Research Article

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Haar wavelet method for vibration analysis of nanobeams

DOI 10.1515/wwfaa-2016-0003
Received Jan 29, 2016; accepted Feb 29, 2016

Abstract: In the current study the Haar wavelet method is adopted for free vibration analysis of nanobeams. The size-dependent behavior of the nanobeams, occurring in nanostructures, is described by Eringen nonlocal elasticity model. The accuracy of the solution is explored. The obtained results are compared with ones computed by finite difference method. The numerical convergence rates determined are found to be in agreement with corresponding convergence theorems.

Keywords: Haar wavelet method; nonlocal elasticity; nanobeams; Richardson extrapolation

AMS: 74S30, 74B99, 65L10

1 Introduction

The Haar wavelet method considered herein has been introduced by Chen and Hsiao for solving lumped and distributed parameter systems [12]. According to approach proposed in [12] the highest order derivative included in the differential equation is expanded into the series of Haar functions. Such an approach allows to overcome shortcomings caused by discontinuities of the Haar functions. An alternate approach has been proposed by Cattani [10], Castro et al. [9], according to which the quadratic waves are “smoothed” with interpolating splines.

The Haar wavelet method introduced in [12] has been adjusted for solving wide class of differential and integral equations covering solid and fluid mechanics [15, 16, 19–22, 29, 46–48], mathematical physics [18, 36, 38–40], evolutionary equations [23, 30], etc. In [22] the Haar wavelet method for solving PDE is developed. In [16, 19, 20, 29, 46–48] the Haar wavelet method is adopted for analysis of composite structures. In [18] the Haar wavelets method is applied for solving differential equations characterizing the dynamics of a current collection system for an electric locomotive. The nuclear reactor dynamics equations are considered in [36, 38–40].

Recent treatments in the area of Haar wavelet method development cover solution of integral and integro-differential equations [6, 7, 17, 25], fractional partial differential equations [26, 41, 43, 44]. The accuracy issues of the Haar wavelet method are studied in [31, 32]. It has been proved in [31] that the order of convergence of the Haar wavelet method is equal to two. In [32] the Richardson extrapolation method is applied and it has been shown that the order of convergence of the extrapolated results is equal to four.

Some review papers covering development of the Haar wavelet method and its application can be referred as [27, 28].

The vibration analysis of nanostructures, based on nonlocal elasticity theory, is performed commonly by applying finite difference (FD) or differential quadrature (DQ) methods [8, 33, 34, 37], also finite element method [13, 42], Rayleigh-Ritz method [11], etc. A number of analytical or semi-analytical solutions are derived.
for particular problems [2, 5]. These solutions can be employed for estimating the accuracy of numerical methods.

In the current study the Haar wavelet method is adopted for free vibration analysis of nanobeams. The obtained numerical results are compared with the results achieved by applying FD method (the study is focused on strong formulation based method). The numerical order of convergence of the results is computed and validated.

2 Basics of Haar wavelets

The Haar function \( h_i(x) \) is defined as [12, 21]

\[
h_i(x) = \begin{cases} 
1 & \text{for } x \in [\xi_1(i), \xi_2(i)] \\
-1 & \text{for } x \in [\xi_2(i), \xi_1(i)] \\
0 & \text{elsewhere}
\end{cases}
\] (1)

In (1) \( i = m + k + 1 \), \( m = 2^j \) (\( M = 2^J \)) stands for a maximum number of square waves that can be sequentially deployed in interval \([A, B]\) and the parameter \( k \) indicates the location of the particular square wave

\[
\xi_1(i) = A + 2k\mu \Delta x, \quad \xi_2(i) = A + (2k + 1)\mu \Delta x, \quad \xi_3(i) = A + 2(k + 1)\mu \Delta x, \quad \mu = M/m, \quad \Delta x = (B - A)/(2M).
\] (2)

The components of the operational matrix of integration can be obtained as n-th order integrals of Haar function (1)

\[
p_{n,i}(x) = \frac{1}{n!} \begin{cases} 
0 & \text{for } x \in [A, \xi_1(i)] \\
(x - \xi_1(i))^n & \text{for } x \in [\xi_1(i), \xi_2(i)] \\
(x - \xi_1(i))^n - 2(x - \xi_2(i))^n & \text{for } x \in [\xi_2(i), \xi_3(i)] \\
(x - \xi_1(i))^n - 2(x - \xi_2(i))^n + (x - \xi_3(i))^n & \text{for } x \in [\xi_3(i), B]
\end{cases}
\] (3)

An integrable and finite function in the interval \([A, B]\) can be expanded into Haar wavelets as

\[
f(x) = \sum_{i=1}^{\infty} a_i h_i(x) = a^T H
\] (4)

In the following sections an approach of the Haar wavelet method proposed in [1] is employed.

3 Governing differential equations for nonlocal Euler-Bernoulli beam

The governing differential equations for Euler-Bernoulli beam are derived in a number of papers [3, 8]. Thus, herein the detailed derivation of governing equations is omitted for conciseness sake. In the following, a short description of the basic concepts, assumptions and principles used is given.

Classical e.g. Euler-Bernoulli beam theory is founded on the following two key assumptions:

- Cross sections of the beam do not deform in a significant manner under the application of transverse or axial loads and can be assumed as rigid,
- During deformation, the cross section of the beam is assumed to remain planar and normal to the deformed axis of the beam.

Based on above assumptions the classical Euler-Bernoulli beam model describing free harmonic vibrations read

\[
EI \frac{d^4 W}{dx^4} - m_0 \Omega^2 W = 0,
\] (5)
According to free harmonic motion, the deflection is assumed in form \( w(x, t) = W(x) \sin(\omega t) \), where \( \omega \) stands for natural frequency of vibration. In (5) \( m_0 \) is mass moment of inertia, \( E \) and \( I \) stand for the Young’s modulus and the second moment of area, respectively.

However, the classical beam governing equation (5) does not cover nonlocal behavior of the structure i.e. “small scale effect”. In order to consider small scale effect the nonlocal elasticity theory should be introduced instead of Hooke’s law. Herein we considered continuum mechanics based on nonlocal elasticity theory and most widely used Eringen model \([46]\) as

\[
(1 - a^2 L^2 \nabla^2)\sigma = C e,
\]

In (6) \( C \) stands for the fourth-order elasticity tensor, \( \sigma \) and \( e \) are the second order stress and strain tensors, respectively. The operator \( \nabla^2 \) is a Laplace operator and \( a = (e_0 a)/L \). The parameter \( e_0 \) describes material properties. The parameters \( L \) and \( a \) stand for external and internal characteristic lengths, respectively.

The nonlocal constitutive equation (6) can be rewritten in terms of moments and deflection as (multiplying (6) by \( z \) and integrating through the cross sectional area)

\[
M - \mu d^2 M/dx^2 = -EI d^2 W/dx^2.
\]

In (7) \( \mu \) stands for nonlocal parameter. Combining the nonlocal constitutive equation (7) and the moment and deflection relation of Euler-Bernoulli beam

\[
\frac{d^4 W}{dx^4} = -m_0 \omega^2 W,
\]

one obtains the governing differential equation for nonlocal Euler-Bernoulli beam in terms of displacement as

\[
EI \frac{d^4 W}{dx^4} + \mu m_0 \omega^2 \frac{d^2 W}{dx^2} = m_0 \omega^2 W.
\]

In non-dimensional variables the governing differential equation for nonlocal Euler-Bernoulli beam reads

\[
\frac{d^4 W}{dx^4} + \mu \lambda^2 \frac{d^2 W}{dx^2} = \lambda^2 W,
\]

where \( X = x/L, \lambda^2 = m_0 \omega^2 L^4/EI \).

In the following section the governing differential equation (10) will be discretized by applying Haar wavelet method.

### 4 Application of the Haar wavelet method

According to the HWDM considered, the highest order derivative existing in differential equation is expanded into Haar wavelets. In the case of considered problem the fourth order derivative can be expanded into Haar wavelets as

\[
\frac{d^4 W}{dx^4} = a^T H.
\]

By integrating (11) four times with respect to \( x \), one obtains the solution \( W(x) \) as

\[
W(X) = a^T P^{(4)} + c_3 \frac{X^3}{6} + c_2 \frac{X^2}{2} + c_1 X + c_0,
\]

where \( P^{(4)} \) is a fourth order operational matrix defined by formula (4) and \( c_0, \ldots, c_3 \) stand for integration constants determined for each particular boundary conditions separately.

Inserting (10), (11) in governing differential equation (9) one obtains

\[
a^T H + \mu \lambda^2 L^2 (a^T P^{(2)} + c_1 X + c_2) = \lambda^2 \left( a^T P^{(4)} + c_3 \frac{X^3}{6} + c_2 \frac{X^2}{2} + c_1 X + c_0 \right).
\]

The following boundary conditions are considered:
a) Pinned-pinned nanobeam

\[ W(0) = 0, \, W(1) = 0, \, W(0) = 0, \, W(1) = 0. \]  

(14)

b) Clamped-clamped nanobeam

\[ W(0) = 0, \, W(1) = 0, \, W(0) = 0, \, W(1) = 0. \]  

(15)

c) Clamped-pinned nanobeam

\[ W(0) = 0, \, W(1) = 0, \, W(0) = 0, \, W(1) = 0. \]  

(16)

In the current study, firstly the integrations constants \( c_0, \ldots, c_3 \) are determined from particular boundary conditions and then the system (13) is solved with respect to frequency parameter \( \lambda \) as an eigenvalue problem.

### 5 Numerical results and convergence analysis

In the following, the three types of boundary conditions are considered (pinned-pinned, clamped-clamped and clamped-pinned) for numerical analysis of nanobeam. It is assumed that the external characteristic length parameter \( L \) is equal to 10 nm and the nonlocal parameter varies in range \([0; 5]\).

Let us denote two numerical solutions found on nested grids \( h_{i-1}, h_i = h_{i-1}/2 \) by \( F_{i-1}, F_i \). According to Richardson extrapolation formulas, the extrapolated values of the solution can be computed as \([35, 45]\)

\[ R_i = F_i + \frac{F_i - F_{i-1}}{2^k - 1}, \]  

(17)

where \( k \) stands for the theoretical order of convergence. However, the theoretical order of convergence of the numerical method may be not preliminarily known. In latter case using the Eq. (16) and three solutions on a sequence of grids \( F_{i-2}, F_{i-1}, F_i, \, h_{i-2}/h_{i-1} = h_{i-1}/h_i = 2 \) one can obtain the estimate on the theoretical order of convergence \( k \) as \([48]\)

\[ k = k_1 = \log \left( \frac{F_{i-2} - F_{i-1}}{F_{i-1} - F_i} \right) / \log(2). \]  

(18)

In (17) \( k_1 \) is the value of the observed order of convergence.

a) Pinned-pinned nanobeam (PP)

The values of the fundamental frequency parameter \( F = \sqrt{\lambda} \) (square root is taken in order to compare with results found in literature), the numerical rates of convergence and absolute errors for different grid levels are presented in Table 1.

It can be seen from Table 1, that the error of the Haar wavelet method is less than that of finite difference method, but remains in the same range. The numerical rate of convergence tends to two in the case of both methods (see columns 4 and 5), but this process is faster in the case of Haar wavelet method. In Table 1 the scaling parameter \( \mu \) is equal to zero and the results are compared with exact solution given in \([8]\) (nanoscale effect is not considered). In Tables 2–3 the value of the scaling parameter \( \mu \) is varied (\( \mu = 3 \) and \( \mu = 5 \)) and instead of absolute error the values of the extrapolated results and their convergence rates are given.

Obviously, the fundamental frequency parameters \( F_j \) corresponding to HWDM and FDM are close and converge to the same value (columns 2–3 in Tables 2–3), but in the case of lower grid \((N = 4; 8; 16)\) the error of the + HWDM is significantly smaller than that of the FDM. Here the error is computed as difference from final value achieved with \( N = 1024 \). The numerically estimated rates of convergence \( k_j \) tend to two in the case of both methods HWDM and FDM (columns 4 and 5). The numerical rates of convergence of the extrapolated results tend to four for both methods (some deviation can be observed in the case of FDM for larger grid, which may be caused by too close values of the frequencies). These results are in agreement with convergence.
Table 1: The values of the fundamental frequencies $F_j$, rates of convergence $k_j$ and errors ($\mu = 0$).

<table>
<thead>
<tr>
<th>Grid</th>
<th>Fundamental frequency $F_j$</th>
<th>Rate of convergence $k_j$</th>
<th>Error</th>
</tr>
</thead>
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<tr>
<td></td>
<td>HWM</td>
<td>FDM</td>
<td>HWM</td>
</tr>
<tr>
<td>4</td>
<td>3.161916</td>
<td>3.061468</td>
<td>0.0203</td>
</tr>
<tr>
<td>8</td>
<td>3.146649</td>
<td>3.121445</td>
<td>0.0051</td>
</tr>
<tr>
<td>16</td>
<td>3.142855</td>
<td>3.136549</td>
<td>2.0085</td>
</tr>
<tr>
<td>32</td>
<td>3.141908</td>
<td>3.140331</td>
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<tr>
<td>64</td>
<td>3.141672</td>
<td>3.141277</td>
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</tr>
<tr>
<td>256</td>
<td>3.141598</td>
<td>3.141588</td>
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</tr>
<tr>
<td>512</td>
<td>3.141593</td>
<td>3.141587</td>
<td>2.0000</td>
</tr>
</tbody>
</table>

Exact 3.14159265

Table 2: The values of the fundamental frequencies $F_j$, rates of convergence $k_j$, extrapolated results and their rates of convergence ($\mu = 3$).

<table>
<thead>
<tr>
<th>Grid</th>
<th>Fundamental frequency $F_j$</th>
<th>Rate of convergence $k_j$</th>
<th>Extrapolated results $R_i$</th>
<th>Rate of convergence of extr. results</th>
</tr>
</thead>
<tbody>
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<td></td>
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<td>FDM</td>
<td>HWM</td>
<td>FDM</td>
</tr>
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<td>2.877579</td>
<td>2.944393</td>
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<td>2.927620</td>
<td>2.944393</td>
<td>2.944300</td>
</tr>
<tr>
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<td>2.940174</td>
<td>1.9911</td>
<td>1.9949</td>
</tr>
<tr>
<td>32</td>
<td>2.944658</td>
<td>2.943315</td>
<td>1.9985</td>
<td>1.9987</td>
</tr>
<tr>
<td>64</td>
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<td>1.9997</td>
<td>1.9997</td>
</tr>
<tr>
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<td>1.9999</td>
<td>1.9999</td>
</tr>
<tr>
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<tr>
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<tr>
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<td>4.0219</td>
</tr>
</tbody>
</table>

Table 3: The values of the fundamental frequencies $F_j$, rates of convergence $k_j$, extrapolated results and their rates of convergence ($\mu = 5$).

<table>
<thead>
<tr>
<th>Grid</th>
<th>Fundamental frequency $F_j$</th>
<th>Rate of convergence $k_j$</th>
<th>Extrapolated results $R_i$</th>
<th>Rate of convergence of extr. results</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>HWM</td>
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<td>HWM</td>
<td>FDM</td>
</tr>
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<tr>
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<td>2.841840</td>
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<td>2.838032</td>
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<td>1.9970</td>
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<tr>
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<td>2.840890</td>
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<td>1.9993</td>
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<tr>
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<tr>
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</tr>
<tr>
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</tr>
<tr>
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</tr>
</tbody>
</table>
Table 4: The values of the fundamental frequencies $F_j$, rates of convergence $k_j$, extrapolated results and their rates of convergence ($\mu = 5$).

<table>
<thead>
<tr>
<th>Grid</th>
<th>Fundamental frequency $F_j$</th>
<th>Rate of convergence $k_j$</th>
<th>Extrapolated results $R_i$</th>
<th>Rate of convergence of extr. results</th>
</tr>
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<tbody>
<tr>
<td>4</td>
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<td>3.502444 3.502445 3.9069</td>
</tr>
</tbody>
</table>

Theorems proved in [31, 32]. Interesting is the small scale effect of the beam or dependency of the solution on scaling parameter $\mu$. Figure 1 shows the relation between non-dimensional frequency parameter and scaling parameter $\mu$.

Increasing the value of the scaling parameter $\mu$ from 0 to 5 causes remarkable reduction of the value of fundamental frequency parameter $\lambda$ (18.2%). Thus, it is reasonable to consider small scale effect for analysis of nanobeams.
b) Pinned-clamped nanobeam

The structure of the Table 4 is the same as Tables 2–3. It can be seen from columns 2 and 3 of Table 4 that the fundamental frequency parameters corresponding to HWDM and FDM are close and converge to the same value. The numerical rates of convergence \( k_j \) tend to two in the case of both methods and the rates of convergence of extrapolated results tend to four in the case of both methods (certain deviations appears in the case of larger grid, which may be caused by too close values of the frequencies). Again, in the case of small grid size \( (N = 4; 8; 16) \) the error of the HWDM is substantially smaller than that of FDM.

c) Clamped-clamped nanobeam

The structure of the Table 5 coincides with that of Table 4. In can be seen from Table 5 that in the case of clamped-clamped nanobeam the behavior of the solutions is similar with the previous two boundary conditions considered above.

Table 5: The values of the fundamental frequencies \( F_j \), rates of convergence \( k_j \), extrapolated results and their rates of convergence \( (\mu = 5) \).

<table>
<thead>
<tr>
<th>Grid</th>
<th>Fundamental frequency ( F_j )</th>
<th>Rate of convergence ( k_j )</th>
<th>Extrapolated results ( R_i )</th>
<th>Rate of conv. of extr. results</th>
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<tbody>
<tr>
<td>( N )</td>
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<td>FDM</td>
<td>HWM</td>
<td>FDM</td>
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</table>

Similarly to pinned-pinned beam, the influence of the scaling parameter \( \mu \) on fundamental frequency parameter \( \lambda \) is significant (21.5%).

6 Conclusions

The Haar wavelet method has been treated for free vibration analysis of nanobeams. The reference solution is realized by FD method (also strong formulation based method). The results obtained by applying HWDM and FDM were similar. The accuracy of the solutions obtained by applying HWDM appears higher in the case of all three boundary conditions considered, especially in the case of small grid size \( (N = 4; 8; 16) \). The numerical rate of convergence has been observed to be equal to two for HWDM and equal to four for extrapolated results. These results are in agreement with convergence theorems proved recently for HWDM in [31, 32]. The obtained results coincide also with the results given in [8] achieved by applying the differential quadrature method (DQM).

Acknowledgement: The research was supported by the Estonian Centre of Excellence in Zero Energy and Resource Efficient Smart Buildings and Districts, ZEBE, grant TK146 funded by the European Regional De-
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