Abstract: On partially ordered set equipped with a partial metric, we study the sufficient conditions for existence of common fixed points of various mappings satisfying generalized weak contractive conditions. These results unify several comparable results in the existing literature. We also study the existence of nonnegative solution of implicit nonlinear integral equation. Furthermore, we study the fractal of finite family of generalized contraction mappings defined on a partial metric space.

Keywords: Common fixed point, dominated maps, dominating maps, weakly compatible, integral equation, iterated function system, partial metric space, partially ordered set

1 Introduction and preliminaries

Alber and Guerre-Delabrere [6] introduced the concept of weakly contractive mappings and proved that weakly contractive mapping defined on a Hilbert space is a Picard operator. Later, Rhoades [42] proved that the corresponding result is also valid when Hilbert space is replaced by a complete metric space. Dutta et al. [19] generalized the weak contractive condition and proved a fixed point theorem for a self map, which in turn generalizes Theorem 1 in [42] and the corresponding result in [6]. The study of common fixed points of mappings satisfying certain contractive conditions has been at the center of rigorous research activity. The study of common fixed point theory involving more than one single valued maps, started with the assumption that all of the maps commuted [17, 37, 49]. Sessa [47] generalized the concept of commuting maps and introducing the weakly commuting maps. Then, Jungck generalized this idea, first to compatible mappings [27] and then to weakly compatible mappings [28]. There are examples that show that each of these generalizations of commutativity is a proper extension of the previous definition. On the other hand, Beg and Abbas [12] obtained a common fixed point theorem extending weak contractive condition for two maps. In this direction, Zhang and Song [51] introduced the concept of a generalized $\varphi$–weak contraction condition and obtained a common fixed point for two maps. Doric [18] proved a common fixed point theorem for generalized $(\psi, \varphi)$–weak contractions. Abbas and Doric [4] obtained a common fixed point theorem for four maps that satisfy contractive condition which is more general than that given in [51].

In 2004, Ran and Reurings [40] investigated the existence of fixed points in partially ordered metric spaces, and then by Nieto and Lopez [35]. Further results in this direction under weak contractive condition were proved, e.g. ([2, 9, 16, 22, 24, 33, 38, 39, 41]).
In 2011, Abbas et al. [2] presented some common fixed point theorems for generalized \((\psi, \varphi)\)–weakly contractive mappings in partially ordered metric spaces. Further, Radenović and Kadelburg [39] proved a result for generalized weak contractive mappings in partially ordered metric spaces.

Partial metric space is a generalized metric space in which each object does not necessarily have to have a zero distance from itself [29]. A motivation behind introducing the concept of a partial metric was to obtain appropriate mathematical models in the theory of computation [23, 30, 46, etc]. Altun and Erduran [7], Oltra and Valero [36] and Valero [50] established some further generalizations of the results in [29], and Romaguera [43] proved a Caristi type fixed point theorem on partial metric spaces. Karapinar [25] proved some fixed point theorems for weak \(\varphi\)–contraction on partial metric spaces in partially ordered sets. Further results in the direction of partial metric space were proved in [1, 3, 5, 10, 14, 15, 44, 48].

It was shown that, in some cases, the results of fixed point in partial metric spaces can be obtained directly from their induced metric counterparts [21, 26, 45]. However, some conclusions important for the application of partial metrics in information sciences cannot be obtained in this way. For example, if \(u\) is a fixed point of map \(f\), then, by using the method from [21], we cannot conclude that \(p(fu, fu) = 0 = p(u, u)\). For further details, we refer the reader to [31, 32].

The aim of this paper is to study common fixed point results for four mappings satisfying generalized contractive conditions in the setup of ordered partial metric spaces.

In the sequel, \(\mathbb{R}, \mathbb{R}^+\) and \(\mathbb{N}\) will denote the set of all real numbers, the set of all nonnegative real numbers and the set of all positive integers, respectively. The usual order on \(\mathbb{R}\) (respectively, on \(\mathbb{R}^+\)) will be indistinctly denoted by \(\leq\) or by \(\geq\).

Consistent with [7] and [29], the following definitions and results will be needed in the sequel.

**Definition 1.1.** Let \(X\) be a nonempty set. A function \(p : X \times X \to \mathbb{R}^+\) is said to be a partial metric on \(X\) if for any \(x, y, z \in X\), the following conditions hold true:

\[
\begin{align*}
(P_1) \quad & p(x, x) = p(y, y) = p(x, y) \text{ if and only if } x = y; \\
(P_2) \quad & p(x, x) \leq p(x, y) \\
(P_3) \quad & p(x, y) = p(y, x) \\
(P_4) \quad & p(x, z) \leq p(x, y) + p(y, z) - p(y, y).
\end{align*}
\]

The pair \((X, p)\) is then called a partial metric space.

If \(p(x, y) = 0\), then \((P_1)\) and \((P_2)\) imply that \(x = y\). But the converse does not hold always.

A trivial example of a partial metric space is the pair \((\mathbb{R}^+, p)\), where \(p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+\) is defined as \(p(x, y) = \max(x, y)\).

**Example 1.2.** [29] If \(X = \{[a, b] : a, b \in \mathbb{R}, a \leq b\}\), then \(p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}\) defines a partial metric \(p\) on \(X\).

For some more examples of partial metric spaces, we refer to [1, 3, 7, 15, 43, 46].

Each partial metric \(p\) on \(X\) generates a \(T_0\) topology \(T_p\) on \(X\) which has as a base the family open \( p\)-balls
\[
B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\},
\]
for all \(x \in X\) and \(\epsilon > 0\).

Observe (see [29, p. 187]) that a sequence \(\{x_n\}\) in a partial metric space \(X\) converges to a point \(x \in X\), with respect to \(T_p\), if and only if \(p(x, x_n) = \lim_{n \to \infty} p(x, x_n)\).

If \(p\) is a partial metric on \(X\), then the function \(p^S : X \times X \to \mathbb{R}^+\) given by \(p^S(x, y) = 2p(x, y) - p(x, x) - p(y, y)\), defines a metric on \(X\).

Furthermore, a sequence \(\{x_n\}\) converges in \((X, p^S)\) to a point \(x \in X\) if and only if

\[
\lim_{n, m \to \infty} p(x_n, x_m) = \lim_{n \to \infty} p(x_n, x) = p(x, x).
\]

**Definition 1.3.** [29] A sequence \(\{x_n\}\) in a partial metric space \(X\) is said to be a Cauchy sequence if \(\lim_{n, m \to \infty} p(x_n, x_m)\) exists and is finite.
A partial metric space $X$ is said to be complete if every Cauchy sequence $\{x_n\}$ in $X$ converges with respect to $\tau_p$ to a point $x \in X$ such that $\lim_{n \to \infty} p(x, x_n) = p(x, x)$. In this case, we say that the partial metric $p$ is complete.

**Lemma 1.4.** [7, 29] Let $X$ be a partial metric space.

(a) A sequence $\{x_n\}$ in $X$ is a Cauchy sequence in $X$ if and only if it is a Cauchy sequence in metric space $(X, p^2)$.

(b) A partial metric space $(X, p)$ is complete if and only if the metric space $(X, p^2)$ is complete.

Two self maps $f$ and $g$ on $X$ are said to be compatible if, whenever $\{x_n\}$ in $X$ such that $\lim_{n \to \infty} p^2(fx_n, x) = 0$ and $\lim_{n \to \infty} p^2(gx_n, x) = 0$ for some $x \in X$, then $\lim_{n \to \infty} p^2(fgx_n, gfx_n) = 0$.

If $fx = gx$ for some $x$ in $X$, then $x$ is called a coincidence point of $f$ and $g$. Furthermore, if the mappings are commuting on their coincidence point, then such mappings are called weakly compatible. [28]

**Definition 1.5.** Let $X$ be a nonempty set. Then $(X, \preceq, p)$ is called an ordered partial metric space iff:

(i) $p$ is a partial metric on $X$, and (ii) $\preceq$ is a partial order on $X$.

We say that the elements $x, y \in X$ are called comparable if either $x \preceq y$ or $y \preceq x$ holds.

Consistent with Abbas et al. ([2]) the following definitions and results will be needed in the sequel.

**Definition 1.6.** [2] Let $(X, \preceq)$ be a partially ordered set and $f$ and $g$ be two self-maps of $X$. Mapping $f$ is said to be dominated if $fx \preceq x$ for each $x$ in $X$. A mapping $g$ is said to be dominating if $x \preceq gx$ for each $x$ in $X$.

**Example 1.7.** Let $X = [0, 1]$ be endowed with usual ordering. Let $f, g : X \to X$ defined by $fx = \frac{x}{a}$ and $gx = ax$ for any positive real number $a \geq 1$. It is easy to see that $f$ is dominated and $g$ is a dominating map.

Zhang and Song [51] obtained the following common fixed point result in metric spaces for a generalized $\varphi$-weak contraction.

**Theorem 1.8.** [51] Let $(X, d)$ be a complete metric space, and let $f, g : X \to X$ be two self-mappings such that for all $x, y \in X$, $d(fx, gy) \leq M(x, y) - \phi(M(x, y))$ holds, where $\phi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function with $\phi(t) > 0$ for $t \in (0, \infty)$, $\phi(0) = 0$, and

$$M(x, y) = \max \left \{d(x, y), d(fx, x), d(gy, y)\right \} + \frac{d(x, y) + d(fx, y)}{2}.$$ 

Then there exists a unique point $u \in X$ such that $u = fu = gu$.

Aydi [10] obtained the following result in partial metric spaces endowed with a partial order.

**Theorem 1.9.** Let $(X, \preceq)$ be an ordered complete partial metric space. Let $f : X \to X$ be a nondecreasing map with respect to $\preceq$. Suppose that the following conditions hold: for $y \preceq x$, we have

(i) $p(fx, fy) \leq p(x, y) - \phi(p(x, y))$, \hspace{1cm} (2)

where $\phi : [0, \infty) \to [0, \infty]$ is a continuous and non-decreasing function such that it is positive in $]0, \infty[$, $\phi(0) = 0$ and $\lim_{t \to \infty} \phi(t) = \infty$;

(ii) there exist $x_0 \in X$ such that $x_0 \preceq fx_0$;

(iii) $f$ is continuous in $(X, p)$, or;

(iii') if a non-decreasing sequence $\{x_n\}$ converges to $x \in X$, then $x_n \preceq x$ for all $n$.

Then $f$ has a fixed point $u \in X$. Moreover, $p(u, u) = 0$.

**Definition 1.10.** [18] The control functions $\psi$ and $\varphi$ are defined as

(a) $\psi : [0, \infty) \to [0, \infty]$ is a continuous nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$, ...
(b) \( \varphi : [0, \infty) \to [0, \infty) \) is a lower semi-continuous function with \( \varphi(t) = 0 \) if and only if \( t = 0 \).

A subset \( W \) of a partially ordered set \( X \) is said to be well ordered if every two elements of \( W \) are comparable.

Recently, Abbas et al. [1] obtained the following result in partial metric spaces.

**Theorem 1.11.** Let \( (X, \preceq) \) be a partially ordered set such that there exist a complete partial metric \( p \) on \( X \) and \( f \) a nondecreasing self map on \( X \). Suppose that for every two elements \( x, y \in X \) with \( y \preceq x \), we have

\[
\psi(p(fx, fy)) \leq \psi(M(x, y)) - \varphi(M(x, y)),
\]

where

\[
M(x, y) = \max \left\{ p(x, y), p(fx, x), p(fy, y), \frac{p(x, fy) + p(y, fx)}{2} \right\},
\]

\( \psi \) and \( \varphi \) are control functions. If there exists \( x_0 \in X \) with \( x_0 \preceq fx_0 \) and one of the following two conditions is satisfied:

(a) \( f \) is continuous self map on \( (X, p^2) \);

(b) for any nondecreasing sequence \( \{x_n\} \) in \( (X, \preceq) \) with \( \lim_{n \to \infty} p^2(z, x_n) = 0 \) it follows \( x_n \preceq z \) for all \( n \in \mathbb{N} \), then \( f \) has a fixed point. Moreover, the set of fixed points of \( f \) is well ordered if and only if \( f \) has one and only one fixed point.

## 2 Common fixed point results

In this section, we obtain common fixed point theorems for four mappings defined on an ordered partial metric space.

We start with the following result.

**Theorem 2.1.** Let \( (X, \preceq, p) \) be an ordered complete partial metric space. Let \( f, g, S \) and \( T \) be self maps on \( X \), \( (f, g) \) be the pair of dominated and \( (S, T) \) be the pair of dominating maps with \( f(X) \subseteq T(X) \) and \( g(X) \subseteq S(X) \). Suppose that, there exists control functions \( \psi \) and \( \varphi \) such that for every two comparable elements \( x, y \in X \),

\[
\psi(p(fx, gy)) \leq \psi(M_p(x, y)) - \varphi(M_p(x, y)),
\]

is satisfied where

\[
M_p(x, y) = \max \left\{ p(Sx, Ty), p(fx, Sx), p(gy, Ty), \frac{p(Sx, gy) + p(fx, Ty)}{2} \right\}.
\]

If for a nonincreasing sequence \( \{x_n\} \) in \( (X, \preceq) \) with \( x_n \preceq y_n \) for all \( n \) and \( \lim_{n \to \infty} p^2(x_n, u) = 0 \) it follows \( u \preceq y_n \) for all \( n \in \mathbb{N} \) and either

(a) \( \{f, S\} \) are compatible, \( f \) or \( S \) is continuous on \( (X, p^2) \) and \( \{g, T\} \) are weakly compatible or

(b) \( \{g, T\} \) are compatible, \( g \) or \( T \) is continuous on \( (X, p^2) \) and \( \{f, S\} \) are weakly compatible,

then \( f, g, S \) and \( T \) have a common fixed point. Moreover, the set of common fixed points of \( f, g, S \) and \( T \) is well ordered if and only if \( f, g, S \) and \( T \) have one and only one common fixed point.

**Proof.** Let \( x_0 \) be an arbitrary point in \( X \). We construct sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that \( y_{2n+1} = fx_{2n} = Tx_{2n+1} \), and \( y_{2n+2} = gx_{2n+1} = Sx_{2n+2} \). By given assumptions, \( x_{2n+2} \preceq Sx_{2n+2} = gx_{2n+1} \preceq x_{2n+1} \), and \( x_{2n+1} \preceq Tx_{2n+1} = fx_{2n} \preceq x_{2n} \). Thus, for all \( n \in \mathbb{N} \) we have \( x_{n+1} \preceq x_n \). We suppose that \( p(y_{2n}, y_{2n+1}) > 0 \), for every \( n \). If not, then \( y_{2n} = x_{2n+1} \), for some \( n \). Further, since \( x_{2n} \) and \( x_{2n+1} \) are comparable, so from (4), we have

\[
\psi(p(y_{2n+1}, y_{2n+2})) = \psi(p(fx_{2n}, gx_{2n+1})) \leq \psi(M_p(x_{2n}, x_{2n+1})) - \varphi(M_p(x_{2n}, x_{2n+1})),
\]

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Hence, \(\psi(p(y_{2n+1}, y_{2n+2})) \leq p(p(y_{2n+1}, y_{2n+2})) - \varphi(p(y_{2n+1}, y_{2n+2}))\), implies that 
\(\varphi(p(y_{2n+1}, y_{2n+2})) = 0\). As, \(\varphi(t) = 0\) if and only if \(t = 0\), it follows that \(y_{2n+1} = y_{2n+2}\). Following the similar arguments, we get \(y_{2n+2} = y_{2n+3}\) and so on. Thus \(y_{2n}\) is the common fixed point of \(f, g, S\) and \(T\) as \(\{y_n\}\) became a constant sequence in \(X\).

Taking \(p(y_{2n}, y_{2n+1}) > 0\) for each \(n\). Since \(x_{2n}\) and \(x_{2n+1}\) are comparable, from (4) we obtain

\[
\psi(p(y_{2n+2}, y_{2n+1})) = \psi(p(x_{2n}, x_{2n+1})) - \varphi(M_p(x_{2n}, x_{2n+1})),
\]

where

\[
M_p(x_{2n}, x_{2n+1}) = \max \left\{ p(Sx_{2n}, Tx_{2n+1}), p(fx_{2n}, Sx_{2n}), p(gx_{2n+1}, Tx_{2n+1}), p(Sx_{2n}, gx_{2n+1}) + p(fx_{2n}, Tx_{2n+1}) \right\}
\]

\[
= \max \left\{ p(y_{2n}, y_{2n+1}), p(y_{2n+1}, y_{2n}), p(y_{2n+2}, y_{2n+1}), p(y_{2n+2}, y_{2n+1}) \right\}
\]

\[
= \max \left\{ p(y_{2n+1}, y_{2n}), p(y_{2n+2}, y_{2n+1}) \right\}.
\]

If \(\max\{p(y_{2n+1}, y_{2n}), p(y_{2n+2}, y_{2n+1})\} = p(y_{2n+2}, y_{2n+1})\), then \(M_p(x_{2n}, x_{2n+1}) \leq p(y_{2n+2}, y_{2n+1})\). But \(M_p(x_{2n}, x_{2n+1}) \geq p(y_{2n+2}, y_{2n+1})\), so

\[M_p(x_{2n}, x_{2n+1}) = p(y_{2n+2}, y_{2n+1}),\]

and (6) give

\[
\psi(p(y_{2n+2}, y_{2n+1})) \leq \psi(M_p(x_{2n}, x_{2n+1})) - \varphi(p(y_{2n+2}, y_{2n+1})),
\]

a contradiction. Hence \(p(y_{2n+2}, y_{2n+1}) \leq p(y_{2n+1}, y_{2n+2}).\) Moreover \(M_p(x_{2n}, x_{2n+1}) \leq p(y_{2n+1}, y_{2n}).\). But, since \(M_p(x_{2n}, x_{2n+1}) \geq p(y_{2n+2}, y_{2n+1}),\) so \(M_p(x_{2n}, x_{2n+1}) = p(y_{2n+2}, y_{2n+1}).\)

Similarly, we have \(p(y_{2n+3}, y_{2n+2}) \leq p(y_{2n+2}, y_{2n+1}).\) Thus the sequence \(\{p(y_{2n+1}, y_{2n})\}\) is nonincreasing. Hence there exists \(c \geq 0\) such that \(\lim_{n \to \infty} p(y_{2n+1}, y_{2n}) = c\). Suppose that \(c > 0\). Then, \(\psi(p(y_{2n+2}, y_{2n+1})) \leq \psi(M_p(x_{2n+1}, x_{2n})) - \varphi(M_p(x_{2n+1}, x_{2n})),\) and by lower semicontinuity of \(\varphi\), we have

\[
\limsup_{n \to \infty} \psi(p(y_{2n+2}, y_{2n+1})) \leq \limsup_{n \to \infty} \psi(p(y_{2n+2}, y_{2n+1})) - \liminf_{n \to \infty} \varphi(p(y_{2n}, y_{2n+1})),
\]

which implies that \(\psi(c) \leq \psi(c) - \varphi(c),\) a contradiction. Therefore \(c = 0\). So we conclude that

\[
\lim_{n \to \infty} p(y_{2n+1}, y_{2n}) = 0.
\]

Now, we show that \(\lim_{n, m \to \infty} p(y_{2n}, y_{2m}) = 0\). If not, there is \(\varepsilon > 0\), and there exist even integers \(2n_k\) and \(2m_k\) with \(2m_k > 2n_k > k\) such that

\[
p(y_{2m_k}, y_{2n_k}) \geq \varepsilon,
\]

and \(p(y_{2m_k}, y_{2n_k}) < \varepsilon\). Since

\[
\varepsilon \leq p(y_{2m_k}, y_{2n_k}) \leq p(y_{2m_k}, y_{2m_k-2}) + p(y_{2m_k-2}, y_{2m_k}) - p(y_{2m_k-2}, y_{2m_k-2})
\]

\[
\leq p(y_{2n_k}, y_{2m_k-2}) + p(y_{2m_k-2}, y_{2m_k-1}) + p(y_{2m_k-1}, y_{2m_k}) - p(y_{2m_k-1}, y_{2m_k-1}) - p(y_{2m_k-2}, y_{2m_k-2}).
\]
From (7) and (8), we have
\[ \lim_{k \to \infty} p(y_{2m_k}, y_{2n_k}) = \varepsilon. \quad (9) \]
Also (9) and inequality \( p(y_{2m_k}, y_{2n_k}) \leq p(y_{2m_k}, y_{2m_k+1}) + p(y_{2m_k+1}, y_{2n_k}) - p(y_{2m_k+1}, y_{2m_k}) \) give that \( \varepsilon \leq \lim_{k \to \infty} p(y_{2m_k+1}, y_{2n_k}) \), while from inequality \( p(y_{2m_k+1}, y_{2n_k}) \leq p(y_{2m_k+1}, y_{2m_k}) + p(y_{2m_k}, y_{2n_k}) - p(y_{2m_k}, y_{2m_k+1}) \) yields \( \lim_{k \to \infty} p(y_{2m_k+1}, y_{2n_k}) \leq \varepsilon \), and hence
\[ \lim_{k \to \infty} p(y_{2m_k+1}, y_{2n_k}) = \varepsilon. \quad (10) \]
Now (10) and inequality \( p(y_{2m_k+1}, y_{2n_k+1}) \leq p(y_{2m_k+1}, y_{2n_k+1}) + p(y_{2n_k+1}, y_{2n_k}) - p(y_{2n_k+1}, y_{2n_k+1}) \) give \( \varepsilon \leq \lim_{k \to \infty} p(y_{2m_k+1}, y_{2n_k+1}) \), while inequality \( p(y_{2m_k+1}, y_{2n_k+1}) \leq p(y_{2m_k+1}, y_{2n_k}) + p(y_{2n_k}, y_{2n_k+1}) - p(y_{2n_k}, y_{2n_k}) \) yields \( \lim_{k \to \infty} p(y_{2m_k+1}, y_{2n_k+1}) \leq \varepsilon \), and so
\[ \lim_{k \to \infty} p(y_{2m_k+1}, y_{2n_k+1}) = \varepsilon. \quad (11) \]
As
\[
M_p(x_{2m_k}, x_{2m_k-1}) = \max \left\{ p(Sx_{2m_k}, Tx_{2m_k-1}), p(fx_{2m_k}, Sx_{2m_k}), p(gx_{2m_k}, T(x_{2m_k-1})), \frac{p(Sx_{2m_k}, gx_{2m_k}) + p(fx_{2m_k}, T(x_{2m_k-1}))}{2} \right\}
\]
\[
= \max \left\{ p(y_{2m_k}, y_{2m_k-1}), p(y_{2m_k+1}, y_{2n_k}), p(y_{2m_k}, y_{2m_k+1}), \frac{p(y_{2n_k}, y_{2m_k}) + p(y_{2m_k+1}, y_{2n_k})}{2} \right\}.
\]
So \( \lim_{k \to \infty} M_p(x_{2m_k}, x_{2m_k-1}) = \max \{ \varepsilon, 0, 0, \varepsilon \} = \varepsilon \). From (4), we obtain
\[
\psi(p(y_{2n_k+1}, y_{2n_k})) = \psi(p(fx_{2n_k}, gx_{2n_k})) \leq \psi(M_p(x_{2n_k}, x_{2m_k-1})) = \psi(M_p(x_{2n_k}, x_{2m_k-1})) - \varphi(M_p(x_{2n_k}, x_{2m_k-1})).
\]
Taking upper limit as \( k \to \infty \) implies that \( \psi(\varepsilon) \leq \psi(\varepsilon) - \varphi(\varepsilon) \), a contradiction as \( \varepsilon > 0 \). Thus, we obtain \( \lim_{n \to \infty} p(y_{2n}, y_{2m}) = 0 \), and it follows that \( \{ y_{2n} \} \) is a Cauchy sequence in \((X, p)\), and hence Cauchy in \((X, p^\delta)\) by Lemma 1.4. Since \((X, p)\) is complete, it follows from Lemma 1A, \((X, p^\delta)\) is also complete, so the sequence \( \{ y_{2n} \} \) is convergent in the metric space \((X, p^\delta)\). Therefore, there exists a point \( z \) in \( X \) such that \( \lim_{n \to \infty} p^\delta(y_{2n}, z) = 0 \). Hence
\[
\lim_{n \to \infty} y_{2n+1} = \lim_{n \to \infty} T(x_{2n+1}) = \lim_{n \to \infty} f(x_{2n}) = z,
\]
and
\[
\lim_{n \to \infty} y_{2n+2} = \lim_{n \to \infty} S(x_{2n+2}) = \lim_{n \to \infty} g(x_{2n+1}) = z.
\]
Equivalently, we have
\[
\lim_{n \to \infty, m \to \infty} p(y_{2n}, y_{2m}) = \lim_{n \to \infty} p(y_{2n}, z) = p(z, z).
\]
Assume that \( S \) is continuous on \((X, p^\delta)\). Then
\[
\lim_{n \to \infty} Sx_{2n+2} = \lim_{n \to \infty} Sf(x_{2n+2}) = Sz.
\]
Also, since \( \{ f, S \} \) are compatible, we have
\[
\lim_{n \to \infty} Sf(x_{2n+2}) = \lim_{n \to \infty} Sf(x_{2n+2}) = Sz.
\]
As, \( Sx_{2n+2} = gx_{2n+1} \leq x_{2n+1} \), so from (4), we have
\[
\psi(p(Sx_{2n+2}, gx_{2n+1})) \leq \psi(M_p(Sx_{2n+2}, x_{2n+1})) - \varphi(M_p(Sx_{2n+2}, x_{2n+1})), \quad (12)
\]
where
\[
M_p(Sx_{2n+2}, x_{2n+1}) = \max \left\{ p(SSx_{2n+2}, T(x_{2n+1})), p(f(Sx_{2n+2}, Sx_{2n+2}), p(gx_{2n+1}, T(x_{2n+1})), \frac{p(SSx_{2n+2}, gx_{2n+1}) + p(f(Sx_{2n+2}, T(x_{2n+1})))}{2} \right\}.
\]
Now we show that \( \lim_{n \to \infty} p(Sx_{2n+2}, gx_{2n+1}) = p(Sz, z) \). Indeed,
\[
p^\delta(Sx_{2n+2}, gx_{2n+1}) = 2p(Sx_{2n+2}, gx_{2n+1}) - p(Sx_{2n+2}, Sx_{2n+2}) - p(gx_{2n+1}, gx_{2n+1}),
\]
implies
\[ p(fSx_{2n+2}, fSx_{2n+2} + gSx_{2n+1}, gSx_{2n+1}) + p^5(fSx_{2n+2}, gSx_{2n+1}) = 2p(fSx_{2n+2}, gSx_{2n+1}), \]
which on taking limit as \( n \to \infty \), implies that
\[ p(Sz, Sz) + p(z, z) + p^5(Sz, z) = 2 \lim_{n \to \infty} p(fSx_{2n+2}, gSx_{2n+1}). \]
This further implies that
\[ p(Sz, Sz) + p(z, z) + [2p(Sz, z) - p(Sz, Sz) - p(z, z)] = 2 \lim_{n \to \infty} p(fSx_{2n+2}, gSx_{2n+1}), \]
that is,
\[ p(Sz, z) = \lim_{n \to \infty} p(fSx_{2n+2}, gSx_{2n+1}). \]

From (12), on taking upper limit as \( n \to \infty \), we obtain \( \psi(p(Sz, z)) \leq \psi(p(Sz, z)) - \varphi(p(Sz, z)) \), and \( Sz = z \).

Now, as \( gSx_{2n+1} \preceq x_{2n+1} \) and \( gSx_{2n+1} \to z \) as \( n \to \infty \), it follows that \( x_{2n+1} \preceq x_{2n+1} \). Hence from (4), we have
\[ \psi(p(fz, gSx_{2n+1})) \leq \psi(M_p(z, x_{2n+1})) - \varphi(M_p(z, x_{2n+1})), \quad (13) \]
where
\[ M_p(z, x_{2n+1}) = \max \left\{ p(Sz, Tx_{2n+1}), p(fz, Sz), p(gx_{2n+1}, Tx_{2n+1}), \frac{p(Sz, gx_{2n+1}) + p(fz, Tx_{2n+1})}{2} \right\} \]

On taking upper limit as \( n \to \infty \), we have \( \psi(p(fz, z)) \leq \psi(p(fz, z)) - \varphi(p(fz, z)) \), and \( fz = z \).

Since \( f(X) \subseteq T(X) \), there exists a point \( w \in X \) such that \( fz = Tw \). Suppose that \( gw \neq Tw \). Since \( w \preceq Tw = fz \preceq z \) implies \( w \preceq z \). From (4), we obtain
\[ \psi(p(Tw, gw)) = \psi(p(fz, gw)) \leq \psi(M_p(z, w)) - \varphi(M_p(z, w)), \quad (14) \]
where \( M_p(z, w) = \max \left\{ p(Sz, Tw), p(fz, Sz), p(gw, Tw), \frac{p(Sz, gw) + p(fz, Tw)}{2} \right\} \]
\[ = \max \left\{ p(z, Tw), p(z, Tw), p(gw, Tw), \frac{p(Tw, gw) + p(Tw, Tw)}{2} \right\} = p(Tw, gw). \]

Now (14) becomes \( \psi(p(Tw, gw)) \leq \psi(p(Tw, gw)) - \varphi(p(Tw, gw)) \), a contradiction. Hence, \( Tw = gw \). Since \( g \) and \( T \) are weakly compatible, \( gz = gfz = gTw = Tgw = Tfw = Tz \). Thus \( z \) is a coincidence point of \( g \) and \( T \).

Now, \( fx_{2n} \preceq x_{2n} \) and \( x_{2n} \to z \) as \( n \to \infty \), imply that \( z \preceq fx_{2n} \). Hence from (4), we get \( \psi(p(fx_{2n}, gz)) \leq \psi(M_p(x_{2n}, z)) - \varphi(M_p(x_{2n}, z)) \), where
\[ M_p(x_{2n}, z) = \max \left\{ p(Sx_{2n}, Tz), p(fx_{2n}, Sx_{2n}), p(gz, Tz), \frac{p(Sx_{2n}, gz) + p(fx_{2n}, Tz)}{2} \right\} \]
\[ = \max \left\{ p(Sx_{2n}, gz), p(fx_{2n}, Sx_{2n}), p(gz, gz), \frac{p(Sx_{2n}, gz) + p(fx_{2n}, gz)}{2} \right\} = p(z, gz) \text{ as } n \to \infty. \]

On taking upper limit as \( n \to \infty \), we have \( \psi(p(z, gz)) \leq \psi(p(z, gz)) - \varphi(p(z, gz)) \), and \( z = gz \). Therefore \( fz = gz = Sz = Tz = z \). The proof is similar when \( f \) is continuous.

Similarly, the result follows when \( b \) holds.

Now, suppose that the set of common fixed points of \( f, g, S \) and \( T \) is well ordered. We are to show that the common fixed point of \( f, g, S \) and \( T \) is unique. Suppose that \( u \) and \( v \) be two fixed points of \( f, g, S \) and \( T \) i.e., \( fu = gu = Su = Tu = u \) and \( fv = gv = Sv = Tv = v \) with \( u \neq v \). Then from (4) we have
\[ \psi(p(u, v)) = \psi(p(fu, gv)) \leq \psi(M_p(u, v)) - \varphi(M_p(u, v)), \quad (15) \]

where

\[ M_p(u, v) = \max \left\{ p(Su, Tv), p(fu, Su), p(gv, Tv), \frac{p(Su, gv) + p(fu, Tv)}{2} \right\} \]

\[ = \max \left\{ p(u, v), p(u, u), p(v, v), \frac{p(u, v) + p(u, v)}{2} \right\} = p(u, v). \]

Thus \( \psi(p(u, v)) \leq \psi(p(u, v)) - \phi(p(u, v)), \) a contradiction. Hence \( u = v. \) Conversely, if \( f, g, S \) and \( T \) have only one common fixed point then the set of common fixed point of \( f, g, S \) and \( T \) is well ordered being singleton.

**Example 2.2.** Let \( X = [0, k] \) for a real number \( k \geq 9/10 \) endowed with usual order \( \leq. \) Let \( p : X \times X \rightarrow \mathbb{R}^+ \) be defined by \( p(x, y) = |x - y| \) if \( x, y \in [0, 1], \) and \( p(x, y) = \max\{x, y\} \) otherwise. It is easily seen that \( (X, p) \) is a complete partial metric space [1]. Consider

\[ \psi(t) = \begin{cases} 3t, & \text{if } 0 \leq t \leq \frac{1}{3} \\ 1, & \text{if } x \in (\frac{1}{3}, 1] \end{cases} \]

and \( \phi(t) = \begin{cases} 0, & \text{if } t = 0 \\ \frac{1}{9}, & \text{if } 0 \leq t < \frac{1}{3} \end{cases} \)

Define the self mappings \( f, g, S \) and \( T \) on \( X \) by

\[ f(x) = \begin{cases} \frac{1}{6}x, & \text{if } x \leq \frac{1}{3} \\ \frac{1}{12}, & \text{if } x \in (\frac{1}{3}, k] \end{cases}, \quad g(x) = \begin{cases} 0, & \text{if } x \leq \frac{1}{3} \\ \frac{1}{3}, & \text{if } x \in (\frac{1}{3}, k] \end{cases}, \quad T(x) = \begin{cases} 0, & \text{if } x \in [0, \frac{1}{3}] \\ x, & \text{if } x \in (\frac{1}{3}, 1] \end{cases}, \quad S(x) = \begin{cases} 0, & \text{if } x = 0 \\ k, & \text{if } x \in (\frac{1}{3}, k] \end{cases} \]

Then \( f(X) \subseteq T(X) \) and \( g(X) \subseteq S(X) \) where \( f \) and \( g \) are dominated and \( S \) and \( T \) are dominating mappings as

\[
\begin{array}{llllll}
\text{for each } x \text{ in } X & & & & & \\
\quad x = 0 & f(0) = 0 & g(0) = 0 & 0 = S(0) & 0 = T(0) \\
\quad x \in [0, \frac{1}{3}] & f(x) = \frac{1}{6}x \leq x & g(x) = 0 < x & x \leq \frac{1}{3} = S(x) & x = T(x) \\
\quad x \in (\frac{1}{3}, k] & f(x) = \frac{1}{12} < x & g(x) = \frac{1}{3} < x & x \leq k = S(x) & x \leq k = T(x) \\
\end{array}
\]

Also note that \( \{f, S\} \) are compatible, \( \{g, T\} \) are weakly compatible with \( f \) is a continuous map.

To show that \( f, g, S \) and \( T \) satisfy (4) for all \( x, y \in X \), we consider the following cases:

(i) if \( x = 0 \) and \( y \in [0, \frac{1}{3}] \), then \( p(fx, gy) = 0 \) and (4) is satisfied.

(ii) For \( x = 0 \) and \( y \in (\frac{1}{3}, k] \), we have

\[
\psi(p(fx, gy)) = \psi(p(0, \frac{1}{3})) = \psi(\frac{1}{3}) = 1 < k - \frac{1}{9} = \psi(k) - \phi(k) = \psi(p(0, k)) - \phi(p(0, k)) = \psi(p(Sx, Ty)) - \phi(p(Sx, Ty)) = \psi(M_p(x, y)) - \phi(M_p(x, y)).
\]

(iii) When \( x \in (0, \frac{1}{3}] \) and \( y \in [0, \frac{1}{3}] \), then

\[
\psi(p(fx, gy)) = \psi(p(\frac{1}{6}x, 0)) = \psi(\frac{1}{2}x) = \frac{1}{2}x \leq 3 \max\{\frac{1}{3} - \frac{1}{6}x, y\} - \frac{1}{3} \max\{\frac{1}{3} - \frac{1}{6}x, y\} = \psi(\max\{\frac{1}{3} - \frac{1}{6}x, y\}) - \phi(\max\{\frac{1}{3} - \frac{1}{6}x, y\}) = \psi(\max\{p(fx, Sx), p(gy, Ty)\}) - \phi(\max\{p(fx, Sx), p(gy, Ty)\}) = \psi(M_p(x, y)) - \phi(M_p(x, y)).
\]
Corollary 2.3. Let $\psi(0) = 0$ is the unique common fixed point of $f, g, S$ and $T$. Suppose that, there exists control functions $\psi$ and $\varphi$ such that $\psi \circ (\psi(p, q)) = \psi(\psi(p, q)) = \psi(0) = 0$ and $\varphi \circ (\varphi(p, q)) = \varphi(\varphi(p, q)) = \varphi(0) = 0$ for every two comparable elements $x, y \in X$.

\[
\psi(p(fx, gy)) = \psi(p(\frac{1}{6}, \frac{1}{3})) = \psi(\frac{1}{3}(1 - \frac{x}{2})) = 1 - \frac{1}{2}x < k - \frac{1}{9} = \psi(\max\{\frac{1}{3}, k\}) - \varphi(\max\{\frac{1}{3}, k\}) = \psi(p(gy, Ty)) - \varphi(p(gy, Ty)) = \psi(M_p(x, y)) - \varphi(M_p(x, y)).
\]

Corollary 2.4. Let $(X, \preceq, p)$ be an ordered complete partial metric space. Let $f, g, S$ and $T$ be self maps on $X$, $(f, g)$ be the pair of dominated and $(S, T)$ be the pair of dominating maps with $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$. Suppose that, there exists control functions $\psi$ and $\varphi$ such that for every two comparable elements $x, y \in X$,

\[
\psi(p(fx, gy)) \leq \psi(p(Sx, Ty)) - \varphi(p(Sx, Ty)).
\]
is satisfied where $\phi \in Y$ and

$$M_p(x, y) = \max\{p(Sx, Ty), p(fx, Sx), p(gy, Ty), \frac{p(Sx, gy) + p(fx, Ty)}{2}\}.$$  

If for a nonincreasing sequence $\{x_n\}$ in $(X, \preceq)$ with $x_n \preceq y_n$ for all $n$ and $\lim_{n \to \infty} p^\phi(x_n, u) = 0$ it follows $u \preceq y_n$ for all $n \in \mathbb{N}$ and either

(a) $\{f, S\}$ are compatible, $f$ or $S$ is continuous on $(X, p^\phi)$ and $\{g, T\}$ are weakly compatible or
(b) $\{g, T\}$ are compatible, $g$ or $T$ is continuous on $(X, p^\phi)$ and $\{f, S\}$ are weakly compatible,

then $f$, $g$, $S$ and $T$ have a common fixed point. Moreover, the set of common fixed points of $f, g, S$ and $T$ is well ordered if and only if, $g, S$ and $T$ have one and only one common fixed point.

**Proof.** Define $\Psi: [0, \infty) \to [0, \infty)$ by $\Psi(x) = \int_0^x \phi(t) dt$, then from (17), we have

$$\Psi(\phi(p(fx, gy))) \leq \Psi(\phi(M_p(x, y))) - \Psi(\phi(M_p(x, y))),$$

which can be written as

$$\psi_1(p(fx, gy)) \leq \psi_1(M_p(x, y)) - \phi_1(M_p(x, y)),$$

where $\psi_1 = \Psi \circ \phi$ and $\phi_1 = \Psi \circ \phi$. Clearly, $\psi_1, \phi_1: \mathbb{R}^+ \to \mathbb{R}^+$, $\psi_1$ is continuous and nondecreasing, $\phi_1$ is a lower semicontinuous, and $\psi_1(t) = \phi_1(t) = 0$ if and only if $t = 0$. Hence by Theorem 2.1, $f, g, S$ and $T$ have a unique common fixed point. \qed

**Remarks 2.5.**

1. If we take $f = g$ and $S = T = I$ (an identity map) in Corollary 2.3, then it extends Theorem 2.1 of [19] to ordered metric spaces.
2. We can not apply Corollary 2.3 in the setup of ordered metric space to the mappings given in Example 2.2. Indeed, if we take $x, y \in (\frac{1}{2}, 2]$ contractive condition in the Corollary 2.3 in the setup of ordered metric space is not satisfied.
3. Theorem 2.1 generalizes Theorem 2.1 of [1], Theorem 2.1 of [3] and Theorem 2.1 of [18] for four maps in the setup of ordered partial metric spaces.

### 3 Application

Let $\Omega = [0, 1]$ be bounded open set in $\mathbb{R}$, and $L^2(\Omega)$ be the set of comparable functions on $\Omega$ whose square is integrable on $\Omega$. Consider an integral equation

$$F(t, x(t)) = \int_\Omega \kappa(t, s, x(s)) ds,$$

(19)

where $F: \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ and $\kappa: \Omega \times \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ be two mappings. Feckan [20] obtained the nonnegative solutions of implicit integral equation (19) as an application of fixed point theorem. We shall study the sufficient condition for existence of solution of integral equation in framework of ordered complete partial metric space. Define $p: X \times X \to \mathbb{R}^+$ by

$$p(x, y) = \max\left(\sup_{t \in \Omega} x(t), \sup_{t \in \Omega} y(t)\right).$$

Then $(X, p)$ is a complete partial metric space endowed with usual order $\leq$. We assume the following:

(i) there exists a positive number $h \in [0, \frac{1}{2})$ connected with the relation that (a) $F(s, u(t)) \leq h \cdot u(t)$ for each $s, t \in \Omega$.
(b) $\int_\Omega \kappa(t, s, v(s)) ds \leq 2h \cdot v(t)$ for each $s, t \in \Omega$.  

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(ii) The control functions \( \psi \) and \( \varphi \) are connected with relation that

\[
\psi(a) + \varphi(2a) \leq \psi(2a),
\]

for every \( a \in \mathbb{R}^+ \).

Thus for every comparable elements \( x, y \in X \),

\[
\psi(p(fx, gy)) \leq \psi(M_p(x, y)) - \varphi(M_p(x, y)),
\]

is satisfies where

\[
M_p(x, y) = \max \{ p(x, y), p(fx, x), p(gy, y), \frac{p(fx, y) + p(gy, x)}{2} \}.
\]

Now we can apply Theorem 2.1 by taking \( S \) and \( T \) as identity maps to obtain the solution of integral equation (19) in \( L^2(\Omega) \).

\[
\square
\]

4 Fractals in partial metric spaces.

Recently, Nazir et al. [34] study the iterated function systems for generalized \( F \)-contraction mappings. In this section, we study the iterated function systems for generalized \((\psi, \varphi)\)-contraction mappings. Consistent with [11], let \( CB^p(X) \) be the family of all non-empty, closed and bounded subsets of the partial metric space \((X, p)\), induced by the partial metric \( p \). Note that closedness is taken from \((X, \tau_p)\) (\( \tau_p \) is the topology induced by \( p \)) and boundedness is given as follows: \( A \) is a bounded subset in \((X, p)\) if there exists an \( x_0 \in X \) and \( M \geq 0 \) such that for all \( a \in A \), we have \( a \in B_p(x_0, M) \), that is, \( p(x_0, a) < p(a, a) + M \). For \( A, B \in CB^p(X) \) and \( x \in X \), define \( \delta_p : CB^p(X) \times CB^p(X) \to [0, \infty) \) and

\[
\begin{align*}
p(x, A) &= \inf \{ p(x, a) : a \in A \}, \\
\delta_p(A, B) &= \sup \{ p(a, B) : a \in A \}, \\
H_p(A, B) &= \max \{ \delta_p(A, B), \delta_p(B, A) \}.
\end{align*}
\]

It can be verified that \( p(x, A) = 0 \) implies \( p^S(x, A) = 0 \), where \( p^S(x, A) = \inf \{ p^S(x, a) : a \in A \} \).
Lemma 4.1. [8] Let \((X, p)\) be a partial metric space and \(A\) be a non-empty subset of \(X\), then \(a \in \overline{A}\) if and only if \(p(a, A) = p(a, a)\).

Proposition 4.2. [11] Let \((X, p)\) be a partial metric space. For any \(A, B, C \in CB^p(X)\) we have the following:

(i) \(\delta_p(A, A) = \sup \{p(a, a) : a \in A\}\);
(ii) \(\delta_p(A, A) \leq \delta_p(A, B)\);
(iii) \(\delta_p(A, B) = 0\) implies \(A \subseteq B\);
(iv) \(\delta_p(A, B) \leq \delta_p(A, C) + \delta_p(C, B) - \inf_{c \in C} p(c, c)\).

Proposition 4.3. [11] Let \((X, p)\) be a partial metric space. For any \(A, B, C \in CB^p(X)\), we have the following:

\((h_1)\) \(H_p(A, A) \leq H_p(A, B)\);
\((h_2)\) \(H_p(A, B) = H_p(B, A)\);
\((h_3)\) \(H_p(A, B) \leq H_p(A, C) + H_p(C, B) - \inf_{c \in C} p(c, c)\);
\((h_4)\) \(H_p(A, B) = 0\) implies that \(A = B\).

The mapping \(H_p : CB^p(X) \times CB^p(X) \rightarrow [0, \infty]\) is called partial Hausdorff metric induced by partial metric \(p\). Every Hausdorff metric is partial Hausdorff metric but converse is not true (see Example 2.6 in [11]).

Theorem 4.4. [11] Let \((X, p)\) be a partial metric space. If \(T : X \rightarrow CB^p(X)\) be a multi-valued mapping such that for all \(x, y \in X\), we have \(H_p(Tx, Ty) \leq kp(x, y)\), where \(k \in (0, 1)\). Then \(T\) has a fixed point.

Definition 4.5. Let \((X, p)\) be a partial metric space and and \(\mathcal{H}_p(X)\) denotes the set of all non-empty compact subsets of \(X\). Let \(\{f_n : n = 1, 2, \ldots, N\}\) be a finite family of self-mappings on \(X\) that satisfy

\[
\psi(p(f_ix, f_jy)) \leq \psi(M_p(x, y)) - \phi(M_p(x, y)),
\]

is satisfied where

\[
M_p(x, y) = \max\{p(x, y), p(f_i(x), x), p(f_j(y), y), p(f_i^2(x), f_j(x)), p(f_i^2(y), f_j(y)), \frac{p(f_i(x), y) + p(f_j(y), x)}{2}\}
\]

for every \(x, y \in X\). We call these maps as a family of generalized \((\psi, \phi)\) – contraction mappings. Define \(T : \mathcal{H}_p(X) \rightarrow \mathcal{H}_p(X)\) by

\[
T(A) = f_1(A) \cup f_2(A) \cup \cdots \cup f_N(A)
= \bigcup_{n=1}^{N} f_n(A), \text{ for each } A \in \mathcal{H}_p(X).
\]

If \(f_n : X \rightarrow X, n = 1, 2, \ldots, N\) are generalized \((\psi, \phi)\) – contraction mappings, then \((X; f_1, f_2, \ldots, f_N)\) is called generalized \((\psi, \phi)\) – iterated function system \(((\psi, \phi) - \text{IFS})\).

Definition 4.6. A nonempty compact set \(A \subseteq X\) is said to be an attractor of the IFS if

(a) \(T(A) = A\) and
(b) there is an open set \(U \subseteq X\) such that \(A \subseteq U\) and \(\lim_{k \rightarrow \infty} T^k(B) = A\) for any compact set \(B \subseteq U\), where the limit is taken with respect to the partial Hausdorff metric.

The largest open set \(U\) satisfying (b) is called a basin of attraction.

Theorem 4.7. Let \((X, p)\) be a complete partial metric space and \((X; f_n, n = 1, 2, \ldots, k)\) a generalized \((\psi, \phi)\) – iterated function system. Let \(T : \mathcal{H}_p(X) \rightarrow \mathcal{H}_p(X)\) be a mapping defined by

\[
T(A) = \cup_{n=1}^{k} f_n(A), \text{ for all } A \in \mathcal{H}_p(X).
\]

Suppose that, there exists control functions \(\psi\) and \(\phi\) such that for every \(A, B \in \mathcal{H}_p(X)\),

\[
\psi(H_p(T(A), T(B))) \leq \psi(M_T(A, B)) - \phi(M_T(A, B))
\]

(20)
is satisfied where
\[
M_T(A, B) = \max\{H_p(A, B), H_p(A, T(A)), H_p(B, T(B)), H_p(T^2(A), T(A)),
H_p(T^2(A), B), H_p(T^2(A), T(B)), \frac{H_p(A, T(B)) + H_p(B, T(A))}{2}\}.
\]

Then \(T\) has a unique fixed point \(U \in \mathcal{H}_p(X)\), that is
\[
U = T(U) = \bigcup_{n=1}^{\infty} f_n(U).
\]

Moreover, for any initial set \(A_0 \in \mathcal{H}_p(X)\), the sequence of compact sets \(\{A_0, T(A_0), T^2(A_0), \ldots\}\) converges to a fixed point of \(T\).

**Proof.** Let \(A_0\) be an arbitrary element in \(\mathcal{H}_p(X)\). If \(A_0 = T(A_0)\), then the proof is finished. So we assume that \(A_0 \neq T(A_0)\). Define
\[
A_1 = T(A_0), A_2 = T(A_1), \ldots, A_{m+1} = T(A_m)
\]
for \(m \in \mathbb{N}\).

We may assume that \(A_m \neq A_{m+1}\) for all \(m \in \mathbb{N}\). If not, then \(A_k = A_{k+1}\) for some \(k\) implies \(A_k = T(A_k)\) and this completes the proof. Take \(A_m \neq A_{m+1}\) for all \(m \in \mathbb{N}\). From (20), we have
\[
\psi(H_p(A_{m+1}, A_{m+2})) = \psi(H_p(T(A_m), T(A_{m+1}))) \\
\leq \psi(M_T(A_m, A_{m+1})) - \varphi(M_T(A_m, A_{m+1}))
\]
where
\[
M_T(A_m, A_{m+1}) = \max\{H_p(A_m, A_{m+1}), H_p(A_m, T(A_m)), H_p(A_{m+1}, T(A_m)),
H_p(T^2(A_m), T(A_m)), H_p(T^2(A_m), A_{m+1}), H_p(T^2(A_m), T(A_m)),
\frac{H_p(A_m, T(A_m)) + H_p(A_{m+1}, T(A_m))}{2}\}
\]
\[
= \max\{H_p(A_m, A_{m+1}), H_p(A_m, A_{m+1}), H_p(A_{m+1}, A_{m+2}),
H_p(A_{m+2}, A_{m+1}), H_p(A_{m+2}, A_{m+1}), H_p(A_{m+2}, A_{m+2}),
\frac{H_p(A_m, A_{m+2}) + H_p(A_{m+1}, A_{m+1})}{2}\}
\]
\[
\leq \max\{H_p(A_m, A_{m+1}), H_p(A_{m+1}, A_{m+2}), \frac{H_p(A_m, A_{m+1}) + H_p(A_{m+1}, A_{m+2})}{2}\}
\]
\[
= \max\{H_p(A_m, A_{m+1}), H_p(A_{m+1}, A_{m+2})\}.
\]

As \(\max\{H_p(A_m, A_{m+1}), H_p(A_{m+1}, A_{m+2})\} \leq M_T(A_m, A_{m+1})\).

Therefore \(M_T(A_m, A_{m+1}) = \max\{H_p(A_m, A_{m+1}), H_p(A_{m+1}, A_{m+2})\}\).

Now if \(M_T(A_m, A_{m+1}) = H_p(A_{m+1}, A_{m+2})\), then (20) gives that
\[
\psi(H_p(A_{m+1}, A_{m+2})) \leq \psi(H_p(A_{m+1}, A_{m+2})) - \varphi(H_p(A_{m+1}, A_{m+2})),
\]
a contradiction. Hence \(M_T(A_m, A_{m+1}) = H_p(A_{m+1}, A_{m+2})\) and
\[
\psi(H_p(A_{m+1}, A_{m+2})) \leq \psi(H_p(A_m, A_{m+1})) - \varphi(H_p(A_m, A_{m+1}))
\]
\[
\leq \psi(H_p(A_m, A_{m+1})),
\]
that is, \(H_p(A_{m+2}, A_{m+2}) \leq H_p(A_m, A_{m+1})\). Thus the sequence \(\{H_p(A_m, A_{m+1})\}\) is nonincreasing. Hence there exists \(c \geq 0\) such that \(\lim_{n \to \infty} H_p(A_n, A_{n+1}) = c\). Suppose that \(c > 0\). Then, \(\psi(H_p(A_{n+2}, A_{n+1})) \leq \psi(H_p(A_{n+1}, A_n)) - \varphi(H_p(A_{n+1}, A_n)),\) and by lower semicontinuity of \(\varphi\), we have
\[
\lim_{n \to \infty} \sup \psi(H_p(A_{n+2}, A_{n+1})) \leq \lim_{n \to \infty} \sup \psi(H_p(A_{n+1}, A_n)) - \lim_{n \to \infty} \inf \varphi(H_p(A_{n+1}, A_n)),
\]
which implies that \( \psi(c) \leq \psi(c) - \varphi(c) \), a contradiction. Therefore \( c = 0 \). So we conclude that

\[
\lim_{n \to \infty} H_p(A_{n+1}, A_n) = 0. \tag{21}
\]

Now, we show that \( \lim_{n, m \to \infty} H_p(A_n, A_m) = 0 \). If not, there is \( \epsilon > 0 \), and there exist even integers \( n_k \) and \( m_k \) with \( m_k > n_k > k \) such that

\[
H_p(A_{n_k}, A_{n_k}) \geq \epsilon, \tag{22}
\]

and \( H_p(A_{m_k-2}, A_{m_k}) < \epsilon \). Since

\[
\epsilon \leq H_p(A_{m_k}, A_{n_k}) \leq H_p(A_{m_k}, A_{m_k-2}) + H_p(A_{m_k-2}, A_{n_k}) - \inf_{a_1 \in A_{m_k-2}} p(a_1, a_1) \\
\leq H_p(A_{m_k}, A_{m_k-2}) + H_p(A_{m_k-2}, A_{m_k-1}) + H_p(A_{m_k-1}, A_{n_k}) \\
- \inf_{a_2 \in A_{m_k-1}} p(a_2, a_2) - \inf_{a_1 \in A_{m_k-2}} p(a_1, a_1).
\]

From (21) and (22), we have

\[
\lim_{k \to \infty} H_p(A_{m_k}, A_{n_k}) = \epsilon. \tag{23}
\]

Also (23) and inequality \( H_p(A_{m_k}, A_{n_k}) \leq H_p(A_{m_k}, A_{m_k-1}) + H_p(A_{m_k-1}, A_{n_k}) - \inf_{a_1 \in A_{m_k-1}} p(a_1, a_1) \) give that \( \epsilon \leq \lim_{k \to \infty} H_p(A_{m_k-1}, A_{n_k}) \), while from inequality \( H_p(A_{m_k-1}, A_{n_k}) \leq H_p(A_{m_k-1}, A_{m_k}) + H_p(A_{m_k}, A_{n_k}) - \inf_{a_2 \in A_{m_k}} p(a_2, a_2) \), we get

\[
\lim_{k \to \infty} H_p(A_{m_k-1}, A_{n_k}) = \epsilon.
\]

Now (26) and inequality \( H_p(A_{m_k-1}, A_{n_k}) \leq H_p(A_{m_k-1}, A_{n_k+1}) + H_p(A_{n_k+1}, A_{n_k}) - \inf_{a_2 \in A_{n_k+1}} p(a_2, a_2) \) give that \( \epsilon \leq \lim_{k \to \infty} H_p(A_{m_k-1}, A_{n_k+1}) \), while inequality \( H_p(A_{m_k-1}, A_{n_k+1}) \leq H_p(A_{m_k-1}, A_{m_k}) + H_p(A_{n_k}, A_{n_k+1}) - \inf_{a_1 \in A_{n_k}} p(a_1, a_1) \)

yields \( \lim_{k \to \infty} H_p(A_{m_k-1}, A_{n_k+1}) \leq \epsilon \), and so

\[
\lim_{k \to \infty} H_p(A_{m_k-1}, A_{n_k+1}) = \epsilon. \tag{25}
\]

As

\[
M_T(A_{m_k}, A_{m_k-1}) = \max\{H_p(A_{n_k}, A_{m_k-1}), H_p(A_{n_k}, A_{m_k}), H_p(A_{m_k-1}, A_{m_k-2}), H_p(A_{m_k-2}, A_{m_k-1}), H_p(A_{m_k-2}, A_{m_k}), H_p(A_{n_k}, A_{m_k-1}) + H_p(A_{n_k}, A_{m_k-1}) \}.
\]

So \( \lim_{k \to \infty} M_T(x_{n_k}, x_{m_k-1}) = \max\{\epsilon, 0, 0, 0, \epsilon, \epsilon, \epsilon\} = \epsilon \). From (20), we obtain

\[
\psi(H_p(A_{n_k+1}, A_{m_k})) = \psi(H_p(A_{n_k}, A_{m_k})) \leq \psi(M_T(A_{n_k}, A_{m_k-1})) - \varphi(M_T(A_{n_k}, A_{m_k-1})).
\]

Taking upper limit as \( k \to \infty \) implies that \( \psi(\epsilon) \leq \psi(\epsilon) - \varphi(\epsilon) \), a contradiction as \( \varphi(\epsilon) > 0 \). Therefore \( \{A_n\} \) is a Cauchy sequence in \( X \). Since \( (\mathcal{F}_p(X), p) \) is complete as \( (X, p) \) is complete, \( \lim_{n \to \infty} H_p(A_n, U) = H_p(U, U) \) for some \( U \in \mathcal{F}_p(X) \), that is, we have \( A_n \to U \) as \( n \to \infty \).

In order to show that \( U \) is the fixed point of \( T \), we contrary assume that \( H_p(U, T(U)) \neq 0 \). Now

\[
\psi \left( H_p(A_{n_k+1}, T(U)) \right) = \psi(H_p(T(A_n), T(U))) \leq \psi(M_T(A_n, U)) - \varphi(M_T(A_n, U)) \tag{26}
\]

where

\[
M_T(A_n, U) = \max\{H_p(A_n, U), H_p(A_n, T(A_n)), H_p(U, T(U)), H_p(T^2(A_n), T(A_n))\},
\]
Now we consider the following cases:

(i) If $M_T(A_n, U) = H_p(A_n, U, )$, then on taking upper limit as $n \to \infty$ in (26), we have

$$\psi \left( H_p(T(U), U) \right) \leq \psi \left( H_p(U, U) \right) - \psi \left( H_p(U, U) \right),$$

a contradiction.

(2) When $M_T(A_n, U) = H_p(A_n, A_{n+1})$, then on taking upper limit as $n \to \infty$ in (26), implies

$$\psi \left( H_p(T(U), U) \right) \leq \psi \left( H_p(U, U) \right) - \varphi \left( H_p(U, U) \right),$$

gives a contradiction.

(3) In case $M_T(A_n, U) = H_p(U, T(U))$, then on taking upper limit as $n \to \infty$ in (26), we get

$$\psi \left( H_p(T(U), U) \right) \leq \psi \left( H_p(U, T(U)) \right) - \varphi \left( H_p(U, T(U)) \right),$$

a contradiction.

(4) If $M_T(A_n, U) = \frac{H_p(A_n, T(U)) + H_p(U, A_{n+1})}{2}$, then on upper taking limit as $n \to \infty$, we have

$$\psi \left( H_p(T(U), U) \right) \leq \psi \left( \frac{H_p(U, T(U)) + H_p(U, U)}{2} \right) - \varphi \left( \frac{H_p(U, T(U)) + H_p(U, U)}{2} \right)$$

$$= \psi \left( \frac{H_p(U, T(U))}{2} \right) - \varphi \left( \frac{H_p(U, T(U))}{2} \right),$$

a contradiction.

(5) When $M_T(A_n, U) = H_p(A_{n+2}, A_{n+1})$, then on taking upper limit as $n \to \infty$ in (26), we get

$$\psi \left( H_p(T(U), U) \right) \leq \psi \left( H_p(U, U) \right) - \varphi \left( H_p(U, U) \right),$$

gives a contradiction.

(6) In case $M_T(A_n, U) = H_p(A_{n+2}, U)$, then on taking upper limit as $n \to \infty$ in (26), we get

$$\psi \left( H_p(T(U), U) \right) \leq \psi \left( H_p(U, U) \right) - \varphi \left( H_p(U, U) \right),$$

a contradiction.

(7) Finally if $M_T(A_n, U) = H_p(A_{n+2}, T(U))$, then on taking upper limit as $n \to \infty$, we have

$$\psi \left( H_p(T(U), U) \right) \leq \psi \left( H_p(U, T(U)) \right) - \varphi \left( H_p(U, U) \right),$$

a contradiction.

Thus, $U$ is the fixed point of $T$.

To show the uniqueness of fixed point of $T$, assume that $U$ and $V$ are two fixed points of $T$ with $H_p(U, V)$ is not zero. From (20), we obtain that

$$\psi(H_p(U, V)) = \psi(H_p(T(U), T(V))$$

$$\leq \psi(M_T(U, V)) - \varphi(M_T(U, V)),$$

where

$$M_T(U, V) = \max \{H_p(U, V), H_p(U, T(U)), H_p(V, T(V)), \frac{H_p(U, T(V)) + H_p(V, T(U))}{2},$$

$$H_p(T^2(U), U), H_p(T^2(U), V), H_p(T^2(U), T(V)) \}.$$
that is,
\[\psi(H_p(U, V)) \leq \psi(H_p(U, V)) - \varphi(H_p(U, V)),\]
a contradiction. Thus \(T\) has a unique fixed point \(U \in \mathcal{H}_p(X)\).

**Remarks 4.8.** In Theorem 4.7, if we take \(S(X)\) the collection of all singleton subsets of \(X\), then clearly \(S(X) \subseteq \mathcal{H}_p(X)\). Moreover, consider \(f_n = f\) for each \(n\), where \(f = f_1\) then the mapping \(T\) becomes

\[T(x) = f(x)\]

With this setting we obtain the following fixed point result.

**Corollary 4.9.** Let \((X, p)\) be a complete partial metric space and \(\{X : f_n, n = 1, 2, \ldots, k\}\) a generalised iterated function system. Let \(f : X \rightarrow X\) be a mapping defined as in Remark 4.8. Suppose that, there exists control functions \(\psi\) and \(\varphi\) such that for any \(x, y \in \mathcal{H}_p(X)\), the following holds:

\[\psi(p(fx, fy)) \leq \psi(M_p(x, y)) - \varphi(M_p(x, y)),\]

where

\[M_p(x, y) = \max \left\{ p(x, y), p(x, fx), p(y, fy), p(f^2x, y), p(f^2x, fx), p(f^2x, fy), \frac{p(x, fx) + p(y, fy)}{2} \right\}.\]

Then \(f\) has a unique fixed point \(x \in X\). Moreover, for any initial set \(x_0 \in X\), the sequence of compact sets \(\{x_0, fx_0, f^2x_0, \ldots\}\) converges to a fixed point of \(f\).

**Acknowledgement:** The first author is grateful to the Erasmus Mundus project FUSION for supporting the research visit to Mälardalen University, Sweden and to the Division of Applied Mathematics at the School of Education, Culture and Communication for creating excellent research environment. The authors thank the referees for their suggestions regarding this work.

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