The Modified (G'/G)-Expansion Method for Nonlinear Evolution Equations

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A modified (G'/G)-expansion method is proposed to construct exact solutions of nonlinear evolution equations. To illustrate the validity and advantages of the method, the (3+1)-dimensional potential Yu-Toda-Sasa-Fukuyama (YTSF) equation is considered and more general travelling wave solutions are obtained. Some of the obtained solutions, namely hyperbolic function solutions, trigonometric function solutions, and rational solutions contain an explicit linear function of the variables in the considered equation. It is shown that the proposed method provides a more powerful mathematical tool for solving nonlinear evolution equations in mathematical physics.

Key words: Nonlinear Evolution Equations; Modified (G'/G)-Expansion Method; Hyperbolic Function Solutions; Trigonometric Function Solutions; Rational Solutions.

1. Introduction

Nonlinear evolution equations (NLEEs) are often presented to describe the motion of isolated waves, localized in a small part of space, in many fields such as hydrodynamic, plasma physics, and nonlinear optic. Seeking exact solutions of NLEEs plays an important role in the study of these nonlinear physical phenomena. In the past several decades, many effective methods for obtaining exact solutions of NLEEs have been presented, such as the inverse scattering method [1], Hirota's bilinear method [2], Bäcklund transformation [3], Painlevé expansion [4], sine-cosine method [5], homogeneous balance method [6], tanh-function method [7 – 9], Jacobi elliptic function expansion method [10 – 12], F-expansion method [13 – 15], auxiliary equation method [16 – 18], rational function expansion method [19 – 21], and exp-function method [22 – 24].

With the development of computer science, recently, directly searching for exact solutions of NLEEs has attracted much attention. This is due to the availability of symbolic computation systems like Mathematica or Maple which enable us to perform the complex and tedious computation on computers. Wang et al. [25] introduced a new direct method called the (G'/G)-expansion method to look for travelling wave solutions of NLEEs. The (G'/G)-expansion method is based on the assumptions that the travelling wave solutions can be expressed by a polynomial in (G'/G), and that $G = G(\xi)$ satisfies a second-order linear ordinary differential equation (LODE):

$$G'' + \lambda G' + \mu G = 0, \tag{1}$$

where $G' = \frac{\mathrm{d}G(\xi)}{\mathrm{d}\xi}$, $G'' = \frac{\mathrm{d}^2G(\xi)}{\mathrm{d}\xi^2}$, $\xi = x - Vt$, V is a constant. The degree of the polynomial can be determined by considering the homogeneous balance between the highest-order derivative and the nonlinear terms appearing in the given NLEE. The coefficients of the polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the method. It was shown that the method present a wider applicability for handling many kinds of NLEEs [26 – 32].

The present paper is motivated by the desire to propose a modified (G'/G)-expansion method for constructing more general exact solutions of NLEEs. In order to illustrate the validity and advantages of the proposed method, we would like to employ it to solve the (3+1)-dimensional potential Yu-Toda-Sasa-Fukuyama (YTSF) equation [33].

The rest of this paper is organized as follows. In Section 2, we describe the modified (G'/G)-expansion method. In Section 3, we use the modified method to solve the (3+1)-dimensional potential YTSF equation. In Section 4, some conclusions are given.

2. Basic Idea of the Modified (G'/G)-Expansion Method

For a given nonlinear PDE, say in four variables x, y, z, and t,

$$P(x,y,z,t,u,u_x,u_y,u_z,u_t,\ldots,)=0, \quad (2)$$

where $u=u(x,y,z,t)$, we use the following transformation:

$$u=u(\xi),\quad \xi=ax+by+cz-\omega t, \quad (3)$$

where a, b, c, and ω are constants. Then (2) is reduced into an ODE:

$$Q(x,y,z,t,u^{(r)},u^{(r+1)},\ldots)=0, \quad (4)$$

where $u^{(r)}=\frac{\mathrm{d}^r u}{\mathrm{d}\xi^r}$, $u^{(r+1)}=\frac{\mathrm{d}^{r+1}u}{\mathrm{d}\xi^{r+1}}$, $r\geqslant 0$ and r is the least order of derivatives in the equation. To keep the solution process as simple as possible, the function Q should not be a total ξ-derivative of another function. Otherwise, taking integration with respect to ξ further reduces the transformed equation [21].

We further introduce

$$u^{(r)}(\xi)=v(\xi)=\sum_{i=1}^{m}\alpha_i\left(\frac{G'}{G}\right)^i+\alpha_0,\quad \alpha_m\neq 0, \quad (5)$$

where $G=G(\xi)$ satisfies (1), while α_0, α_i ($i=1,2,\cdots,m$) are constants to be determined later. Then a direct computation gives

$$u^{(r+1)}(\xi)=v'(\xi)=-\sum_{i=1}^{m}i\alpha_i\left[\left(\frac{G'}{G}\right)^{i+1}+\lambda\left(\frac{G'}{G}\right)^{i}+\mu\left(\frac{G'}{G}\right)^{i-1}\right], \quad (6)$$

$$\begin{aligned}u^{(r+2)}(\xi)=v''(\xi)=\sum_{i=1}^{m}i\alpha_i\Bigg[&(i+1)\left(\frac{G'}{G}\right)^{i+2}\\&+(2i+1)\lambda\left(\frac{G'}{G}\right)^{i+1}+i(\lambda^2+2\mu)\left(\frac{G'}{G}\right)^{i}\\&+(2i-1)\lambda\mu\left(\frac{G'}{G}\right)^{i-1}+(i-1)\mu^2\left(\frac{G'}{G}\right)^{i-2}\Bigg],\end{aligned} \quad (7)$$

and so on, here the prime denotes the derivative with respective to ξ.

To determine u explicitly, we take the following four steps:

Step 1. Determine the integer m by substituting (5) along with (1) into (4), and balancing the highest-order nonlinear term(s) and the highest-order partial derivative.

Step 2. Substitute the value of m determined in Step 1 along with (1) into (4) and collect all terms with the same order of (G'/G) together, thus the left-hand side of (4) is converted into a polynomial in (G'/G). Then set each coefficient of this polynomial to zero to derive a set of algebraic equations for a, b, c, ω, α_0, and α_i.

Step 3. Solve the system of algebraic equations obtained in Step 2 for a, b, c, ω, α_0, and α_i by use of Mathematica.

Step 4. Use the results obtained in above steps to derive a series of fundamental solutions $v(\xi)$ of (4) depending on (G'/G). Since the solutions of (1) have been well known for us, we can obtain exact solutions of (2) by integrating each of the obtained fundamental solutions $v(\xi)$ with respect to ξ, r times:

$$\begin{aligned}u=u(\xi)=&\int^{\xi}\int^{\xi_r}\cdots\int^{\xi_2}v(\xi_1)\mathrm{d}\xi_1\cdots\mathrm{d}\xi_{r-1}\mathrm{d}\xi_r\\&+\sum_{j=1}^{r}d_j\xi^{r-j},\end{aligned} \quad (8)$$

where d_j are arbitrary constants.

Remark 1. It can be easily found that when $r=0$, $u(\xi)=v(\xi)$ then (5) becomes Wang et al.'s ansatz solution (2.4) in [25]. Under this circumstance, the method proposed in the present paper is the same as that of [25]. However, when $r\geqslant 1$, solution (8) maybe contain an explicit polynomial in ξ even if it is simplified. Such a solution cannot be obtained by the method [25], see the next section for more details. Therefore, the proposed method can be seen as a modified version or a note of Wang et al.'s method [25].

3. Application to the (3+1)-Dimensional Potential YTSF Equation

Let us consider in this section the (3+1)-dimensional potential YTSF equation [33],

$$-u_{xt}+u_{xxxz}+4u_xu_{xz}+2u_{xx}u_z+3u_{yy}=0, \quad (9)$$

which can be derived from the (3+1)-dimensional YTSF equation

$$\begin{aligned}&(-4v_t+\Phi(v)v_z)_x+3v_{yy}=0,\\&\Phi(v)=\partial_x^2+4v+2v_x\partial_x^{-1},\end{aligned} \quad (10)$$

by using the potential $v = u_x$. It was Yu et al. [34] who extended the (2+1)-dimensional Bogoyavlenskii-Schiff equation [35],

$$v_t + \Phi(v)v_z = 0, \quad \Phi(v) = \partial_x^2 + 4v + 2v_x\partial_x^{-1}, \quad (11)$$

to the (3+1)-dimensional nonlinear evolution equation in the form of (9).

Using the transformation (3), we reduce (9) into an ODE of the form

$$a^3cu^{(4)} + 6a^2cu'u'' + (4a\omega + 3b^2)u'' = 0. \quad (12)$$

Integrating (12) once with respect to ξ and setting the integration constant to zero yields

$$a^3cu^{(3)} + 3a^2c(u')^2 + (4a\omega + 3b^2)u' = 0. \quad (13)$$

Further setting $r = 1$ and $u' = v$, we have

$$a^3cv'' + 3a^2cv^2 + (4a\omega + 3b^2)v = 0. \quad (14)$$

According to Step 1, we get $m+2 = 2m$, hence $m = 2$. We then suppose that (14) has the formal solution

$$v = \alpha_2\left(\frac{G'}{G}\right)^2 + \alpha_1\left(\frac{G'}{G}\right) + \alpha_0, \quad \alpha_2 \neq 0. \quad (15)$$

Substituting (15) along with (1) into (14) and collecting all terms with the same order of (G'/G) together, the left-hand side of (14) is converted into a polynomial in (G'/G). Setting each coefficient of the polynomial to zero, we derive a set of algebraic equations for a, b, c, ω, α_0, α_1, and α_2 as follows:

$$\left(\frac{G'}{G}\right)^0 : 3b^2\alpha_0 + 4a\omega\alpha_0 + 3a^2c\alpha_0^2 + a^3c\alpha_1\lambda\mu + 2a^3c\alpha_2\mu^2 = 0,$$

$$\left(\frac{G'}{G}\right)^1 : 3b^2\alpha_1 + 4a\omega\alpha_1 + 6a^2c\alpha_0\alpha_1 + a^3c\alpha_1\lambda^2 + 2a^3c\alpha_1\mu + 6a^3c\alpha_2\lambda\mu = 0,$$

$$\left(\frac{G'}{G}\right)^2 : 3a^2c\alpha_1^2 + 3b^2\alpha_2 + 4a\omega\alpha_2 + 6a^2c\alpha_0\alpha_2 + 3a^3c\alpha_1\lambda + 4a^3c\alpha_2\lambda^2 + 8a^3c\alpha_2\mu = 0,$$

$$\left(\frac{G'}{G}\right)^3 : 2a^3c\alpha_1 + 6a^2c\alpha_1\alpha_2 + 10a^3c\alpha_2\lambda = 0,$$

$$\left(\frac{G'}{G}\right)^4 : 6a^3c\alpha_2 + 3a^2c\alpha_2^2 = 0.$$

Solving this set of algebraic equations by the use of Mathematica, we have

$$\alpha_2 = -2a, \quad \alpha_1 = -2a\lambda, \quad \alpha_0 = -\frac{1}{3}a(\lambda^2 + 2\mu), \quad \omega = \frac{a^3c(\lambda^2 - 4\mu) - 3b^2}{4a} \quad (16)$$

and

$$\alpha_2 = -2a, \quad \alpha_1 = -2a\lambda, \quad \alpha_0 = -2a\mu, \quad \omega = -\frac{a^3c(\lambda^2 - 4\mu) + 3b^2}{4a}. \quad (17)$$

We, therefore, obtain

$$v = -2a\left(\frac{G'}{G}\right)^2 - 2a\lambda\left(\frac{G'}{G}\right) - \frac{1}{3}a(\lambda^2 + 2\mu), \quad \omega = \frac{a^3c(\lambda^2 - 4\mu) - 3b^2}{4a} \quad (18)$$

and

$$v = -2a\left(\frac{G'}{G}\right)^2 - 2a\lambda\left(\frac{G'}{G}\right) - 2a\mu, \quad \omega = -\frac{a^3c(\lambda^2 - 4\mu) + 3b^2}{4a}. \quad (19)$$

Substituting the general solutions of (1) into (18) and (19), respectively, and using (8), we obtain three types of travelling wave solutions of (9). When $\lambda^2 - 4\mu > 0$, we obtain a hyperbolic function solution:

$$u = -\frac{1}{2}a(\lambda^2 - 4\mu)\int^{\xi}\left(\frac{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_1\right) + C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_1\right)}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_1\right) + C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_1\right)}\right)^2 \mathrm{d}\xi_1 + \frac{1}{6}a(\lambda^2 - 4\mu)\xi + d_1, \quad (20)$$

where $\xi = ax + by + cz - \frac{a^3c(\lambda^2-4\mu)-3b^2}{4a}t$, C_1, C_2, and d_1 are arbitrary constants, and

$$u=-\frac{1}{2}a(\lambda^2-4\mu)\int^{\xi}\left(\frac{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_1\right)+C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_1\right)}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_1\right)+C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_1\right)}\right)^2\mathrm{d}\xi_1+\frac{1}{2}a(\lambda^2-4\mu)\xi+d_1, \quad (21)$$

where $\xi=ax+by+cz+\frac{a^3c(\lambda^2-4\mu)+3b^2}{4a}t$, C_1, C_2, and d_1 are arbitrary constants. When $\lambda^2-4\mu<0$, we obtain a trigonometric function solution:

$$u=-\frac{1}{2}a(4\mu-\lambda^2)\int^{\xi}\left(\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_1\right)+C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_1\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_1\right)+C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_1\right)}\right)^2\mathrm{d}\xi_1+\frac{1}{6}a(\lambda^2-4\mu)\xi+d_1, \quad (22)$$

where $\xi=ax+by+cz-\frac{a^3c(\lambda^2-4\mu)-3b^2}{4a}t$, C_1, C_2, and d_1 are constants, and

$$u=-\frac{1}{2}a(4\mu-\lambda^2)\int^{\xi}\left(\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_1\right)+C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_1\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_1\right)+C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_1\right)}\right)^2\mathrm{d}\xi_1+\frac{1}{2}a(\lambda^2-4\mu)\xi+d_1, \quad (23)$$

where $\xi=ax+by+cz+\frac{a^3c(\lambda^2-4\mu)+3b^2}{4a}t$, C_1, C_2, and d_1 are constants.

When $\lambda^2-4\mu=0$, we obtain a rational solution:

$$u=\frac{2aC_2}{C_1+C_2\xi}+d_1, \quad (24)$$

where $\xi=ax+by+cz+\frac{3b^2}{4a}t$, C_1, C_2, and d_1 are arbitrary constants.

If we use Wang et al.'s method [25] to solve the potential YTSF equation (9), usually we will balance $u^{(3)}$ and $(u')^2$ in (13) and obtain $m+3=2(m+1)$, namely $m=1$. We then suppose that (13) has solution in the following form:

$$u=\alpha_1\left(\frac{G'}{G}\right)+\alpha_0, \quad \alpha_1\neq 0. \quad (25)$$

Substituting (25) along with (1) into (13) and collecting all terms with the same order of (G'/G) together, the left-hand side of (13) is converted into a polynomial in (G'/G). Setting each coefficient of the polynomial to zero, we derive a set of algebraic equations for a, b, c, ω, α_0, and α_1 as follows:

$$\left(\frac{G'}{G}\right)^0: \quad -3b^2\alpha_1\mu-4a\omega\alpha_1\mu-a^3c\alpha_1\lambda^2\mu-2a^3c\alpha_1\mu^2+3a^2c\alpha_1^2\mu^2=0,$$

$$\left(\frac{G'}{G}\right)^1: \quad -3b^2\alpha_1\lambda-4a\omega\alpha_1\lambda-a^3c\alpha_1\lambda^3-8a^3c\alpha_1\lambda\mu+6a^2c\alpha_1^2\lambda\mu=0,$$

$$\left(\frac{G'}{G}\right)^2: \quad -3b^2\alpha_1-4a\omega\alpha_1-7a^3c\alpha_1\lambda^2+3a^2c\alpha_1^2\lambda^2-8a^3c\alpha_1\mu+6a^2c\alpha_1^2\mu=0,$$

$$\left(\frac{G'}{G}\right)^3: \quad -12a^3c\alpha_1\lambda+6a^2c\alpha_1^2\lambda=0,$$

$$\left(\frac{G'}{G}\right)^4: \quad -6a^3c\alpha_1+3a^2c\alpha_1^2=0.$$

Solving the set of algebraic equations by the use of Mathematica, we have

$$\alpha_1=2a, \quad \alpha_0=\alpha_0, \quad \omega=-\frac{a^3c(\lambda^2-4\mu)+3b^2}{4a}. \quad (26)$$

We, therefore, obtain the following formal hyperbolic function solution, trigonometric function solution, and rational solution of (9):

$$u = a\sqrt{\lambda^2-4\mu}\left(\frac{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}\right) - a\lambda + \alpha_0, \quad \lambda^2-4\mu>0, \tag{27}$$

where $\xi = ax+by+cz+\frac{a^3c(\lambda^2-4\mu)+3b^2}{4a}t$, C_1, C_2, and α_0 are arbitrary constants;

$$u = a\sqrt{4\mu-\lambda^2}\left(\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}\right) - a\lambda + \alpha_0, \quad \lambda^2-4\mu<0, \tag{28}$$

where $\xi = ax+by+cz-\frac{a^3c(\lambda^2-4\mu)-3b^2}{4a}t$, C_1, C_2, and α_0 are constants;

$$u = \frac{2aC_2}{C_1+C_2\xi} - a\lambda + \alpha_0, \quad \lambda^2-4\mu=0, \tag{29}$$

where $\xi = ax+by+cz+\frac{3b^2}{4a}t$, C_1, C_2, and α_0 are arbitrary constants.

Obviously, Solution (29) is equivalent to Solution (24). In what follows, we would like to compare Solutions (20) – (23) and Solutions (27) and (28).

When $\lambda^2-4\mu>0$, $C_1C_2>0$, (20) and (21) can be respectively simplified as

$$u = a\sqrt{\lambda^2-4\mu}\cdot\tanh\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+\frac{1}{2}\ln\left(\frac{C_2}{C_1}\right)\right] - \frac{1}{3}a(\lambda^2-4\mu)\xi + d_1, \tag{30}$$

where $\xi = ax+by+cz-\frac{a^3c(\lambda^2-4\mu)-3b^2}{4a}t$, and

$$u = a\sqrt{\lambda^2-4\mu}\cdot\tanh\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+\frac{1}{2}\ln\left(\frac{C_2}{C_1}\right)\right] + d_1, \tag{31}$$

where $\xi = ax+by+cz+\frac{a^3c(\lambda^2-4\mu)+3b^2}{4a}t$.

Solution (27) can be simplified as

$$u = a\sqrt{\lambda^2-4\mu}\cdot\tanh\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+\frac{1}{2}\ln\left(-\frac{C_2}{C_1}\right)\right] - a\lambda + \alpha_0, \tag{32}$$

where $\xi = ax+by+cz+\frac{a^3c(\lambda^2-4\mu)+3b^2}{4a}t$. It is easy to see that (32) is equivalent to (31), however, (30) cannot be obtained from (27).

When $\lambda^2-4\mu>0$, $C_1C_2<0$, (20) and (21) can be respectively simplified as

$$u = a\sqrt{\lambda^2-4\mu}\coth\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+\frac{1}{2}\ln\left(-\frac{C_2}{C_1}\right)\right] - \frac{1}{3}a(\lambda^2-4\mu)\xi + d_1, \tag{33}$$

where $\xi = ax+by+cz-\frac{a^3c(\lambda^2-4\mu)-3b^2}{4a}t$, and

$$u = a\sqrt{\lambda^2-4\mu}\coth\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+\frac{1}{2}\ln\left(-\frac{C_2}{2C_1}\right)\right] + d_1, \tag{34}$$

where $\xi = ax+by+cz+\frac{a^3c(\lambda^2-4\mu)+3b^2}{4a}t$.

Solution (27) can be simplified as

$$u = a\sqrt{\lambda^2-4\mu}\coth\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+\frac{1}{2}\ln\left(-\frac{C_2}{C_1}\right)\right] - a\lambda + \alpha_0, \tag{35}$$

where $\xi = ax+by+cz+\frac{a^3c(\lambda^2-4\mu)+3b^2}{4a}t$. It is obvious that (35) is equivalent to (34), however, (33) cannot be obtained from (27).

When $\lambda^2-4\mu<0$, from (22) we have

$$u = -a\sqrt{4\mu-\lambda^2}\tan\left[\frac{\sqrt{4\mu-\lambda^2}}{2}\xi-\arctan\left(\frac{C_2}{C_1}\right)\right] - \frac{1}{3}a(\lambda^2-4\mu)\xi + d_1 \tag{36}$$

or

$$u = a\sqrt{4\mu-\lambda^2}\cot\left[\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + \arctan\left(\frac{C_1}{C_2}\right)\right] - \frac{1}{3}a(\lambda^2-4\mu)\xi + d_1, \tag{37}$$

where $\xi = ax+by+cz-\frac{a^3c(\lambda^2-4\mu)-3b^2}{4a}t$.

From (23) we have

$$u = -a\sqrt{4\mu-\lambda^2}\tan\left[\frac{\sqrt{4\mu-\lambda^2}}{2}\xi - \arctan\left(\frac{C_2}{C_1}\right)\right] + d_1 \tag{38}$$

or

$$u = -a\sqrt{4\mu-\lambda^2}\cot\left[\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + \arctan\left(\frac{C_1}{C_2}\right)\right] + d_1, \tag{39}$$

where $\xi = ax+by+cz+\frac{a^3c(\lambda^2-4\mu)+3b^2}{4a}t$.

From (28) we have

$$u = -a\sqrt{4\mu-\lambda^2}\tan\left[\frac{\sqrt{4\mu-\lambda^2}}{2}\xi - \arctan\left(\frac{C_2}{C_1}\right)\right] - a\lambda + \alpha_0 \tag{40}$$

or

$$u = a\sqrt{4\mu-\lambda^2}\cot\left[\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + \arctan\left(\frac{C_1}{C_2}\right)\right] - a\lambda + \alpha_0, \tag{41}$$

where $\xi = ax+by+cz+\frac{a^3c(\lambda^2-4\mu)+3b^2}{4a}t$. We can easily see that (40) and (41) are equivalent to (38) and (39), respectively. However, (36) and (37) cannot be obtained from (28).

The above comparisons show that (20) – (24) conclude (27) – (29) as special cases. Among of them, (21), (23), and (24) are equivalent to (27), (28), and (29), respectively. However, as demonstrated above, (20) and (22) with an explicit linear function in ξ are different from (27) – (29) and cannot be obtained by the (G'/G)-expansion method [25, 26, 28 – 31] and its improvements [27, 36] if we don't transform (13) into (14) but directly solving (13). In this sense, we may conclude that the modified version proposed in this paper is different from and superior to Wang et al.'s method [25] if and only if the reduced ODE (4) possesses the property $r \geqslant 1$. Zhang et al.'s ansatz solution in [27] has its advantages over the one in this paper and those in [25, 26, 28 – 31, 36] for solving NLEEs with variable coefficients. The generalized (G'/G)-expansion method proposed more recently by Yu [36] shows its advantages by introducing a more general ansatz solution than that used in [25] so that more general solutions can be obtained, which cannot be obtained by the modified version in this paper and those in [25 – 31].

Remark 2. All solutions obtained above have been checked with Mathematica by putting them back into the original Equation (9).

4. Conclusions

In summary, more general travelling wave solutions of the (3+1)-dimensional potential YTSF equation are obtained owing to the modified (G'/G)-expansion method proposed in this paper. Some of the obtained hyperbolic function solutions and trigonometric function solutions contain an explicit linear function of the variables in the potential YTSF equation. It may be important to explain some physical phenomena. The paper shows the effectiveness and advantages of the proposed method in handling the solution process of NLEEs. Employing it to study other NLEEs is our task in the future.

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