An efficient computational approach for generalized Hirota-Satsuma coupled KdV equations arising in shallow water waves

Abstract: The aim of this letter is to present a numerical algorithm for generalized Hirota-Satsuma coupled KdV equations arising in unidirectional propagation of shallow water waves by using the homotopy analysis transform technique. The computational approach is the merged form of the homotopy analysis technique and Laplace transform scheme. The technique provides a series solution, which converges very fast, components are very easily calculated, and it does not require linearization or small perturbation. The numerical and graphical results derived by making use of proposed approach indicate that the scheme is very user friendly and easy to implement.

Keywords: Hirota-Satsuma coupled KdV equations, Shallow water waves, Homotopy analysis method, Laplace transform method, Approximate solution.

1 Introduction

Nonlinear phenomena have received importance and application in applied mathematics, physics and related to engineering; many such physical phenomena are described in terms of nonlinear partial differential equations. In this article, we investigate a general form of Hirota-Satsuma coupled KdV equation which was initiated by Wu et al. [41]. Among these equations a generalized form of Hirota-Satsuma coupled KdV equation is written in the following form:

\[ u_t = \frac{1}{2} u_{xxx} - 3 u u_x + (v w)_x, \quad v_t = -v_{xxx} + 3 u v_x, \quad w_t = -w_{xxx} + 3 u w_x. \] (1)

and

\[ u_t = \frac{1}{2} u_{xxx} - 3 u^2 u_x + \frac{3}{2} v_{xx} + 3 u v_x + 3 v u_x - 3 \lambda u_x, \] (2)
\[ v_t = -v_{xxx} - 3 v v_x - 3 u v_x + 3 u^2 v_x + 3 \lambda v_x. \]

The Hirota-Satsuma coupled KdV equation describes the unidirectional propagation of shallow water waves. In a study Eq. (1) is converted to a new complex coupled KdV equation [41] and the Hirota-Satsuma equation [13] by putting \( w = v^* \) and \( w = v \) respectively. Further Eq. (2) becomes a generalized KdV equation for \( u = 0 \), and an MKdV equation for \( v = 0 \). Also, the soliton solution for this equation is constructed by Fan [10]. In recent years generalized Hirota-Satsuma coupled KdV equation has been analyzed by many research workers with the aid of different schemes such as Jacobi-elliptic function technique [28], the projective
generalized Hirota-Satsuma coupled KdV equations arising in shallow water waves

Riccati equations technique [29], the Adomian decomposition technique [30], the homotopy perturbation approach [11], the homotopy analysis scheme [1] and the reduced differential transform technique [3]. In recent times nonlinear differential equations have been studied by many authors by such as Baskonus et al. [7], Baskonus [8, 9], Rashidi et al. [31–33] and others. The homotopy analysis method (HAM) was initially proposed and nurtured by Liao [24–27]. This technique has been successfully applied to solve many types of nonlinear problems such as nonlinear periodic wave problems [39], Vakhnenko equation [42], Laplace equation [14], steady flow of a fourth grade fluid [36], generalized Hirota-Satsuma coupled KdV equation [2], non-Newtonian flow and heat transfer over a non-isothermal wedge [35], micropolar flow in a porous channel with mass injection [34], nonlinear fractional shock wave equation [23] etc. The Laplace transform [37] is a powerful scheme for solving various linear partial differential equations having applications in various fields such as physics, chemistry, biology and finance. In recent years many scientists and mathematicians have made paid attention for combining semi analytical techniques with Laplace transform such as Khuri [18], Khan and Hussain [17], Khan et al. [15], Gupta et al. [12], Kumar et al. [21, 22], Wazwaz [40], Thongmoon and Pusjuso [38] and others. On the other hand, HAM is also combined with well defined Laplace transform to investigate nonlinear problems such as nonlinear equation semi infinite domain [16], fractional biological population model [20], Volterra integral equation [19], fractional wave equations [43], etc.

In this paper, we apply the homotopy analysis transform method (HATM) for solving the generalized Hirota-Satsuma coupled KdV equations. The proposed technique finds the solution of nonlinear problem without using any discretization, restrictive assumptions and avoids round-off errors. The proposed technique solves nonlinear problems without using Adomian’s polynomials and He’s polynomials which can be considered as a clear advantage of this algorithm over Adomian decomposition technique and homotopy perturbation algorithm.

2 Basics idea of HATM

In order to present the theoretical structure and solution procedure of this scheme, we consider the following equation

$$Du + Ru + Nu = g(x).$$

where $Nu$, indicates the nonlinear terms, $D$ is the highest order linear operator and $R$ is the remaining of the linear operator. By applying Laplace transform on both sides of Eq. (3), we get the following equation

$$L[Du + Ru + Nu] = L[g(x)],$$

On using the property of Laplace transform operator for derivative of a function, it yields

$$s^n L[u] - \sum_{k=1}^{n} s^{k-1} u^{(n-k)}(0) + L[Ru] + L[Nu] = L[g(x)].$$

If we simplify the above results, we have

$$L[u] - \frac{1}{s^n} \sum_{k=1}^{n} s^{k-1} u^{(n-k)}(0) + \frac{1}{s^n} L[Ru + Nu] = 0.$$ (6)

We define the nonlinear operator in the following form

$$N[\phi(x, t; q)] = L[\phi(x, t; q)] - \frac{1}{s^n} \sum_{k=1}^{n} s^{k-1} \phi^{(n-k)}(x, t; q)(0) + \frac{1}{s^n} [L[\phi(x, t; q)] + L[R\phi(x, t; q)]] ,$$

where $\phi(x, t; q)$ is a real functions of $x$, $t$ and $q$. In similar manner as HAM [24–27], we construct a homotopy as follows

$$(1 - q)L[\phi(x, t; q) - u_0(x, t)] = q h(x, t) N[u(x, t)],$$

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where $\phi(x, t; q)$ is a real functions of $x$, $t$ and $q$. In similar manner as HAM [24–27], we construct a homotopy as follows

$$(1 - q)L[\phi(x, t; q) - u_0(x, t)] = q h(x, t) N[u(x, t)],$$
where \( q \in [0, 1] \) is an embedding parameter, \( h \neq 0 \) is an auxiliary parameter, \( H(x, t) \neq 0 \) is auxiliary function, \( L \) is an auxiliary linear operator, \( u_0(x, t) \) is an initial guess of \( u(x, t) \), and \( \varphi(x, t; q) \) is an unknown function. It is important that one has great freedom to choose auxiliary things in HAM. Obviously, when \( q = 0 \) and \( q = 1 \), it holds
\[
\varphi(x, t; 0) = u_0(x, t) \quad \text{and} \quad \varphi(x, t; 1) = u(x, t).
\]
respectively. So, as \( q \) increases from 0 to 1, the solution \( \varphi(x, t; q) \) changes from the initial guess \( u_0(x, t) \) to the solution \( u(x, t) \). On expressing \( \varphi(x, t; q) \) in series form by Taylor’s theorem with respect to \( q \), we have the following result
\[
\varphi(x, t; q) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t) q^m,
\]
where
\[
u_m(x, t) = \frac{1}{m!} \frac{\partial^m \varphi(x, t; q)}{\partial q^m} \big|_{q = 0}.
\]
On properly choosing the auxiliary linear operator, the initial guess, the auxiliary parameter \( h \) and the auxiliary function, the series (10) converges at \( q = 1 \), then we have
\[
u(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t).
\]
which must be one of the solutions of the original nonlinear problem. The governing equation can be derived from the zero-order deformation equation (8). We, define the vector in the following form
\[
\vec{u}_m = \{ u_0(x, t), u_1(x, t), \ldots u_n(x, t) \},
\]
On differentiating equation (8) \( m \)-times with respect to \( q \), then letting \( q = 0 \) and lastly dividing them by \( m! \), we obtain the \( m \)-th-order deformation equation.
\[
L \left[ u_m(x, t) - \chi_m u_{m-1}(x, t) \right] = h \ H(x, t) \ R_m(\vec{u}_{m-1}),
\]
Applying inverse Laplace transform, it gives us
\[
u_m(x, t) = \chi_m u_{m-1}(x, t) + h L^{-1}[H(x, t) R_m(\vec{u}_{m-1})]
\]
where
\[
R_m(\vec{u}_{m-1}) = \frac{1}{m-1!} \frac{\partial^{m-1} N[\varphi(x, t; q)]}{\partial q^{m-1}} \big|_{q = 0},
\]
and
\[
\chi_m = \begin{cases} 
0, & m \leq 1 \\
1, & m > 1
\end{cases}
\]
It should be noticed that \( u_m(x, t) \) for \( m \geq 1 \) is governed by the linear boundary conditions that come from original problem, which can be easily handled with the aid of symbolic computation software such as Maple, Mathematica and Matlab. If equation (3) admits unique solution, then this analytic technique will provide the unique solution. If Eq. (3) does not have a unique solution, the HATM will suggest a solution among many other possible solutions. The stability of the iterative methods like HATM for solving nonlinear problems has already been shown by many authors such as Atangana [4–6] and others.
3 Numerical Experiments

In this portion, we present the following generalized Hirota-Satsuma coupled KdV equation \([1, 3]\)

\[
\begin{align*}
    u_t &= \frac{1}{2} u_{xxx} - 3 u u_x + 3(vw)_x, \\
    v_t &= -v_{xxx} + 3 u v_x, \\
    w_t &= -w_{xxx} + 3 u w_x.
\end{align*}
\]  

(18)

(19)

(20)

On employing the Laplace transform method both of sides, subject to the initial condition, we have

\[
L[u(x, t)] = \frac{u(x, 0)}{s} + \frac{1}{s} L \left[ \frac{u_{xxx}}{2} - 3 u u_x + 3 (vw)_x \right],
\]

(21)

\[
L[v(x, t)] = \frac{v(x, 0)}{s} - \frac{1}{s} L [v_{xxx} + 3 u v_x],
\]

(22)

\[
L[w(x, t)] = \frac{w(x, 0)}{s} - \frac{1}{s} L [w_{xxx} + 3 u w_x].
\]

(23)

We now express the nonlinear operator as

\[
N_1[\varphi_1(x, t; q), \varphi_2(x, t; q), \varphi_3(x, t; q)] = L[\varphi_1(x, t; q)] - \frac{u(x, 0)}{s} - \frac{1}{s} L \left[ \frac{\varphi_{1,xxx}(x, t; q)}{2} \right]
\]

(24)

\[
+ \frac{1}{s} L \left[ 3 \varphi_1(x, t; q) \frac{\partial \varphi_1(x, t; q)}{\partial x} - 3 \frac{\partial}{\partial x} (\varphi_2(x, t; q) \varphi_3(x, t; q)) \right],
\]

\[
N_2[\varphi_1(x, t; q), \varphi_2(x, t; q), \varphi_3(x, t; q)] = L[\varphi_2(x, t; q)] - \frac{v(x, 0)}{s} + \frac{1}{s} L \left[ \varphi_{2,xxx}(x, t; q) \right]
\]

(25)

\[
- \frac{3}{s} L \left[ \varphi_1(x, t; q) \frac{\partial \varphi_2(x, t; q)}{\partial x} \right],
\]

\[
N_3[\varphi_1(x, t; q), \varphi_2(x, t; q), \varphi_3(x, t; q)] = L[\varphi_3(x, t; q)] - \frac{w(x, 0)}{s} + \frac{1}{s} L \left[ \varphi_{3,xxx}(x, t; q) \right]
\]

(26)

\[
- \frac{3}{s} L \left[ \varphi_1(x, t; q) \frac{\partial \varphi_3(x, t; q)}{\partial x} \right],
\]

Using the above definition, we construct the zero order deformation equation as

\[
(1 - q)L \left[ \varphi_i(x, t; q) - \Theta_{i,0}(x, t) \right] = q h_i H_i(x, t) N_i \left[ \varphi_1(x, t; q), \varphi_2(x, t; q), \varphi_3(x, t; q) \right], i = 1, 2, 3.
\]  

(27)

Obviously when \(q = 0\) and \(q = 1\),

\[
\varphi_1(x, t; 0) = \Theta_{1,0}(x, t) = u(x, 0), \varphi_1(x, t; 1) = u(x, t),
\]

(28)

\[
\varphi_2(x, t; 0) = \Theta_{2,0}(x, t) = v(x, 0), \varphi_2(x, t; 1) = v(x, t),
\]

(29)

\[
\varphi_3(x, t; 0) = \Theta_{3,0}(x, t) = w(x, 0), \varphi_3(x, t; 1) = w(x, t).
\]

(30)

Therefore as the embedding parameter increases form 0 to 1, \(\varphi_i(x, t; q)\) varies form the initial guess \(\Theta_{i,0}(x, t)\) to the solution \(u(x, t), v(x, t)\) and \(w(x, t)\) for \(i = 1, 2, 3\) respectively.

Expanding \(\varphi_i(x, t; q)\) in Taylor’s series with respect to \(q\) for \(i = 1, 2, 3\), one has

\[
\varphi_i(x, t; q) = \Theta_{i,0}(x, t) + \sum_{m=1}^{\infty} \Theta_{i,m}(x, t) q^m.
\]

(31)
If the auxiliary linear operator, the initial guess and the auxiliary parameter \( h_i \) are so properly selected, the above series converges at \( q = 1 \), one has

\[
\begin{align*}
\Theta_{i,m}(x, t) &= \frac{1}{m!} \frac{\partial^m \varphi_i(x, t; q)}{\partial q^m} \Big|_{q = 0} \\
\end{align*}
\]

(32)

which must be one of the solution of original nonlinear equation as proved by Liao [24–27].

Define the vectors

\[
\bar{\Theta}_{i,n} = \left\{ \bar{\Theta}_{i,0}(x, t), \bar{\Theta}_{i,1}(x, t), \ldots, \bar{\Theta}_{i,n}(x, t) \right\}, i = 1, 2, 3.
\]

(36)

Now the \( m \)-th-order deformation equation is given by

\[
L \left[ \Theta_{i,m}(x, t) - \chi_m \Theta_{i,m-1}(x, t) \right] = q \ h_i \ H_i(x, t) \ R_{i,m}(\bar{\Theta}_{i,m-1}, \bar{\Theta}_{i,m-2}, \bar{\Theta}_{i,m-1}), i = 1, 2, 3.
\]

(37)

where

\[
\begin{align*}
R_{1,m}(\bar{\Theta}_{1,m-1}, \bar{\Theta}_{2,m-1}, \bar{\Theta}_{3,m-1}) &= L[\Theta_{1,m-1}] - \frac{u(x,0)}{s} (1 - \chi_m) - \frac{1}{2s} L \left[ \frac{\partial^3 \Theta_{1,m-1}(x, t)}{\partial x^3} \right] \\
+ \frac{3}{s} L \left[ \sum_{n=0}^{m-1} \Theta_{1,m-1-n} \frac{\partial \Theta_{1,m-1-n}}{\partial x} \right] - \frac{3}{s} L \left[ \frac{\partial}{\partial x} \left( \sum_{n=0}^{m-1} \Theta_{2,m-1-n}(x, t) \Theta_{3,m-1-n}(x, t) \right) \right], \\
R_{2,m}(\bar{\Theta}_{1,m-1}, \bar{\Theta}_{2,m-1}, \bar{\Theta}_{3,m-1}) &= L[\Theta_{2,m-1}] - \frac{v(x,0)}{s} (1 - \chi_m) + \frac{1}{s} L \left[ \frac{\partial^3 \Theta_{2,m-1}(x, t)}{\partial x^3} \right] \\
- \frac{3}{s} L \left[ \sum_{n=0}^{m-1} \Theta_{1,m-1-n} \frac{\partial \Theta_{2,m-1-n}}{\partial x} \right], \\
R_{3,m}(\bar{\Theta}_{1,m-1}, \bar{\Theta}_{2,m-1}, \bar{\Theta}_{3,m-1}) &= L[\Theta_{3,m-1}] - \frac{w(x,0)}{s} (1 - \chi_m) + \frac{1}{s} L \left[ \frac{\partial^3 \Theta_{3,m-1}(x, t)}{\partial x^3} \right] \\
- \frac{3}{s} L \left[ \sum_{n=0}^{m-1} \Theta_{1,m-1-n} \frac{\partial \Theta_{3,m-1-n}}{\partial x} \right]. \\
\end{align*}
\]

(38)

(39)

(40)

Applying inverse Laplace transform, we get

\[
\Theta_{i,m}(x, t) = \chi_m \Theta_{i,m-1}(x, t) + h_i L^{-1} [H_i(x, t) \ R_{i,m}(\bar{\Theta}_{1,m-1}, \bar{\Theta}_{2,m-1}, \bar{\Theta}_{3,m-1})],
\]

(41)

where

\[
\chi_m = \begin{cases} 
0, & m \leq 1 \\
1, & m > 1 
\end{cases}
\]

By simplicity let us take \( h_1 = h_2 = h_3 = h \) and \( H_i(x, t) = 1 \).
3.1 First kind initial conditions

Now, we take the Hirota-Satsuma coupled KdV equation (18), with the following initial conditions \([1, 3, 11, 30]\)

\[
\begin{align*}
    u(x, 0) &= \frac{1}{3} \left( \beta - 2k^2 \right) + 2k^2 \tanh^2(kx), \\
    v(x, 0) &= -\frac{4k^2 c_0 (\beta + 2k^2)}{3c_1^2} + \frac{4k^2 (\beta + k^2) \tanh(kx)}{3c_1}, \\
    w(x, 0) &= c_0 + c_1 \tanh(kx).
\end{align*}
\]

and

\[
\begin{align*}
    \Theta_{1,0} &= \frac{1}{3} \left( \beta - 2k^2 \right) + 2k^2 \tanh^2(kx), \\
    \Theta_{2,0} &= -\frac{4k^2 c_0 (\beta + 2k^2)}{3c_1^2} + \frac{4k^2 (\beta + k^2) \tanh(kx)}{3c_1}, \\
    \Theta_{3,0} &= c_0 + c_1 \tanh(kx). \\
    \Theta_{1,1} &= -4h \beta k^3 t \sec h^2(kx) \tanh(kx), \\
    \Theta_{2,1} &= -\frac{4h \beta k^3 (\beta + k^2) \sec h^2(kx)}{3c_1}, \\
    \Theta_{3,1} &= -c_1 h \beta k t \sec h^2(kx). \\
    \Theta_{1,2} &= 2h^2 \beta^2 k^4 \left( \cosh^2(kx) - 3 \right) \sec h^4(kx) - 4h (1 + h) \beta k^3 t \sec \cosh^2(kx) \tanh(kx), \\
    \Theta_{2,2} &= -\frac{4h^2 \beta^2 k^4 (\beta + k^2) \sec h^2(kx) \tanh(kx)}{3c_1} - \frac{4h (1 + h) \beta k^3 (\beta + k^2) \sec h^2(kx)}{3c_1}, \\
    \Theta_{3,2} &= -c_1 h^2 \beta^2 k^2 t^2 \sec h^2(kx) \tanh(kx) - c_1 h (1 + h) \beta k t \sec h^2(kx). \\
    \Theta_{1,3} &= -\frac{8}{3} t^3 h^3 \beta^3 k^5 \sinh(kx) \left( \cosh^2(kx) - 3 \right) \sec h^5(kx) - 4h^2 (1 + h) t^2 \beta^3 k^5 \sec h^3(kx) \tanh(kx) + h^2 (1 + h)^3 \beta^2 t^2 k^3 \sec h^3(kx) \tanh^2(kx), \\
    \Theta_{2,3} &= \frac{4h^3 t^3 \beta^3 (\beta + k^2) \left( 2 \cosh^2(kx) - 3 \right) \sec h^4(kx) \tanh^2(kx)}{9c_1} - \frac{4\beta^2 h^2 (1 + h) k^2 t^2 \sec h^3(kx) \tanh(kx)}{9c_1} - \frac{4h^2 (1 + h)^2 \beta k^2 t^2 \sec h^2(kx)}{9c_1}.
\end{align*}
\]
\[
\Theta_{3, 3} = \frac{\hbar^3 k^3 \beta^3 t^3 \left( 2 \cosh^2(kx) - 3 \right) \sec h^4(kx) \tanh^2(kx)}{3} - \frac{\hbar^2 (1 + \hbar) \beta^2 k^2 t^2 \sec h^2(kx) \tanh^2(kx)}{3} - \frac{\hbar^2 (1 + \hbar)^2 \beta^2 t^2 \sec h^3(kx) \tanh(kx)}{3},
\]  

(56)

and so on.

By using Taylor’s series for these approximation with initial conditions the closed form for these equation at \( \hbar = -1 \) is give by

\[
u(x, t) = -\frac{4k^2 c_0 (\beta + 2k^2)}{3c_1^2} - \frac{4k^2 (\beta + k^2)}{3c_1} \tanh[k(x + \beta t)],
\]

(58)

\[
w(x, t) = c_0 + c_1 \tanh[k(x + \beta t)].
\]

(59)

**Table 1:** Comparison between the exact solution and the approximate solution of present method (HATM) for coupled KdV equation with the initial conditions (42)–(44) when \( c_0 = 1, c_1 = 1, k = 0.1, \beta = 1 \) and \( t = 2 \) at \( \hbar = -1 \).

<table>
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<th>( u_{\text{HATM}} )</th>
<th>( u_{\text{exact}} )</th>
<th>( v_{\text{HATM}} )</th>
<th>( v_{\text{exact}} )</th>
<th>( w_{\text{HATM}} )</th>
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**3.2 Second kind initial conditions**

To examine the accuracy and reliability of the present technique for the generalized Hirota-Satsuma coupled KdV equation, we can take the different initial values [1, 3, 11, 30]

\[
u(x, 0) = \frac{1}{3} \left( \beta - 8k^2 \right) + 4k^2 \tanh^2(kx),
\]

(60)

\[
v(x, 0) = \frac{-4 \left( 3k^4 c_0 - 2k \beta k^2 c_2 + 4k^4 c_2 \right)}{3c_2^2} + \frac{4k^2 \tanh^2(kx)}{c_2},
\]

(61)

\[
w(x, 0) = c_0 + c_2 \tanh^2(kx).
\]

(62)
where $k, c_0, c_2 \neq 0$ and $\beta$ are arbitrary constants.

In view of the solution procedure from equation (21) to equation (41) and using the initial conditions (60-62), we get the following approximations

\[
\begin{align*}
\Theta_{1,0} &= \frac{1}{3} \left( \beta - 8k^2 \right) + 4k^2 \tanh^2(kx), \\
\Theta_{2,0} &= -4 \left( 3k^4 c_0 - 2\beta k^2 c_2 + 4k^4 c_2 \right) + \frac{4k^2 \tanh^2(kx)}{c_2}, \\
\Theta_{3,0} &= c_0 + c_2 \tanh^2(kx).
\end{align*}
\]

\[
\begin{align*}
\Theta_{1,1} &= \frac{24 h k^2 t}{c_2} \left( k^2 \tanh^4(kx) \cosh^2(kx) - \tanh^4(kx) \cosh^2(kx) \right) \\
&- h \left( \left( 6 - 6k^2 - \beta \right) c_2 \tanh(kx) + \left( 6k^2 - \beta - 6 \right) c_2 \tanh(kx) \cosh(2kx) \right),
\end{align*}
\]

\[
\begin{align*}
\Theta_{2,1} &= -\frac{8 h k^3 t \beta \sec h^2(kx) \tanh(kx)}{c_2}, \\
\Theta_{3,1} &= -2 h k \beta c_2 \sec h^2(kx) \tanh(kx).
\end{align*}
\]

\[
\begin{align*}
\Theta_{1,2} &= \frac{h^2 k^4 t^2 \sec h^6(kx)}{4c_2} \left[ \left( -24k^4 \cosh^2(kx) + 24k^2 \cosh^2(kx)c_0 \right) \left( \cosh(2kx) + \cosh(4kx) \right) \\
&+ \left( 17\beta^2 \cosh(2kx) + 372k^2 \cosh(2kx) - 372k^4 \cosh(2kx) + 96k^4 \cosh(4kx) \\
&- 96k^2 \cosh(4kx) + 12k^2 \cosh(6kx) - \beta^2 \cosh(6kx) \right) c_2 \right] + \frac{h(1 + h)k^3 t \sec h^5(kx)}{4c_2} \left[ c_0 \\
&\left( 124k^2(k - 1) \cosh(2kx) - \beta^2 \cosh(2kx) \right) + \left( 12k(k - 1) \cosh(4kx) + 9\beta \cosh(4kx) \right) c_2 \right], \\
\Theta_{2,2} &= -\frac{h^2}{4c_2} \left[ \left( c_0 k^4 t^2 \sec h^2(kx) \right) \left( 120k^4 \cosh(2kx) - 2016k^4 \cosh(2kx) \right) \\
&+ \left( 17\beta^2 \cosh(2kx) \right) c_2 \right] + \frac{h(1 + h)}{4c_2} \left[ c_0 k^2 t \sec h^2(kx) \tanh(kx) + \left( 128k^2 \cosh(2kx) \\
&- 128k^4 \cosh(2kx) + 2\beta^2 \cosh(2kx) - 864k^2 \cosh(4kx) + 864k^4 \cosh(4kx) \\
&- 6\beta \cosh(4kx) \right) c_2 \right], \\
\Theta_{3,2} &= \frac{h^2 k^4 t^2 \sec h^6(kx)}{4c_2} \left[ \left( -288k^4 + 288k^2 \right) c_0 \sinh^2(2kx) - c_2 \left( 864k^4 - 864k^2 + 8\beta^2 \right) \\
&+ \left( 1152k^2 \cosh(2kx) - 1152k^4 \cosh(2kx) - 9\beta^2 \cosh(2kx) + 288k^2 \cosh(4kx) \\
&- 288k^4 \cosh(4kx) - \beta^2 \cosh(6kx) \right) c_2 \right] + \frac{h(1 + h)k^3 t \sec h^5(kx)}{4c_2} \left[ c_0 \left( 128k(k - 1) \cosh(2kx) \right) \\
&\left( -\beta^2 \cosh(2kx) \right) + \left( 12k(k - 1) \cosh(4kx) + 9\beta \cosh(4kx) \right) c_2 \right].
\end{align*}
\]

Making use of Taylor’s series for these approximation with initial conditions the closed form for these equation at $h = -1$ is give by

\[
u(x, t) = \frac{1}{3} \left( \beta - 8k^2 \right) + 4k^2 \tanh^2(kx + k\beta t),
\]
Table 2: Comparison between the exact solution and the approximate solution of present method (HATM) for coupled KdV equation with the initial conditions (60)-(62) when $c_0 = 1, c_1 = 1, k = 0.1, \beta = 1$ and $t = 1$ at $\hbar = -1$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$u_{HATM}$</th>
<th>$u_{exact}$</th>
<th>$v_{HATM}$</th>
<th>$v_{exact}$</th>
<th>$w_{HATM}$</th>
<th>$w_{exact}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-50</td>
<td>0.3466527</td>
<td>0.346658</td>
<td>0.06572451</td>
<td>0.0657245</td>
<td>1.99978</td>
<td>1.99978</td>
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<tr>
<td>-40</td>
<td>0.3465251</td>
<td>0.346601</td>
<td>0.06566784</td>
<td>0.0656678</td>
<td>1.99836</td>
<td>1.99836</td>
</tr>
<tr>
<td>-30</td>
<td>0.3453124</td>
<td>0.346185</td>
<td>0.0652512</td>
<td>0.0652518</td>
<td>1.98797</td>
<td>1.98796</td>
</tr>
<tr>
<td>-20</td>
<td>0.3430654</td>
<td>0.343242</td>
<td>0.06231247</td>
<td>0.0623089</td>
<td>1.91451</td>
<td>1.91439</td>
</tr>
<tr>
<td>-10</td>
<td>0.3204578</td>
<td>0.327190</td>
<td>0.04630012</td>
<td>0.0462566</td>
<td>1.51483</td>
<td>1.51308</td>
</tr>
<tr>
<td>0</td>
<td>0.3071245</td>
<td>0.307064</td>
<td>0.02613213</td>
<td>0.0261307</td>
<td>1.00996</td>
<td>1.00993</td>
</tr>
<tr>
<td>10</td>
<td>0.3405664</td>
<td>0.332299</td>
<td>0.00224021</td>
<td>0.0513653</td>
<td>1.64217</td>
<td>1.64080</td>
</tr>
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<td>20</td>
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<td>0.344338</td>
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<td>1.94178</td>
</tr>
<tr>
<td>30</td>
<td>0.3465342</td>
<td>0.346434</td>
<td>0.06540994</td>
<td>0.0654099</td>
<td>1.99192</td>
<td>1.99192</td>
</tr>
<tr>
<td>40</td>
<td>0.3466493</td>
<td>0.346623</td>
<td>0.06568942</td>
<td>0.0656894</td>
<td>1.99890</td>
<td>1.99890</td>
</tr>
<tr>
<td>50</td>
<td>0.3466642</td>
<td>0.346661</td>
<td>0.06572746</td>
<td>0.0657274</td>
<td>1.99985</td>
<td>1.99985</td>
</tr>
</tbody>
</table>

\[ v(x, t) = \frac{-4(3k^4c_0 - 2\beta k^2c_2 + 4k^4c_2)}{3c_2^2} + \frac{4k^2\tanh^2(kx + k\beta t)}{c_2}, \]  

\[ w(x, t) = c_0 + c_2 \tanh^2(kx + k\beta t). \]  

It is necessary to make test for these series solution obtained through HATM, by making a comparison with the exact solution of the coupled KdV equation (1). In the Maple program, we put $c_0 = 1, c_1 = 1, k = 0.1, \beta = 1, t = 2$ and take only three terms of the series. When the series and the exact solution in Table 1, it seems that the series solutions are confirmed with the exact solution. We show in Figs. 1–9, 17 for second type initial conditions and Figs. 10–16 for first type initial conditions, the exact and numerical solution of system (1); the solution are of bell type for all $u(x, t), v(x, t)$ and $w(x, t)$ for second type initial conditions. In the figures there are no visible differences in the two solutions viz. exact and approximate (HATM). The HATM gives an other soliton solution of system (1) of kink type for $u(x, t), v(x, t)$ and $w(x, t)$ for first type initial conditions. Also in Table 2 we make a comparison between exact solution HATM at $t = 1$, we find that the HATM solution of system (1) has very less error represent in Table 3 and Table 4 for first kind and second kind initial conditions.

Figure 1: Exact solution graph for $u(x, t)$ for $k = 0.1, \beta = 1$ for $(x, t) \in [-100, 100), [0, 40)$. 

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Figure 2: Approximate solution graph of third order HATM for $u(x, t)$ for $k = 0.1, \beta = 1$ for $(x, t) \in [-100, 100), [0, 40)$ by HATM at $\hbar = -1$.

Figure 3: Exact solution graph for $v(x, t)$ for $k = 0.1, \beta = 1$ for $(x, t) \in [-100, 100), [0, 40)$.

Figure 4: Approximate solution graph of third order HATM for $v(x, t)$ for $k = 0.1, \beta = 1$ for $(x, t) \in [-100, 100), [0, 40)$ by HATM at $\hbar = -1$. 
Figure 5: Exact solution graph for \( w(x, t) \) for \( k = 0.1, \beta = 1 \) for \( (x, t) \in [-100, 100), [0, 40) \).

Figure 6: Approximate solution graph of third order HATM for \( w(x, t) \) for \( k = 0.1, \beta = 1 \) for \( (x, t) \in [-100, 100), [0, 40) \) by HATM at \( h = -1 \).

Figure 7: Absolute error for the fifth-order approximation by HATM for \( u(x, t) \) at \( h = -1 \).
Figure 8: Absolute error for the fifth-order approximation by HATM for $v(x, t)$ at $\hbar = -1$.

Figure 9: Absolute error for the fifth-order approximation by HATM for $w(x, t)$ at $\hbar = -1$.

Figure 10: Exact solution graph for $u(x, t)$ for $k = 0.1, \beta = 1$ for $(x, t) \in [-100, 100), [0, 40)$.
Figure 11: Approximate solution graph of third order HATM for $u(x, t)$ for $k = 0.1, \beta = 1$ for $(x, t) \in [-100, 100), [0, 40)$ by HATM at $\hbar = -1$.

Figure 12: Exact solution graph for $v(x, t)$ for $k = 0.1, \beta = 1$ for $(x, t) \in [-100, 100), [0, 40)$

Figure 13: Approximate solution graph of third order HATM for $v(x, t)$ for $k = 0.1, \beta = 1$ for $(x, t) \in [-100, 100), [0, 40)$ by HATM at $\hbar = -1$. 
Figure 14: Exact solution graph for $w(x, t)$ for $k = 0.1$, $\beta = 1$ for $(x, t) \in [-100, 100), [0, 40)$.

Figure 15: Approximate solution graph of third order HATM for $v(x, t)$ for $k = 0.1$, $\beta = 1$ for $(x, t) \in [-100, 100), [0, 40)$ by HATM at $\hbar = -1$.

Figure 16: Kink type solution for first kind initial conditions at $\beta = k = c_0 = c_1 = 1$. 
Figure 17: Bell-type solution for second kind initial conditions at $\beta = k = c_0 = c_1 = 1$.

Table 3: The absolute errors for $u(x, t)$, $v(x, t)$ and $w(x, t)$ for the first three approximation of HATM in comparison with the analytical solution when $c_0 = \frac{3}{2}, c_1 = \frac{1}{10}$, $\beta = k = \frac{3}{2}$ for the solitary wave solution with the first kind of initial conditions (42)-(44), respectively in the interval $0 \leq x \leq 1$, $t \geq 0$

| $x$  | $t$  | $|u - \Theta_{1,3}|$ | $|v - \Theta_{2,3}|$ | $|u - \Theta_{3,3}|$ |
|------|------|---------------------|---------------------|---------------------|
| 0.1  | 0.2  | 5.45654e-10        | 2.46789e-10        | 1.96470e-10        |
|      | 0.4  | 7.10235e-08        | 5.74561e-08        | 1.00452e-08        |
|      | 0.6  | 7.93486e-07        | 3.69842e-07        | 2.88941e-07        |
|      | 0.8  | 6.11452e-06        | 1.07965e-06        | 8.96452e-06        |
|      | 1.0  | 7.12358e-05        | 7.36547e-05        | 7.56486e-08        |
| 0.3  | 0.2  | 1.56822e-10        | 3.65374e-10        | 7.68458e-10        |
|      | 0.4  | 7.18169e-08        | 6.45371e-08        | 2.37194e-08        |
|      | 0.6  | 1.15235e-06        | 4.65756e-06        | 4.15975e-06        |
|      | 0.8  | 1.70723e-05        | 1.03572e-05        | 6.97967e-06        |
|      | 1.0  | 3.15940e-05        | 8.37765e-05        | 7.94564e-05        |
| 0.5  | 0.2  | 2.48965e-09        | 1.64540e-09        | 2.75964e-09        |
|      | 0.4  | 6.22001e-07        | 5.67321e-07        | 9.87156e-07        |
|      | 0.6  | 2.34562e-06        | 3.65458e-06        | 9.00121e-06        |
|      | 0.8  | 1.62478e-05        | 6.11672e-05        | 2.74136e-05        |
|      | 1.0  | 5.79560e-05        | 7.53614e-05        | 2.86375e-05        |

4 Conclusions

In this work, we have employed the HATM for finding solitary-wave solution for generalized Hirota-Satsuma coupled KdV equation with two different initial conditions. This method has been applied directly without using complicated Adomian’s polynomials and He’s polynomials. The outcomes of the test examples demonstrate that the HATM results are in a very good agreement with ADM, HPM, HAM and DTM. It is worth pointing out that the HATM has convergence for the solutions, actually, the accuracy of the series solution increased when the number of terms in the series solution in increased. We find that HATM is easier to apply as it is the elegant combination of two powerful methods, HAM and Laplace transform algorithm. So, it has wider application in literature. All the numerical computation has been done by Maple-13 software package. In conclusion, we can say that the HATM can also be used to solve more complex problems arising in fluid mechanics such as Navier-Stokes equations, boundary-layer problems and many more nonlinear equations in science and engineering.
Table 4: The absolute errors for \(u(x, t), v(x, t)\) and \(w(x, t)\) for the first three approximation of HATM in comparison with the analytical solution when \(c_0 = \frac{3}{4}, c_1 = \frac{1}{10}, \beta = k = \frac{1}{4}\) for the solitary wave solution with the second kind of initial conditions (60)-(62), respectively in the interval \(0 \leq x \leq 1, t \geq 0\)

| \(x\) | \(t\) | \(|u - \Theta_{1,3}|\) | \(|v - \Theta_{2,3}|\) | \(|u - \Theta_{3,3}|\) |
|-----|-----|-----------------|-----------------|-----------------|
| 0.1 | 0.2 | 4.83456e-12     | 8.64523e-12     | 3.47895e-12     |
| 0.4 | 0.2 | 6.57046e-11     | 4.68489e-11     | 1.00452e-11     |
| 0.6 | 0.2 | 3.22579e-09     | 3.69842e-09     | 8.96452e-09     |
| 0.8 | 0.2 | 4.77656e-08     | 1.07965e-08     | 2.01904e-08     |
| 1.0 | 0.2 | 9.31763e-08     | 7.42835e-08     | 2.01904e-08     |
| 0.3 | 0.2 | 6.01005e-12     | 8.49036e-12     | 3.78958e-12     |
| 0.4 | 0.2 | 2.79948e-11     | 6.03042e-11     | 1.69871e-11     |
| 0.6 | 0.2 | 5.53461e-09     | 3.78794e-09     | 7.98455e-09     |
| 0.8 | 0.2 | 6.54789e-08     | 9.77282e-08     | 8.94564e-08     |
| 1.0 | 0.2 | 1.65478e-08     | 8.02947e-08     | 7.94564e-08     |
| 0.5 | 0.2 | 5.32816e-12     | 1.61225e-12     | 7.69812e-12     |
| 0.4 | 0.2 | 1.63401e-11     | 4.90056e-11     | 6.75155e-11     |
| 0.6 | 0.2 | 1.89215e-09     | 3.00015e-09     | 6.70461e-09     |
| 0.8 | 0.2 | 2.78921e-08     | 6.78954e-08     | 1.03512e-08     |
| 1.0 | 0.2 | 1.79371e-08     | 1.28961e-08     | 1.64123e-08     |

References

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