Relativistic Aspects of Moving Charges Interaction based on Lienard-Wiechert Potentials

Abstract: This study presents some basic aspects of electrostatic forces based on wave-particle interaction described by Lienard-Wiechert potentials. Using the scalar and vector potentials the force obtained for a certain point in space is analyzed. Dynamical aspects connected to mass dependence of velocity are obtained. Hypothetical possibilities corresponding to the case when velocity exceeds light speed are also presented.

1 Introduction

Starting from Maxwell equations, the Lienard-Wiechert retarded potentials present the electromagnetic field created in a certain point of space by a moving electrical charge situated at a certain distance. Next the electric and magnetic field are determined, so the force acting upon another electrical charge in this point of space can be easily computed.

However, less attention was given to the fact that for estimating position and mass of the source of the propagating electromagnetic field this force can be considered as a basic physical quantity. Unlike energy and distances, the force considered along a certain axis can be either positive or negative without any intuitive problems. This allows an interpretation of relativistic dynamics based on electromagnetic force to avoid intuitive difficulties generated by an imaginary square root when velocity exceeds light speed.

As was shown in papers as [1], a moving electrical charge generates inside a certain reference system a propagating electromagnetic field. The reference system is considered to be inertial, despite the fact that the speed $v$ of this moving charge is not supposed to be a constant quantity (as shown in [2]). This electromagnetic field is described by the scalar potential $\Phi$ and by the vector potential $\vec{A}$. For simplicity, the velocity of this field is considered to be $c$ (the light speed) inside this reference system along its entire trajectory (in the most general case this quantity would depend on the electrical permittivity and magnetic permeability specific to the material medium considered).

According to classical electrodynamics, the charge density $\rho$ for for a point charge $q$ moving on a trajectory $R_q(t)$ can be written as

$$\rho(\vec{r}, t) = q \delta^3(\vec{r} - \vec{R}_q(t))$$

(1)

and the current density $\vec{J}$ can be written as

$$\vec{J}(\vec{r}, t) = q \vec{\dot{R}}_q(t) \delta^3(\vec{r} - \vec{R}_q(t))$$

(2)

where

$$\vec{\dot{R}}_q(t) = \frac{d\vec{R}_q(t)}{dt}$$

(3)
For practicality, it is useful to define the retarded time $t_r$ as
\[ t_r = t - \frac{|\vec{r} - \vec{R}_q(t_r)|}{c} \] (4)
(the time moment for which the emitted field reaches the observer situated in $\vec{r}$ at time $t$).

Under Lorentz gauge, the scalar potential $\Phi$ and vector potential $\vec{A}$ can be described as
\[ \Phi(\vec{r}, t) = \frac{q}{4\pi\varepsilon_0} \frac{1}{R - \frac{\vec{v}R}{c}} \] (5)
\[ \vec{A}(\vec{r}, t) = \frac{q}{4\pi\varepsilon_0 c^2} \frac{\vec{v}}{R - \frac{\vec{v}R}{c}} \] (6)

where
\[ \vec{R} = \vec{r} - \vec{R}_q(t_r), \quad \vec{v} = \vec{R}_q(t_r) \] (7)

The electric field $\vec{E}$ and the magnetic field $\vec{B}$ can be evaluated using
\[ \vec{E}(\vec{r}, t) = -\nabla \Phi(\vec{r}, t) - \frac{\partial \vec{A}(\vec{r}, t)}{\partial t} \] (8)
\[ \vec{B}(\vec{r}, t) = \nabla \times \vec{A}(\vec{r}, t) \] (9)

Finally we conclude that:
(i) the electric field $\vec{E}$ as a sum of two terms
\[ \vec{E} = \vec{E}_1 + \vec{E}_2 \] (10)
\[ \vec{E}_1(\vec{r}, t) = \frac{q}{4\pi\varepsilon_0} \left( \frac{\vec{R} - \frac{\vec{v}R}{c}}{R - \frac{\vec{v}R}{c}} \right) \left( 1 - \frac{\vec{v}^2}{c^2} \right) \] (11)
\[ \vec{E}_2(\vec{r}, t) = \frac{q}{4\pi\varepsilon_0} \left( \vec{R} \times \left( \left( \frac{\vec{R} - \frac{\vec{v}R}{c}}{R - \frac{\vec{v}R}{c}} \right) \times \frac{\vec{v}}{c} \right) \right) \] (12)
(ii) the magnetic field $\vec{B}$ as a sum of three terms
\[ \vec{B} = \vec{B}_1 + \vec{B}_{2a} + \vec{B}_{2b} \] (13)
\[ \vec{B}_1(\vec{r}, t) = \frac{q}{4\pi\varepsilon_0 c^2} \left( -\vec{R} \times \vec{v} \right) \left( 1 - \frac{\vec{v}^2}{c^2} \right) \] (14)
\[ \vec{B}_{2a}(\vec{r}, t) = \frac{q}{4\pi\varepsilon_0 c^2} \left( -\vec{R} \times \frac{\vec{v}}{c} \right) \left( \frac{\vec{v}R}{c^2} \right) \] (15)
\[ \vec{B}_{2b}(\vec{r}, t) = \frac{q}{4\pi\varepsilon_0 c^2} \left( -\vec{R} \times \frac{\vec{v}}{c} \right) \left( R - \frac{\vec{v}R}{c} \right) \] (16)

It should be emphasized that the relativistic factor $1 - \frac{v^2}{c^2}$ has appeared without using the Lorentz formulae; moreover, the square root of this expression is missing. As will be shown in next paragraph, the restriction regarding the magnitude of velocity $v$ as related to light speed $c$ is no more required [1–3, 6, 7].
2 Relativistic Aspects Connected to Electrostatic Forces Generated by Moving Charges

The force $F$ generated by the electric and magnetic fields which acts upon an electrical charge $Q$ moving with speed $\vec{V}$ in the point of space $\vec{r}$ at time moment $t$ corresponds to

$$\vec{F}(\vec{r}, t) = Q(\vec{r}, t)\vec{E}(\vec{r}, t) + Q(\vec{r}, t)\vec{V}(\vec{r}, t)\vec{B}(\vec{r}, t)$$ \hspace{1cm} (17)

In order to demonstrate the role of basic relativistic aspects implied by the use of electromagnetical force that determines the source of the received electromagnetic field, the simplified case when the speed $v$ is constant along Ox axis (null acceleration and null vector product $\vec{R} \times \vec{v}$) will be analyzed.

Consequently, all terms of the magnetic field $\vec{B}(\vec{r}, t)$ and the second term $E_2$ of the electric field vanish. The single nonzero term will be the first term $E_1$ of the electric field (corresponding to the electrostatic field).

In this case, an observer situated in the point of space $\vec{r}$ at time moment $t$ could use for estimating the source of the received electromagnetic field the electrostatic force $F_{el}$ acting upon an electrical charge $Q$ on a very short time interval around this $t$ moment of time. By equalizing this force with the force emitted by a supposed motionless body possessing the same electrical charge situated in a certain point of $R'$ coordinate along Ox axis, we conclude that

$$Q\vec{E}(\vec{r}, t) = \frac{Qq}{4\pi\varepsilon_0} \frac{\vec{R'}}{R'^3}$$ \hspace{1cm} (18)

By substituting $\vec{E}$ with $E_1$ (the single nonzero term), we conclude that

$$\frac{Qq}{4\pi\varepsilon_0} \frac{\vec{R'}}{R'^3} = \frac{Qq}{4\pi\varepsilon_0} \frac{(\vec{R} - \vec{vR}/c)}{(\vec{R} - \vec{vR}/c)^3} \left(1 - \frac{v^2}{c^2}\right)$$ \hspace{1cm} (19)

Simplifying with $Qq/4\pi\varepsilon_0$ we conclude that

$$\frac{\vec{R'}}{R'^3} = \frac{(\vec{R} - \vec{vR}/c)}{(\vec{R} - \vec{vR}/c)^3} \left(1 - \frac{v^2}{c^2}\right)$$ \hspace{1cm} (20)

Since $\vec{R}, \vec{R'}, \vec{v}$ are vectors oriented along Ox axis, we can substitute

$$\vec{R} \leftrightarrow x\vec{i}, \vec{R'} \leftrightarrow x'\vec{i}, \vec{v} \leftrightarrow v\vec{i}$$ \hspace{1cm} (21)

where $\vec{i}$ represents the unit vector for the Ox axis. Thus

$$\frac{x'\vec{i}}{|x'|^2} = \frac{(x\vec{i} - \frac{vx}{c}\vec{i}) \left(1 - \frac{v^2}{c^2}\right)}{(x - \frac{vx}{c})^2}$$ \hspace{1cm} (22)

If Right Hand Side of previous equation is a positive quantity, then $x'$ (the numerator of Left Hand Side) should be a positive quantity also ($x' = |x'|$). We conclude that

$$\frac{1}{x'^2} = \frac{\left(1 - \frac{v^2}{c^2}\right)}{(x - \frac{vx}{c})^2}$$ \hspace{1cm} (23)

Thus quantity $x'$ corresponding to the supposed position for the source (estimated using the electrostatic force $E_1$, the only available) would be

$$x' = \frac{(x - \frac{vx}{c})}{\sqrt{1 - \frac{v^2}{c^2}}}$$ \hspace{1cm} (24)
If the synchronization moment of time between the reference system of the moving charge and the reference system of the observer would be considered when the electromagnetic field is emitted by the moving charge (this means the time is set to zero in both inertial reference systems), quantities \( x \) and \( \tau = x/c \) would correspond to the position and time moment for the source of the electromagnetic field (generated by the moving charge) for case \( v = 0 \) (\( x \) is considered to be positive). Substituting

\[
\frac{x}{c} \leftrightarrow \tau
\]

in previous equation, we conclude that

\[
x' = \frac{x - v\tau}{\sqrt{1 - v^2/c^2}}
\]  

(26)

The time moment \( \tau' \) for the moment of time when the electromagnetic field was emitted can be further determined by dividing \( x' \) quantity (the supposed distance) to \( c \) (the speed of the propagating electromagnetic field). We conclude that

\[
\tau' = \frac{(\frac{x}{c} - \frac{v}{c}\tau)}{\sqrt{1 - v^2/c^2}}
\]  

(27)

By substituting

\[
\frac{x}{c} \leftrightarrow \tau
\]

(28)

in the first term at numerator, and by substituting

\[
\tau \leftrightarrow \frac{x}{c}
\]

(29)

in the second term at numerator, we conclude that

\[
\tau' = \frac{(\tau - \frac{x}{c})}{\sqrt{1 - v^2/c^2}}
\]  

(30)

It can be easily noticed that final expressions obtained for supposed quantities \( x' \), \( \tau' \) as related to quantities \( x \), \( \tau \) corresponding to the motionless case are identical to Lorentz formulae from the Special Relativity Theory. They were derived using exclusively classical electrodynamics and classical space-time relations.

At first view this attempt seems that it only justifies Lorentz formulae by using a classical approach. However, it should be emphasized that no square root for \( 1 - v^2/c^2 \) has appeared for the electromagnetic field. The electrostatic field corresponding to the simplified case when the speed \( v \) is constant along \( Oz \) axis (null acceleration and null vector product \( \vec{R} \times \vec{v} \)) has been used for determining a supposed position. So, the previous case can be analyzed without any restrictions if we consider that speed \( v \) would exceed light speed \( c \) (this means \( v > c \), contrary to the assumptions of Special Relativity Theory).

The most convenient method for applying previously mentioned aspects when \( v > c \) (under the same assumptions of null acceleration and null vector product \( \vec{R} \times \vec{v} \)) requires determining the electrostatic force using the same formula for electric field \( \vec{E} \) based on Lienard-Wiechert potentials, supposing the source of the electromagnetic field is situated in a certain position \( x' \) along \( Ox \) axis.

By simplifying with \( Qq/(4\pi\epsilon_0) \) the same equation is obtained

\[
\frac{x'}{|x'|^3} = \frac{(x - \frac{vx}{c^2}) \left(1 - \frac{v^2}{c^2}\right)}{(x - \frac{vx}{c^2})^3}
\]

(31)

but the Right Hand Side is negative. By simplifying with \( (x - vx/c) \) (supposed to be a nonzero value) in Right Hand Side and substituting \( x' \leftrightarrow -|x'| \) (as for a negative quantity) we conclude that

\[
\frac{-|x'|}{|x'|^3} = \frac{(\frac{v^2}{c^2} - 1)}{(x - \frac{vx}{c^2})^2}
\]

(32)
Thus $|x'|$ would be

$$
|x'| = \frac{|x - \frac{v x}{c}|}{\sqrt{\frac{v^2}{c^2} - 1}}
$$

(33)

and finally

$$
 x' = -|x'| = -\frac{|x - \frac{v x}{c}|}{\sqrt{\frac{v^2}{c^2} - 1}}
$$

(34)

Depending on direction of $\vec{v}$ as related to $\vec{R}$, quantity $x - vx/c$ can be either a positive or negative when $v$ exceeds $c$, so the modulus symbol must be kept in previous formula.

Substituting $x/c \leftrightarrow \tau$ (the propagating time, $x$ is considered positive) we conclude that

$$
 x' = -|x'| = -\frac{|x - v \tau|}{\sqrt{\frac{v^2}{c^2} - 1}}
$$

(35)

The time moment $\tau'$ for the moment of time when the electromagnetic field was emitted can be determined also by dividing $|x'|$ quantity (the supposed distance) to $c$ (the speed of the propagating electromagnetic field). We conclude that

$$
 \tau' = \frac{|\frac{x}{c} - \frac{v}{c} \tau|}{\sqrt{\frac{v^2}{c^2} - 1}}
$$

(36)

By substituting

$$
 \frac{x}{c} \leftrightarrow \tau
$$

(37)

in the first term at numerator, and by substituting

$$
 \tau \leftrightarrow \frac{x}{c}
$$

(38)

in the second term at numerator, we conclude that

$$
 \tau' = \frac{|\tau - \frac{v}{c} \tau|}{\sqrt{\frac{v^2}{c^2} - 1}}
$$

(39)

Thus the Lorentz formulae have been extended to the case when $v > c$.

### 3 Conclusions

The study has presented spatial and temporal aspects connected to the electrostatic forces generated by Lienard-Wiechert potentials. The scalar and vector potentials emitted by an electrical charge $q$ are generating specific forces which are compared to the forces generated by a motionless particle (with the same electrical charge) situated in a supposed point of space. When these values are set to be equal, specific spatial and temporal aspects from Special Relativity are obtained. Moreover, these relativistic formula can be extended to the case when the particle emitting the electromagnetic potentials moves with speed $v > c$. Dynamical aspects connected to mass dependence of velocity can be easily obtained since for $v = c$ the electrostatic force generated by these Lienard-Wiechert potentials would present a null value. Thus the resulting acceleration would be zero, as for the case when the electrostatic force generated by a motionless electrical charge $q$ is divided to an infinite mass (there is no action of the emitted electromagnetic field upon an electrical charge $Q$ receiving this field).
References