lemniscatic functions*, Acta Univ. Latviensis **688** (2005), 39–50.

[3] GRITSANS, A.—SADYRBAEV, F.: *Lemniscatic functions in the theory of the Emden–Fowler diferential equation*, in: Proceedings of IMCS of University of Latvia, Vol. 3, Riga, 2003, pp. 69–82, http://www.lumii.lv/Pages/sbornik/s3f3v1.pdf

[4] GRITSANS, A.—SADYRBAEV, F.: *Characteristic numbers of nonautonomous Emden–Fowler type equations*, Math. Modell. Anal. **11** (2006), 243–252.

[5] MOORE, R.—NEHARI, Z.: *Nonoscillation theorems for a class of nonlinear differential equations,* Trans. Amer. Math. Soc. **93** (1959), 30–52.

[6] NEHARI, Z.: *Characteristic values associated with a class of nonlinear second-order differential equations,* Acta Math. **105** (1961), 141–175.

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