Research Article

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Fractals of generalized $F-$ Hutchinson operator

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Abstract: The aim of this paper is to construct a fractal with the help of a finite family of $F-$ contraction mappings, a class of mappings more general than contraction mappings, defined on a complete metric space. Consequently, we obtain a variety of results for iterated function systems satisfying a different set of contractive conditions. Some examples are presented to support the results proved herein. Our results unify, generalize and extend various results in the existing literature.

Keywords: Iterated function system, set-valued mapping, domain of sets, fixed point, $F$-contraction

1 Introduction and preliminaries

Iterated function systems are based on the mathematical foundations laid by Hutchinson [10]. He showed that the Hutchinson operator constructed with the help of a finite system of contraction mappings defined on an Euclidean space $\mathbb{R}^n$ has closed and bounded subset of $\mathbb{R}^n$ as its fixed point, called attractor of iterated function system (see also in [6]). In this context, fixed point theory plays significant and vital role to help in construction of fractals.

Fixed point theory is studied in an environment created with appropriate mappings satisfying certain conditions. Recently, many researchers have obtained fixed point results for single and multi-valued mappings defined on metric spaces.

Banach contraction principle [5], one of the basic and the most widely applied fixed point theorem in all of analysis reads as follows:

Theorem 1.1. Let $(X, d)$ be a complete metric space and $f : X \to X$ a contraction on $X$ with contraction constant $\alpha \in [0,1)$, that is, for any $x, y \in X$, the following holds:

$$d(fx, fy) \leq \alpha d(x, y).$$

Then $f$ has a unique fixed point in $X$. Furthermore, for any initial guess $x_0 \in X$ the sequence of simple iterates \{x_0, fx_0, f^2x_0, f^3x_0, \ldots \} converges to a fixed point of $f$.

Banach contraction principle [5] is of paramount importance in metrical fixed point theory with a wide range of applications, including iterative methods for solving linear, nonlinear, differential, integral, and difference equations. This initiated several researchers to extend and enhance the scope of metric fixed point theory. As a result, Banach contraction principle has been extended either by generalizing the domain of the mapping [2, 3, 9, 11, 12, 19, 20] or by extending the contractive condition on the mappings [7, 8, 13, 14, 16, 18].

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There are certain cases when the range $X$ of a mapping is replaced with a family of sets possessing some topological structure and consequently a single valued mapping is replaced with a multivalued mapping. Nadler [15] was the first who combined the ideas of multivalued mappings and contractions and hence initiated the study of metric fixed point theory of multivalued operators, see also [1, 4, 1]. The fixed point theory of multivalued operators provides important tools and techniques to solve the problems of pure, applied and computational mathematics which can be restructured as an inclusion equation for an appropriate multivalued operator.

The purpose of this paper is to construct a fractal set of iterated function system, a certain finite collection of mappings defined on a metric space which induce compact valued mappings defined on a family of compact subsets of a metric space. We prove that Hutchinson operator defined with the help of a finite family of $F$– contraction mappings on a complete metric space is itself generalized $F$– contraction mapping on a family of compact subsets of $X$. We then obtain a final fractal obtained by successive application of a generalized $F$– Hutchinson operator. A nontrivial example is presented to support the result proved herein.

In what follows, the letters $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^+$, $\mathbb{N}$ will denote the set of all real numbers, the set of all positive real numbers and the set of all natural numbers, respectively.

**Definition 1.2.** Let $(X, d)$ be a metric space and $C \subseteq X$. Then $C$ is compact if every sequence $\{x_n\}$ in $C$ contains a subsequence having a limit in $C$. Note that closed and bounded subsets of $\mathbb{R}^n$ are compact. Also, every finite set in $\mathbb{R}^n$ is compact. On the other hands, $(0, 1] \subseteq \mathbb{R}$ is not compact as $\left\{1 + \frac{1}{2^n}, \ldots\right\} \subset (0, 1]$ does not have any convergent subsequence. Also, $\mathbb{Z} \subset \mathbb{R}$ is not compact.

Let $(X, d)$ be a metric space and $\mathcal{H}(X)$ denotes the set of all non-empty compact subsets of $X$. For $A, B \in \mathcal{H}(X)$, let

$$H(A, B) = \max\{\sup_{a \in A} d(b, a), \sup_{b \in B} d(a, B)\},$$

where $d(x, B) = \inf\{d(x, b) : b \in B\}$ is the distance of a point $x$ from the set $B$. The mapping $H$ is said to be the Pompeiu-Hausdorff metric induced by $d$. If $(X, d)$ is a complete metric space, then $(\mathcal{H}(X), H)$ is also a complete metric space.

For the sake of completeness, we state and prove the following Lemma.

**Lemma 1.3.** Let $(X, d)$ be a metric space. For all $A, B, C, D \in \mathcal{H}(X)$, the following hold:

(i) If $B \subseteq C$, then $\sup_{a \in A} d(a, C) \leq \sup_{a \in A} d(a, B)$.

(ii) $\sup_{x \in A \cup B} d(x, C) = \max\{\sup_{a \in A} d(a, C), \sup_{b \in B} d(b, C)\}$.

(iii) $H(A \cup B, C \cup D) \leq \max\{H(A, C), H(B, D)\}$.

**Proof.** To prove (i): Since $B \subseteq C$, for all $a \in A$, we have

$$d(a, C) = \inf\{d(a, c) : c \in C\} \leq \inf\{d(a, b) : b \in B\} = d(a, B),$$

which implies that

$$\sup_{a \in A} d(a, C) \leq \sup_{a \in A} d(a, B).$$

To prove (ii):

$$\sup_{x \in A \cup B} d(x, C) = \sup_{x \in A \cup B} d(x, C) : x \in A \cup B = \max\{\sup_{a \in A} d(a, C) : a \in A\}, \sup_{b \in B} d(b, C) : b \in B\}

= \max\{\sup_{a \in A} d(a, C), \sup_{b \in B} d(b, C)\}.$$

To prove (iii): Note that

$$\sup_{x \in A \cup B} d(x, C \cup D) \leq \max\{\sup_{a \in A} d(a, C \cup D), \sup_{b \in B} d(b, C \cup D)\} \text{ (by using (ii))}$$

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In the similar way, we obtain that

\[ d(y, A \cup B) \leq \max \{ H(A, C), H(B, D) \}. \]

Hence it follows that

\[ H(A \cup B, C \cup D) = \max \{ \sup_{x \in A \cup B} d(x, C \cup D), \sup_{y \in C \cup D} d(y, A \cup B) \} \leq \max \{ H(A, C), H(B, D) \}. \]

\[ \square \]

Wardowski [21] introduced a new contraction called \( F \)-contraction and proved a fixed point result as an interesting generalization of the Banach contraction principle.

Consistent with [21], the following definition and examples are needed in the sequel.

Let \( F \) be the collection of all continuous mappings \( F : \mathbb{R}^+ \to \mathbb{R} \) that satisfy the following conditions:

\begin{enumerate}
  \item \( F \) is strictly increasing, that is, for all \( \alpha, \beta \in \mathbb{R}^+ \) such that \( \alpha < \beta \) implies that \( F(\alpha) < F(\beta) \).
  \item For every sequence \( \{ a_n \} \) of positive real numbers, \( \lim_{n \to \infty} a_n = 0 \) and \( \lim_{n \to \infty} F(a_n) = -\infty \) are equivalent.
  \item There exists \( k \in (0, 1) \) such that \( \lim_{a \to 0^+} a^k F(a) = 0 \).
\end{enumerate}

**Definition 1.4.** [21] Let \( (X, d) \) be a metric space. A self-mapping \( f \) on \( X \) is called an \( F \)-contraction if for any \( x, y \in X \), there exists \( F \in F \) and \( \tau > 0 \) such that

\[ \tau + F(d(fx, fy)) \leq F(d(x, y)), \quad (2) \]

whenever \( d(fx, fy) > 0 \).

From \((F_1)\) and \((2)\), we conclude that

\[ d(fx, fy) < d(x, y) \quad \text{for all } x, y \in X, \ fx \neq fy. \]

Indeed from \((2)\), for all \( x, y \in X \) with \( d(fx, fy) > 0 \), we have

\[ F(d(fx, fy)) < F(d(x, y)). \]

Since \( F \) is strictly increasing \((F_1)\), it follows that

\[ d(fx, fy) < d(x, y) \quad \text{for all } x, y \in X \text{ whenever } fx \neq fy. \]

Thus, every \( F \)-contraction mapping is contractive, and in particular, every \( F \)-contraction mapping is continuous.

Following examples show that there are variety of contractive conditions corresponding to different choices of elements in \( F \).

**Example 1.5.** Let \( F : \mathbb{R}^+ \to \mathbb{R} \) be defined by \( F(\lambda) = \ln(\lambda) \) for \( \lambda > 0 \). Then \( F \) satisfies \((F_1)\) to \((F_3)\). A mapping \( f : X \to X \) satisfying \((2)\) is a contraction with contractive factor \( e^{-\tau} \), that is,

\[ d(fx, fy) \leq e^{-\tau} d(x, y), \quad \text{for all } x, y \in X, \ fx \neq fy. \quad (3) \]

It is clear that for \( x, y \in X \) such that \( fx = fy \), the inequality \( d(fx, fy) \leq e^{-\tau} d(x, y) \) holds.

**Example 1.6.** If we take \( F(\lambda) = \ln(\lambda + \lambda), \lambda > 0 \), then \( F \) satisfies \((F_1)\) to \((F_3)\) and \((2)\) is of the form

\[ \frac{d(fx, fy)}{d(x, y)} e^{d(fx, fy) - d(x, y)} \leq e^{-\tau}, \quad \text{for all } x, y \in X, \ fx \neq fy. \quad (4) \]
Example 1.7. Consider $F(\lambda) = -1/\sqrt{\lambda}$ for $\lambda > 0$, then $F \in \mathcal{F}$. In this case, $F$-contraction mapping $f$ satisfies
\[ d(fx, fy) \leq \frac{1}{(1 + \tau \sqrt{d(x, y)})^2} d(x, y), \quad \text{for all } x, y \in X, \; fx \neq fy. \] (5)

Note that, the above is a special case of nonlinear contraction of the type $d(fx, fy) \leq \psi(d(x, y))d(x, y)$ for all $x, y \in X, \; fx \neq fy$. For details, see [7, 16].

Example 1.8. Let $F(\lambda) = \ln(\lambda^2 + \lambda)$, $\lambda > 0$. Then $F$ satisfies $(F_1)$-$(F_3)$ and the mapping $f$ satisfies the following condition
\[ \frac{d(fx, fy)(d(fx, fy) + 1)}{d(x, y)(d(x, y) + 1)} \leq e^{-\tau}, \quad \text{for all } x, y \in X, \; fx \neq fy. \] (6)

In all above Examples, conditions (3–6) are satisfied for any $x, y \in X$ with $fx = fy$.

Theorem 1.9. [21] Let $(X, d)$ be a complete metric space and $f : X \to X$ an $F$-contraction mapping. Then $f$ has a unique fixed point in $X$ and for every $x_0 \in X$ a sequence of iterates $\{x_0, fx_0, f^2x_0, \ldots\}$ converges to the fixed point of $f$.

Theorem 1.10. Let $(X, d)$ be a metric space and $f : X \to X$ an $F$-contraction. Then

1. $f$ maps elements in $\mathcal{H}(X)$ to elements in $\mathcal{H}(X)$.
2. If for any $A \in \mathcal{H}(X)$,
\[ f(A) = \{f(x) : x \in A\}. \]

Then $f : \mathcal{H}(X) \to \mathcal{H}(X)$ is a $F$-contraction mapping on $(\mathcal{H}(X), H)$.

Proof. As $F$-contraction mapping is continuous. The image of a compact set under $f : X \to X$ is compact, that is,
\[ A \in \mathcal{H}(X) \; \text{implies} \; f(A) \in \mathcal{H}(X). \]

To prove (2): Let $A, B \in \mathcal{H}(X)$ with $H(f(A), f(B)) \neq 0$. Since $f : X \to X$ is $F$-contraction, we obtain that
\[ 0 < d(fx, fy) < d(x, y) \quad \text{for all } x, y \in X, \; x \neq y. \]

Thus we have
\[ d(fx, fy) = \inf_{y \in B} d(fx, fy) < \inf_{y \in B} d(x, y) = d(x, B). \]

Also
\[ d(fy, f(A)) = \inf_{x \in A} d(fy, fx) < \inf_{x \in A} d(y, x) = d(y, A). \]

Now
\[ H(f(A), f(B)) = \max\{\sup_{x \in A} d(fx, f(B)), \sup_{y \in B} d(fy, f(A))\} < \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\} = H(A, B). \]

Since $F$ is strictly increasing,
\[ F(H(f(A), f(B))) < F(H(A, B)). \]

Consequently, there exists some $\tau^* > 0$ such that
\[ \tau^* + F(H(f(A), f(B))) \leq F(H(A, B)). \]

Hence $f : \mathcal{H}(X) \to \mathcal{H}(X)$ is a $F$-contraction. \qed
**Theorem 1.11.** Let \((X, d)\) be a metric space and \(\{f_n : n = 1, 2, \ldots, N\}\) a finite family of F-contraction self-mappings on \(X\). Define \(T : \mathcal{H}(X) \to \mathcal{H}(X)\) by

\[
T(A) = f_1(A) \cup f_2(A) \cup \cdots \cup f_N(A) = \bigcup_{n=1}^{N} f_n(A), \quad \text{for each } A \in \mathcal{H}(X).
\]

Then \(T\) is F-contraction on \(\mathcal{H}(X)\).

**Proof.** We demonstrate the claim for \(N = 2\). Let \(f_1, f_2 : X \to X\) be two F-contractions. Take \(A, B \in \mathcal{H}(X)\) with \(H(T(A), T(B)) \neq 0\). From Lemma 1.3 (iii), it follows that

\[
\tau + F(\{H(T(A), T(B))\}) = \tau + F\left(H(f_1(A), f_2(B))\right) \leq \tau + F\left(\max\{H(f_1(A), f_2(B)), H(f_2(A), f_2(B))\}\right) \leq F(H(A, B)).
\]

\(\square\)

**Theorem 1.12.** Let \((X, d)\) be a complete metric space and \(\{f_n : n = 1, 2, \ldots, N\}\) a finite family of F-contractions on \(X\). Define a mapping on \(\mathcal{H}(X)\) as

\[
T(A) = f_1(A) \cup f_2(A) \cup \cdots \cup f_N(A) = \bigcup_{n=1}^{N} f_n(A), \quad \text{for each } A \in \mathcal{H}(X).
\]

Then

1. \(T : \mathcal{H}(X) \to \mathcal{H}(X)\);
2. \(T\) has a unique fixed point \(U \in \mathcal{H}(X)\), that is \(U = T(U) = \bigcup_{n=1}^{N} f_n(U)\);
3. for any initial set \(A_0 \in \mathcal{H}(X)\), the sequence of compact sets \(\{A_0, T(A_0), T^2(A_0), \ldots\}\) converges to a fixed point of \(T\).

**Proof.** (1) Since each \(f_i\) is F-contraction, therefore from definition of \(T\) and Theorem 1.10 conclusion follows immediately. (2) From Theorem 1.11 \(T : \mathcal{H}(X) \to \mathcal{H}(X)\) is F-contraction. Moreover the completeness of \((X, d)\) implies that \((\mathcal{H}(X), H)\) is complete. Consequently (2) and (3) follow from Theorem 1.9. \(\square\)

**Definition 1.13.** Let \((X, d)\) be a metric space. A mapping \(T : \mathcal{H}(X) \to \mathcal{H}(X)\) is said to be a generalized F-contraction if there exists \(F \in \mathcal{F}\) and \(\tau > 0\) such that for any \(A, B \in \mathcal{H}(X)\) with \(H(T(A), T(B)) \neq 0\), the following holds:

\[
\tau + F(\{H(T(A), T(B))\}) \leq F(M_T(A, B)),
\]

where

\[
M_T(A, B) = \max\{H(A, B), H(A, T(A)), H(B, T(B)), \frac{H(A, T(B)) + H(B, T(A))}{2}, H(T^2(A), T(A)), H(T^2(A), B), H(T^2(A), T(B))\}.
\]

The operator \(T\) defined above is also called generalized F- Hutchinson operator. Note that if \(T\) defined in Theorem 1.11 is F-contraction, then it is trivially generalized F-contraction and so \(T\) is generalized F- Hutchinson operator. The converse does not hold, see [22].

**Definition 1.14.** Let \(X\) be a metric space. If \(f_n : X \to X, n = 1, 2, \ldots, N\) are F-contraction mappings, then \((X; f_1, f_2, \ldots, f_N)\) is called generalized (F-contractive) iterated function system (IFS).

Thus the generalized iterated function system consists of a metric space and finite family of F-contraction mappings on \(X\).

**Definition 1.15.** A nonempty compact set \(A \subseteq X\) is said to be an attractor of the generalized F-contractive IFS if

1. \(T(A) = A\)
2. there is an open set \(V \subseteq X\) such that \(A \subseteq V\) and \(\lim_{k \to \infty} T^k(B) = A\) for any compact set \(B \subseteq V\), where the limit is taken with respect to the Hausdorff metric.

The largest open set \(V\) satisfying (b) is called a basin of attraction.
2 Main Results

We start with the following result. In this result we prove the existence of fixed point of generalized $F$– contraction operator $T$.

**Theorem 2.1.** Let $(X,d)$ be a complete metric space and \( \{X : f_n, n = 1, 2, \cdots, k\} \) a generalized iterated function system. Let \( T : \mathcal{H}(X) \rightarrow \mathcal{H}(X) \) be defined by

\[
T(A) = f_1(A) \cup f_2(A) \cup \cdots \cup f_n(A) = \bigcup_{n=1}^{N} f_n(A), \quad \text{for each } A \in \mathcal{H}(X).
\]

If $T$ is a generalized $F$– Hutchinson operator, then $T$ has a unique fixed point $U \in \mathcal{H}(X)$, that is

\[
U = T(U) = \bigcup_{n=1}^{k} f_n(U).
\]

Moreover, for any initial set $A_0 \in \mathcal{H}(X)$, the sequence of compact sets \( \{A_0, T(A_0), T^2(A_0), \ldots\} \) converges to a fixed point of $T$.

**Proof.** Let $A_0$ be an arbitrary element in $\mathcal{H}(X)$. If $A_0 = T(A_0)$, then the proof is finished. So we assume that $A_0 \neq T(A_0)$. Define

\[
A_1 = T(A_0), A_2 = T(A_1), \ldots, A_m = T(A_{m-1})
\]

for $m \in \mathbb{N}$.

We may assume that $A_m \neq A_{m+1}$ for all $m \in \mathbb{N}$. If not, then $A_k = A_{k+1}$ for some $k$ implies $A_k = T(A_k)$ and this completes the proof. Take $A_m \neq A_{m+1}$ for all $m \in \mathbb{N}$. From (7), we have

\[
\tau + F(H(A_{m+1}, A_{m+2})) = \tau + F(H(T(A_m), T(A_{m+1}))) \leq F(M_T(A_m, A_{m+1})),
\]

where

\[
M_T(A_m, A_{m+1}) = \max\{H(A_m, A_{m+1}), H(A_m, T(A_m)), H(A_{m+1}, T(A_{m+1}))\},
\]

\[
H(A_m, T(A_{m+1})) = H(T^2(A_m), T(A_m)), H\left( T^2(A_m), A_{m+1}\right), H\left( T^2(A_m), T(A_{m+1})\right)\}
\]

\[
= \max\{H(A_m, A_{m+1}), H(A_m, A_{m+1}), H(A_{m+1}, A_{m+2}), H(A_{m+2}, A_{m+1})\},
\]

\[
H(A_{m+1}, A_{m+2}), H(A_{m+1}, A_{m+2}), H(A_m, A_{m+1}) + H(A_{m+1}, A_{m+2})\}
\]

\[
= \max\{H(A_m, A_{m+1}), H(A_{m+1}, A_{m+2}), H(A_m, A_{m+1}) + H(A_{m+1}, A_{m+2})\}.
\]

Thus, we have

\[
\tau + F(H(A_{m+1}, A_{m+2})) \leq F(\max\{H(A_m, A_{m+1}), H(A_{m+1}, A_{m+2})\}) = F(H(A_m, A_{m+1})),
\]

that is,

\[
F(H(A_{m+1}, A_{m+2})) \leq F(H(A_m, A_{m+1})) - \tau
\]

for all $m \in \mathbb{N}$. Therefore

\[
F(H(A_n, A_{n+1})) \leq F(H(A_{n-1}, A_n)) - \tau \leq F(H(A_{n-2}, A_{n-1})) - 2\tau \leq \cdots \leq F(H(A_0, A_1)) - n\tau
\]

and we obtain that $\lim_{n \to \infty} F(H(A_n, A_{n+1})) = -\infty$ that together with $(F_2)$ implies that

\[
\lim_{n \to \infty} H(A_n, A_{n+1}) = 0.
\]
Now by \((F_3)\), there exists \(h \in (0, 1)\) such that

\[
\lim_{n \to \infty} [H(A_n, A_{n+1})]^h F \left( H(A_n, A_{n+1}) \right) = 0.
\]

Thus we have

\[
[H(A_n, A_{n+1})]^h F \left( H(A_n, A_{n+1}) \right) - [H(A_n, A_{n+1})]^h F \left( H(A_0, A_{n+1}) \right) \leq -n\tau[H(A_n, A_{n+1})]^h \leq 0.
\]

On taking limit as \(n \to \infty\) we obtain

\[
\lim_{n \to \infty} n[H(A_n, A_{n+1})]^h = 0.
\]

As \(\lim_{n \to \infty} n^\frac{1}{h} H(A_n, A_{n+1}) = 0\), so there exists \(n_1 \in \mathbb{N}\) such that

\[
n^\frac{1}{h} H(A_n, A_{n+1}) \leq 1
\]

for all \(n \geq n_1\). So we have

\[
H(A_n, A_{n+1}) \leq \frac{1}{n^{1/h}}
\]

for all \(n \geq n_1\). For \(m, n \in \mathbb{N}\) with \(m > n \geq n_1\), we have

\[
H(A_n, A_m) \leq H(A_n, A_{n+1}) + H(A_{n+1}, A_{n+2}) + \ldots + H(A_{m-1}, A_m) \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/h}}.
\]

By the convergence of the series \(\sum_{i=n}^{\infty} \frac{1}{i^{1/h}}\), we get \(H(A_n, A_m) \to 0\) as \(n, m \to \infty\). Therefore \(\{A_n\}\) is a Cauchy sequence in \(X\). Since \((\mathcal{C}_c(X), H)\) is complete, we have \(A_n \to U\) as \(n \to \infty\) for some \(U \in \mathcal{C}_c(X)\).

In order to show that \(U\) is the fixed point of \(T\), we contrary assume that Pompeiu-Hausdorff weight assigned to the \(U\) and \(T(U)\) is not zero. Now

\[
\tau + F \left( H(A_{n+1}, T(U)) \right) = \tau + F(H(T(A_n), T(U))) \leq F \left( M_T(A_n, U) \right), \quad (8)
\]

where

\[
M_T(A_n, U) = \max \{H(A_n, U), H(A_n, T(A_n)), H(U, T(U)), H(A_n, T(U)) + H(U, T(A_n)) , H(T^2(A_n), T(A_n)), H(T(A_n), T(U)), H(T^2(A_n), T(U)) \}
\]

\[
= \max \{H(A_n, U), H(A_n, A_{n+1}), H(U, T(U)), H(A_n, T(U)) + H(U, A_{n+1}) , H(A_{n+2}, A_{n+1}), H(A_{n+2}, U), H(A_{n+2}, T(U)) \}.
\]

Now we consider the following cases:

1. If \(M_T(A_n, U) = H(A_n, U)\), then on taking limit as \(n \to \infty\) in (8), we have

\[
\tau + F \left( H(T(U), U) \right) \leq F \left( H(U, U) \right),
\]

a contradiction.

2. When \(M_T(A_n, U) = H(A_n, A_{n+1})\), then

\[
\tau + F \left( H(T(U), U) \right) \leq F \left( H(U, U) \right),
\]

gives a contradiction.

3. In case \(M_T(A_n, U) = H(U, T(U))\), then on taking limit as \(n \to \infty\) in (8), we get

\[
\tau + F \left( H(T(U), U) \right) \leq F \left( H(U, T(U)) \right),
\]

a contradiction.
Let each
\[ \{a\} \]

Then
\[ f \]

Corollary 2.3.

Let
\[ M_T (A_n, U) = \frac{H(A_n, T(U)) + H(U, A_{n+1})}{2} \]

(4) If
\[ M_T (A_n, U) = \frac{H(A_n, T(U)) + H(U, A_{n+1})}{2} \]

on taking limit as
\[ n \to \infty \]

we have
\[ \tau + F (H(T(U), U)) \leq F (H(U, T(U))) = F \left( \frac{H(U, T(U)) + H(V, T(U))}{2} \right) \]

a contradiction.

(5) When
\[ M_T (A_n, U) = H(A_{n+2}, A_{n+1}) \]

then
\[ \tau + F (H(T(U), U)) \leq F (H(U, U)) \]

gives a contradiction.

(6) In case
\[ M_T (A_n, U) = H(A_{n+2}, U) \]

then on taking limit as
\[ n \to \infty \]

we get
\[ \tau + F (H(T(U), U)) \leq F (H(U, U)) \]

a contradiction.

(7) Finally if
\[ M_T (A_n, U) = H(A_{n+2}, T(U)) \]

then on taking limit as
\[ n \to \infty \]

we have
\[ \tau + F (H(T(U), U)) \leq F (H(U, T(U))) \]

a contradiction.

Thus,
\[ U \]

is the fixed point of
\[ T \]

To show the uniqueness of fixed point of
\[ T \]

assume that
\[ U \]

and
\[ V \]

are two fixed points of
\[ T \]

with
\[ H(U, V) \]

is not zero. Since
\[ T \]

is a
\[ F \]

-contraction map, we obtain that
\[ \tau + F (H(U, V)) = \tau + F (H(T(U), T(V))) \]

\[ \leq F (\max \{H(U, V), H(U, T(U)), H(V, T(V))\}, H(U, T(V)), H(U, V), \frac{H(U, T(V)) + H(V, T(U))}{2}, H(T^2(U), U), H(T^2(U), V), H(T^2(U), T(V))) \]

\[ = F (\max \{H(U, V), H(U, U), H(V, V), H(U, V), \frac{H(U, V) + H(V, U)}{2}, H(U, U), H(U, V), H(U, V)\}) = F (H(U, V)) \]

a contradiction as \( \tau > 0 \). Thus
\[ T \]

has a unique fixed point
\[ U \in \mathcal{H}(X) \].

\[ \square \]

Remark 2.2.

In Theorem 2.1, if we take
\[ \mathcal{S}(X) \]

the collection of all singleton subsets of
\[ X \]

then clearly
\[ \mathcal{S}(X) \subseteq \mathcal{H}(X) \]. Moreover, consider
\[ f_n = f \]

for each
\[ n \]

where
\[ f = f_1 \]

then the mapping
\[ T \]

becomes
\[ T(x) = f(x) \].

With this setting we obtain the following fixed point result.

Corollary 2.3.

Let
\[ (X, d) \]

be a complete metric space and
\[ \{X : f_n, n = 1, 2, \cdots, k\} \]

a generalized iterated function system. Let
\[ f : X \to X \]

be a mapping defined as in Remark 2.2. If there exists some
\[ F \in F \]

and
\[ \tau > 0 \]

such that for any
\[ x, y \in \mathcal{H}(X) \]

with
\[ d(f(x), f(y)) \neq 0 \]

the following holds:
\[ \tau + F (d(f(x), f(y))) \leq F (M_f (x, y)) \]

where
\[ M_f (x, y) = \max \left\{ d(x, y), d(x, f(x)), d(y, f(y)), \frac{d(x, f(x)) + d(y, f(y))}{2}, d(f^2(x), y), d(f^2(x), f(x)), d(f^2(x), f(y)) \right\} \]

Then
\[ f \]

has a unique fixed point
\[ x \in X \]. Moreover, for any initial set
\[ x_0 \in X \]

the sequence of compact sets
\[ \{x_0, f_0 x_0, f^2 x_0, \cdots\} \]

converges to a fixed point of
\[ f \].

Corollary 2.4.

Let
\[ (X, d) \]

be a complete metric space and
\[ \{X : f_n, n = 1, 2, \cdots, k\} \]

be iterated function system where each
\[ f_i \]

for
\[ i = 1, 2, \cdots, k \]

is a contraction self-mapping on
\[ X \]

Then
\[ \mathcal{H}(X) \to \mathcal{H}(X) \]

defined in Theorem
2.1 has a unique fixed point in $\mathcal{H}(X)$. Furthermore, for any set $A_0 \in \mathcal{H}(X)$, the sequence of compact sets $(A_0, T(A_0), T^2(A_0), \ldots)$ converges to a fixed point of $T$.

**Proof.** It follows from Theorem 1.13 that if each $f_i$ for $i = 1, 2, \ldots, k$ is a contraction mapping on $X$, then the mapping $T : \mathcal{H}(X) \to \mathcal{H}(X)$ defined by

$$T(A) = \cup_{n=1}^{k} f_n(A), \text{ for all } A \in \mathcal{H}(X)$$

is contraction on $\mathcal{H}(X)$. Using Theorem 2.1, the result follows.

**Corollary 2.5.** Let $(X, d)$ be a complete metric space and $(X; f_n, n = 1, 2, \cdots, k)$ an iterated function system where each $f_i$ for $i = 1, 2, \cdots, k$ is a mapping on $X$ satisfying

$$d(f_i x, f_i y) e^{d(f_i x, f_i y) - d(x,y)} \leq e^{-\tau} d(x, y), \text{ for all } x, y \in X, f_i x \neq f_i y,$$

where $\tau > 0$. Then the mapping $T : \mathcal{H}(X) \to \mathcal{H}(X)$ defined in Theorem 2.1 has a unique fixed point in $\mathcal{H}(X)$. Furthermore, for any set $A_0 \in \mathcal{H}(X)$, the sequence of compact sets $(A_0, T(A_0), T^2(A_0), \ldots)$ converges to a fixed point of $T$.

**Proof.** Take $F(\lambda) = \ln(\lambda) + \lambda, \lambda > 0$ in Theorem 1.11, then each mapping $f_i$ for $i = 1, 2, \cdots, k$ on $X$ satisfies

$$d(f_i x, f_i y) e^{d(f_i x, f_i y) - d(x,y)} \leq e^{-\tau} d(x, y), \text{ for all } x, y \in X, f_i x \neq f_i y,$$

where $\tau > 0$. Again form Theorem 1.11, the mapping $T : \mathcal{H}(X) \to \mathcal{H}(X)$ defined by

$$T(A) = \cup_{n=1}^{k} f_n(A), \text{ for all } A \in \mathcal{H}(X)$$

satisfies

$$H(T(A), T(B)) e^{H(T(A), T(B)) - H(A, B)} \leq e^{-\tau} H(A, B),$$

for all $A, B \in \mathcal{H}(X), H(T(A), T(B)) \neq 0$. Using Theorem 2.1, the result follows.

**Corollary 2.6.** Let $(X, d)$ be a complete metric space and $(X; f_n, n = 1, 2, \cdots, k)$ be iterated function system such that each $f_i$ for $i = 1, 2, \cdots, k$ is a mapping on $X$ satisfying

$$d(f_i x, f_i y) (d(f_i x, f_i y) + 1) \leq e^{-\tau} d(x, y) (d(x, y) + 1), \text{ for all } x, y \in X, f_i x \neq f_i y,$$

where $\tau > 0$. Then the mapping $T : \mathcal{H}(X) \to \mathcal{H}(X)$ defined in Theorem 2.1 has a unique fixed point in $\mathcal{H}(X)$. Furthermore, for any set $A_0 \in \mathcal{H}(X)$, the sequence of compact sets $(A_0, T(A_0), T^2(A_0), \ldots)$ converges to a fixed point of $T$.

**Proof.** By taking $F(\lambda) = \ln(\lambda^2 + \lambda + \lambda, \lambda > 0$ in Theorem 1.11, we obtain that each mapping $f_i$ for $i = 1, 2, \cdots, k$ on $X$ satisfies

$$d(f_i x, f_i y) (d(f_i x, f_i y) + 1) \leq e^{-\tau} d(x, y) (d(x, y) + 1), \text{ for all } x, y \in X, f_i x \neq f_i y,$$

where $\tau > 0$. Again it follows from Theorem 1.11 that the mapping $T : \mathcal{H}(X) \to \mathcal{H}(X)$ defined by

$$T(A) = \cup_{n=1}^{k} f_n(A), \text{ for all } A \in \mathcal{H}(X)$$

satisfies

$$H(T(A), T(B)) (H(T(A), T(B)) + 1) \leq e^{-\tau} H(A, B) (H(A, B) + 1),$$

for all $A, B \in \mathcal{H}(X), H(T(A), T(B)) \neq 0$. Using Theorem 2.1, the result follows.

**Corollary 2.7.** Let $(X, d)$ be a complete metric space and $(X; f_n, n = 1, 2, \cdots, k)$ be iterated function system such that each $f_i$ for $i = 1, 2, \cdots, k$ is a mapping on $X$ satisfying

$$d(f_i x, f_i y) \leq \frac{1}{(1 + \tau \sqrt{d(x,y)})} d(x, y), \text{ for all } x, y \in X, f_i x \neq f_i y.$$
where $\tau > 0$. Then the mapping $T : \mathcal{H}(X) \to \mathcal{H}(X)$ defined in Theorem 2.1 has a unique fixed point $\mathcal{H}(X)$. Furthermore, for any set $A_0 \in \mathcal{H}(X)$, the sequence of compact sets $\{A_0, T(A_0), T^2(A_0), \ldots\}$ converges to a fixed point of $T$.

**Proof.** Take $F(\lambda) = -1/\sqrt{\lambda}$, $\lambda > 0$ in Theorem 1.11, then each mapping $f_i$ for $i = 1, 2, \ldots, k$ on $X$ satisfies

$$d(f_i(x), f_i(y)) \leq \frac{1}{(1 + \tau \sqrt{d(x, y)})^2} d(x, y),$$

for all $x, y \in X$, $f_i x \neq f_i y$.

where $\tau > 0$. Again it follows form Theorem 1.11 that the mapping $T : \mathcal{H}(X) \to \mathcal{H}(X)$ defined by

$$T(A) = \bigcup_{n=1}^{k} f_n(A),$$

for all $A \in \mathcal{H}(X)$, $H(T(A), T(B)) \neq 0$. Using Theorem 2.1, the result follows.

**Example 2.8.** Let $X = [0, 1] \times [0, 1]$ and $d$ be a Euclidean metric on $X$. Define $f_1, f_2 : X \to X$ as

$$f_1(x, y) = \left(\frac{1}{x + 1}, \frac{y}{y + 1}\right)$$

and $f_2(x, y) = \left(\frac{\sin x}{\sin x + 1}, \frac{1}{\sin y + 1}\right)$.

Note that, for all $x = (x_1, y_1), y = (x_2, y_2) \in X$ with $x \neq y$,

$$d(f_1(x), f_1(y)) = d((\frac{1}{x_1 + 1}, \frac{y_1}{y_1 + 1}), (\frac{1}{x_2 + 1}, \frac{y_2}{y_2 + 1})) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} < \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = d((x_1, y_1), (x_2, y_2)) = d(x, y).$$

Also

$$d(f_2(x), f_2(y)) = d((\frac{\sin x_1}{\sin x_1 + 1}, \frac{1}{\sin y_1 + 1}), (\frac{\sin x_2}{\sin x_2 + 1}, \frac{1}{\sin y_2 + 1}))$$

$$= \sqrt{(\frac{\sin x_1 - \sin x_2}{\sin x_1 + 1} - \frac{\sin y_1 - \sin y_2}{\sin y_1 + 1})^2} < \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = d((x_1, y_1), (x_2, y_2)) = d(x, y).$$

Now there exists $\tau > 0$ such that

$$d(f_1(x), f_1(y))(1 + \tau \sqrt{d(x, y)})^2 \leq d(x, y) \quad \text{and} \quad d(f_2(x), f_2(y))(1 + \tau \sqrt{d(x, y)})^2 \leq d(x, y)$$

are satisfied. Consider the iterated function system $\{\mathbb{R}^2; f_1, f_2\}$ with mapping $T : \mathcal{H}([0, 1]^2) \to \mathcal{H}([0, 1]^2)$ given as

$$T(A) = f_1(A) \cup f_2(A)$$

for all $A \in \mathcal{H}([0, 1]^2)$.

For all $A, B \in \mathcal{H}([0, 1]^2)$ with $H(T(A), T(B)) \neq 0$, by Theorem 1.10,

$$H(T(A), T(B))(1 + \tau \sqrt{H(A, B)})^2 \leq H(A, B)$$

holds. Furthermore, we can analyze the convergence of $T$ to the attractor of iterated function system in the Figure 1.
Fractals of generalized $F$–Hutchinson operator

Figure 1: Convergence of $T$ to the attractor of IFS.

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