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Weakly Noncollapsed RCD Spaces with Upper Curvature Bounds

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Abstract: We show that if a $CD(K, n)$ space $(X, d, f\mathcal{H}^n)$ with $n \geq 2$ has curvature bounded above by κ in the sense of Alexandrov then f is constant.**Keywords:** Riemannian curvature-dimension condition, upper curvature bound, Alexandrov space, optimal transport**MSC:** Primary 53C20, 53C21

1 Introduction

In [16] Gigli and De Philippis introduced the following notion of a *noncollapsed* $RCD(K, n)$ space. An $RCD(K, n)$ space (X, d, m) is noncollapsed if n is a natural number and $m = \mathcal{H}^n$. A similar notion was considered by Kitabeppu in [23].

Noncollapsed $RCD(K, n)$ spaces give a natural intrinsic generalization of noncollapsing limits of manifolds with lower Ricci curvature bounds which are noncollapsed in the above sense by work of Cheeger–Colding [13].

In [16] Gigli and De Philippis also considered the following a-priori weaker notion. An $RCD(K, n)$ space (X, d, m) is *weakly noncollapsed* if n is a natural number and $m \ll \mathcal{H}^n$. Gigli and De Philippis gave several equivalent characterizations of weakly noncollapsed $RCD(K, n)$ spaces and studied their properties. By work of Mondino–Naber [31], Kell–Mondino [25] and Brué–Semola [11] it follows that an $RCD(K, n)$ space is weakly noncollapsed iff $\mathcal{R}_n \neq \emptyset$ where $\mathcal{R}_n$ is the rectifiable set of n -regular points in X .

It is well-known that if $(X, d, m) = (M^n, g, e^{-f} d \text{vol}_g)$ where (M^n, g) is a smooth n -dimensional Riemannian manifold and f is a smooth function on M then (X, d, m) is $RCD(K, n)$ iff $f = \text{const}$. More precisely, the classical Bakry–Emery condition $BE(K, N)$, $K \in \mathbb{R}$ and $N \geq n$, for a (compact) smooth metric measure space $(M^n, g, e^{-f} d \text{vol}_g)$, $f \in C^\infty(M)$, is

$$\frac{1}{2} L |\nabla u|_g^2 \geq \langle \nabla Lu, \nabla u \rangle_g + \frac{1}{N} (Lu)^2 + K |\nabla u|_g^2, \quad \forall u \in C^\infty(M),$$

where $L = \Delta - \nabla f$. In [8, Proposition 6.2] Bakry shows that $BE(K, N)$ holds if and only if

$$\nabla f \otimes \nabla f \leq (N - n) \left(\text{ric}_g + \nabla^2 f - Kg \right).$$

In particular, if $N = n$, then f is locally constant.

On the other hand, in [5, 6, 18] it was proven that a metric measure space (X, d, m) satisfies $RCD(K, N)$ if and only if the corresponding Cheeger energy satisfies a weak version of $BE(K, N)$ that is equivalent to the classical version for $(M, g, e^{-f} d \text{vol}_g)$ from above.

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In [16] Gigli and De Philippis conjectured that a weakly noncollapsed $RCD(K, n)$ space is already noncollapsed up to rescaling of the measure by a constant. Our main result is that this conjecture holds if a weakly noncollapsed space has curvature bounded above in the sense of Alexandrov.

Theorem 1.1. *Let $n \geq 2$ and let $(X, d, f\mathcal{H}^n)$ (where f is L^1_{loc} with respect to $\mathcal{H}^n$ and $\text{supp}(f\mathcal{H}^n) = X$) be a complete metric measure space which is $CBA(\kappa)$ (has curvature bounded above by κ in the sense of Alexandrov) and satisfies $CD(K, n)$. Then $f = \text{const}$ ¹.*

Since smooth Riemannian manifolds locally have curvature bounded above this immediately implies

Corollary 1.2. *Let (M^n, g) be a smooth Riemannian manifold and suppose $(M^n, g, f\mathcal{H}^n)$ is $CD(K, n)$ where K is finite and $f \geq 0$ is L^1_{loc} with respect to $\mathcal{H}^n$ and $\text{supp}(f\mathcal{H}^n) = M$. Then $f = \text{const}$.*

As was mentioned above, this corollary was well-known in case of smooth f but was not known in case of general locally integrable f .

In [24] it was shown that if a space (X, d, m) is $CD(K, n)$ and has curvature bounded above then X is $RCD(K, n)$ and if in addition $m = \mathcal{H}^n$ then X is Alexandrov with two sided curvature bounds. Combined with Theorem 1.1 this implies that the same remains true if the assumption on the measure is weakened to $m \ll \mathcal{H}^n$.

Corollary 1.3. *Let $n \geq 2$ and let $(X, d, f\mathcal{H}^n)$ where f is L^1_{loc} with respect to $\mathcal{H}^n$ and $\text{supp}(f\mathcal{H}^n) = X$ be a complete metric measure space which is $CBA(\kappa)$ (has curvature bounded above by κ in the sense of Alexandrov) and satisfies $CD(K, n)$. Then X is $RCD(K, n)$, $f = \text{const}$, $\kappa(n-1) \geq K$, and (X, d) is an Alexandrov space of curvature bounded below by $K - \kappa(n-2)$.*

Remark 1.4. Note that since a space $(X, d, f\mathcal{H}^n)$ satisfying the assumptions of Theorem 1.1 is automatically $RCD(K, n)$, as was remarked in [16] it follows from the results of [25] that n must be an integer.

Bakry's proof for smooth manifolds does not easily generalize to a non-smooth context. But let us describe a strategy that does generalize to $RCD + CAT$ spaces.

Assume that (X, d) is induced by a smooth manifold (M^n, g) and the density function f is smooth and positive such that (X, d, fm) satisfies $RCD(K, n)$. Then, by integration by parts on (M, g) the induced Laplace operator L is given by

$$Lu = \Delta u - \langle \nabla \log f, \nabla u \rangle, \quad u \in C^\infty(M), \quad (1.1)$$

where Δu is the classical Laplace-Beltrami operator of (M, g) for smooth functions. By a recent result of Han one has for any $RCD(K, n)$ space that the operator L is equal to the trace of Gigli's Hessian [20] on the set of n -regular points $\mathcal{R}_n$. Hence, after one identifies the trace of Gigli's Hessian with the Laplace-Beltrami operator Δ of M (what is true on (M^n, g)), one obtains immediately that $\nabla \log f = 0$. If M is connected, this yields the claim.

The advantage of this approach is that it does not involve the Ricci curvature tensor and in non-smooth context one might follow the same strategy. However, we have to overcome several difficulties that arise from the non-smoothness of the density function f and of the space (X, d, m) .

In particular, we apply the recently developed DC calculus by Lytchak-Nagano for spaces with upper curvature bounds to show that on the regular part of X the Laplace operator with respect to $\mathcal{H}^n$ is equal to the trace of the Hessian. By [24] the combination of CD and CAT condition implies that $f^{1/N}$ is locally semiconcave and hence f is locally Lipschitz on the regular part of X . This allows us to generalize the above argument for smooth Riemannian manifolds to the general case.

¹ Here and in all applications by $f = \text{const}$ we mean $f = \text{const}$ a.e. with respect to $\mathcal{H}^n$.

In section 2 we provide necessary preliminaries. We present the setting of *RCD* spaces and the calculus for them. We state important results by Mondino-Cavalletti (Theorem 2.4), Han (Theorem 2.11) and Gigli (Theorem 2.7, Proposition 2.9). We also give a brief introduction to the calculus of *BV* and *DC* functions for spaces with upper curvature bounds.

In section 3 we develop a structure theory for general *RCD* + *CAT* spaces where we adapt the *DC* calculus of Lytchak-Nagano [27]. This might be of independent interest.

Finally, in section 4 we prove our main theorem following the above idea.

2 Preliminaries

2.1 Curvature-dimension condition

A *metric measure space* is a triple (X, d, m) where (X, d) is a complete and separable metric space and m is a locally finite measure.

$\mathcal{P}^2(X)$ denotes the set of Borel probability measures μ on (X, d) such that $\int_X d(x_0, x)^2 d\mu(x) < \infty$ for some $x_0 \in X$ equipped with L^2 -Wasserstein distance W_2 . The sub-space of m -absolutely continuous probability measures in $\mathcal{P}^2(X)$ is denoted by $\mathcal{P}^2(X, m)$.

The *N-Renyi entropy* is

$$S_N(\cdot|m) : \mathcal{P}_b^2(X) \rightarrow (-\infty, 0], \quad S_N(\mu|m) = - \int \rho^{1-\frac{1}{N}} dm \text{ if } \mu = \rho m, \text{ and } 0 \text{ otherwise.}$$

Note that $\rho m \in \mathcal{P}^2(X, m)$ implies $\rho \in L^1(m)$. The functional S_N is lower semi-continuous, and $S_N(\mu) \geq -m(\text{supp } \mu)^{\frac{1}{N}}$ by Jensen's inequality.

For $\kappa \in \mathbb{R}$ we define

$$\cos_\kappa(x) = \begin{cases} \cosh(\sqrt{|\kappa|x}) & \text{if } \kappa < 0 \\ 1 & \text{if } \kappa = 0 \\ \cos(\sqrt{\kappa}x) & \text{if } \kappa > 0 \end{cases} \quad \sin_\kappa(x) = \begin{cases} \frac{\sinh(\sqrt{|\kappa|x})}{\sqrt{|\kappa|}} & \text{if } \kappa < 0 \\ x & \text{if } \kappa = 0 \\ \frac{\sin(\sqrt{\kappa}x)}{\sqrt{\kappa}} & \text{if } \kappa > 0 \end{cases}$$

Let π_κ be the diameter of a simply connected space form $\mathbb{S}_\kappa^2$ of constant curvature κ , i.e.

$$\pi_\kappa = \begin{cases} \infty & \text{if } \kappa \leq 0 \\ \frac{\pi}{\sqrt{\kappa}} & \text{if } \kappa > 0 \end{cases}$$

For $K \in \mathbb{R}$, $N \in (0, \infty)$ and $\theta \geq 0$ we define the *distortion coefficient* as

$$t \in [0, 1] \mapsto \sigma_{K,N}^{(t)}(\theta) = \begin{cases} \frac{\sin_{K/N}(t\theta)}{\sin_{K/N}(\theta)} & \text{if } \theta \in [0, \pi_{K/N}), \\ \infty & \text{otherwise.} \end{cases}$$

Note that $\sigma_{K,N}^{(t)}(0) = t$. For $K \in \mathbb{R}$, $N \in [1, \infty)$ and $\theta \geq 0$ the *modified distortion coefficient* is

$$t \in [0, 1] \mapsto \tau_{K,N}^{(t)}(\theta) = \begin{cases} \theta \cdot \infty & \text{if } K > 0 \text{ and } N = 1, \\ t^{\frac{1}{N}} \left[\sigma_{K,N-1}^{(t)}(\theta) \right]^{1-\frac{1}{N}} & \text{otherwise.} \end{cases}$$

Definition 2.1 ([29, 35]). We say (X, d, m) satisfies the *curvature-dimension condition* $CD(K, N)$ for $K \in \mathbb{R}$ and $N \in [1, \infty)$ if for every $\mu_0, \mu_1 \in \mathcal{P}_b^2(X, m)$ there exists an L^2 -Wasserstein geodesic $(\mu_t)_{t \in [0,1]}$ and an optimal coupling π between μ_0 and μ_1 such that

$$S_N(\mu_t|m) \leq - \int \left[\tau_{K,N}^{(1-t)}(d(x, y)) \rho_0(x)^{-\frac{1}{N}} + \tau_{K,N}^{(t)}(d(x, y)) \rho_1(y)^{-\frac{1}{N}} \right] d\pi(x, y)$$

where $\mu_i = \rho_i dm$, $i = 0, 1$.

Remark 2.2. If (X, d, m) is complete and satisfies the condition $CD(K, N)$ for $N < \infty$, then $(\text{supp } m, d)$ is a geodesic space and $(\text{supp } m, d, m)$ is $CD(K, N)$.

In the following we always assume that $\text{supp } m = X$.

Remark 2.3. For the variants $CD^*(K, N)$ and $CD^e(K, N)$ of the curvature-dimension condition we refer to [12, 18].

2.2 Calculus on metric measure spaces

For further details about this section we refer to [2–4, 19].

Let (X, d, m) be a metric measure space, and let $\text{Lip}(X)$ be the space of Lipschitz functions. For $f \in \text{Lip}(X)$ the local slope is

$$\text{Lip}(f)(x) = \limsup_{y \rightarrow x} \frac{|f(x) - f(y)|}{d(x, y)}, \quad x \in X.$$

If $f \in L^2(m)$, a function $g \in L^2(m)$ is called *relaxed gradient* if there exists sequence of Lipschitz functions f_n which L^2 -converges to f , and there exists h such that $\text{Lip} f_n$ weakly converges to h in $L^2(m)$ and $h \leq g$ m-a.e.. $g \in L^2(m)$ is called the *minimal relaxed gradient* of f and denoted by $|\nabla f|$ if it is a relaxed gradient and minimal w.r.t. the L^2 -norm amongst all relaxed gradients. The space of L^2 -Sobolev functions is then

$$W^{1,2}(X) := D(\text{Ch}^X) := \left\{ f \in L^2(m) : \int |\nabla f|^2 dm < \infty \right\}.$$

$W^{1,2}(X)$ equipped with the norm $\|f\|_{W^{1,2}(X)}^2 = \|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2$ is a Banach space. If $W^{1,2}(X)$ is a Hilbert space, we say (X, d, m) is *infinitesimally Hilbertian*. In this case we can define

$$(f, g) \in W^{1,2}(X)^2 \mapsto \langle \nabla f, \nabla g \rangle := \frac{1}{4} |\nabla(f+g)|^2 - \frac{1}{4} |\nabla(f-g)|^2 \in L^1(m).$$

Assuming X is locally compact, if U is an open subset of X , we say $f \in W^{1,2}(X)$ is in the domain $D(\Delta, U)$ of the *measure valued Laplace* Δ on U if there exists a signed Radon functional Δf on the set of Lipschitz functions g with bounded support in U such that

$$\int \langle \nabla g, \nabla f \rangle dm = - \int g d\Delta f. \quad (2.1)$$

If $U = X$ and $\Delta f = [\Delta f]_{ac} m$ with $[\Delta f]_{ac} \in L^2(m)$, we write $[\Delta f]_{ac} =: \Delta f$ and $D(\Delta, X) = D_{L^2(m)}(\Delta)$. μ_{ac} denotes the m -absolutely continuous part in the Lebesgue decomposition of a Borel measure μ . If $\mathbb{V}$ is any subspace of $L^2(m)$ and $f \in D_{L^2(m)}(\Delta)$ with $\Delta f \in \mathbb{V}$, we write $f \in D_{\mathbb{V}}(\Delta)$.

Theorem 2.4 (Cavalletti-Mondino, [15]). *Let (X, d, m) be an essentially non-branching $CD(K, N)$ space for some $K \in \mathbb{R}$ and $N > 1$. For $p \in X$ consider $d_p = d(p, \cdot)$ and the associated disintegration $m = \int_Q h_\alpha \mathcal{H}^1|_{X_\alpha} q(d\alpha)$.*

Then $d_p \in D(\Delta, X \setminus \{p\})$ and $\Delta d_p|_{X \setminus \{p\}}$ has the following representation formula:

$$\Delta d_p|_{X \setminus \{p\}} = -(\log h_\alpha)' m - \int_Q h_\alpha \delta_{a(X_\alpha)} q(d\alpha).$$

Moreover

$$\Delta d_p|_{X \setminus \{p\}} \leq (N-1) \frac{\sin'_{K/(N-1)}(d_p(x))}{\sin_{K/(N-1)}(d_p(x))} m \quad \& \quad [\Delta d_p|_{X \setminus \{p\}}]^{reg} \geq -(N-1) \frac{\sin'_{K/(N-1)}(d_p(x))}{\sin_{K/(N-1)}(d_p(x))} m.$$

Remark 2.5. The sets X_α in the previous disintegration are geodesic segments $[a(X_\alpha), p]$ with initial point $a(X_\alpha)$ and endpoint p . In particular, the set of points $q \in X$ such that there exists a geodesic connecting p and q that is extendible beyond q , is a set of full measure.

Definition 2.6. A metric measure space (X, d, m) satisfies the Riemannian curvature-dimension condition $RCD(K, N)$ for $K \in \mathbb{R}$ and $N \in [1, \infty]$ if it satisfies a curvature-dimension conditions $CD(K, N)$ and is infinitesimally Hilbertian.

The previous definition was first proposed in [4] for $N = \infty$. For a brief overview on the historical development in the case $N < \infty$ we refer to [24, Remark 2.15].

In [20] Gigli introduced a notion of $\text{Hess } f$ in the context of RCD spaces. $\text{Hess } f$ is tensorial and defined for $f \in W^{2,2}(X)$ that is the second order Sobolev space. An important property of $W^{2,2}(X)$ that we will need in the following is

Theorem 2.7 (Corollary 3.3.9 in [20], [34]). $D_{L^2(m)}(\Delta) \subset W^{2,2}(X)$.

Remark 2.8. The closure of $D_{L^2(m)}(\Delta)$ in $W^{2,2}(X)$ is denoted $H^{2,2}(X)$ [20, Proposition 3.3.18].

The next proposition [20, Proposition 3.3.22 i)] allows to compute $\text{Hess } f$ explicitly.

Proposition 2.9. Let $f, g_1, g_2 \in H^{2,2}(X)$. Then $\langle \nabla f, \nabla g_i \rangle \in W^{1,2}(X)$, and

$$2 \text{Hess } f(\nabla g_1, \nabla g_2) = \langle \nabla g_1, \nabla \langle \nabla f, \nabla g_2 \rangle \rangle + \langle \nabla g_2, \nabla \langle \nabla f, \nabla g_1 \rangle \rangle + \langle \nabla f, \nabla \langle \nabla g_1, \nabla g_2 \rangle \rangle \quad (2.2)$$

holds m -a.e. where the two sides in this expression are well-defined in $L^0(m)$.

Theorem 2.10 ([11]). Let (X, d, m) be a metric measure space satisfying $RCD(K, N)$ with $N < \infty$. Then, there exist $n \in \mathbb{N}$ and such that the set of n -regular points $\mathcal{R}_n$ has full measure.

Theorem 2.11 ([22]). Let (X, d, m) be as in the previous theorem. If $N = n \in \mathbb{N}$, then for any $f \in \mathbb{D}_\infty$ we have that $\Delta f = \text{tr Hess } f$ m -a.e. . More precisely, if $B \subset \mathcal{R}_n$ is a set of finite measure and $(e_i)_{i=1, \dots, n}$ is a unit orthogonal basis on B , then

$$\Delta f|_B = \sum_{i=1}^n \text{Hess } f(e_i, e_i)|_B =: [\text{tr Hess } f]|_B.$$

Corollary 2.12. Let (X, d, m) be a metric measure space as before. If $f \in D_{L^2(m)}(\Delta)$, we have that $\Delta f = \text{tr Hess } f$ m -a.e. in the sense of the previous theorem.

2.3 Spaces with upper curvature bounds

We will assume familiarity with the notion of $CAT(\kappa)$ spaces. We refer to [9, 10] or [24] for the basics of the theory.

Definition 2.13. Given a point p in a $CAT(\kappa)$ space X we say that two unit speed geodesics starting at p define the same direction if the angle between them is zero. This is an equivalence relation by the triangle inequality for angles and the angle induces a metric on the set $S_p^g(X)$ of equivalence classes. The metric completion $\Sigma_p^g X$ of $S_p^g X$ is called the *space of geodesic directions* at p . The Euclidean cone $C(\Sigma_p^g X)$ is called the *geodesic tangent cone* at p and will be denoted by $T_p^g X$.

The following theorem is due to Nikolaev [10, Theorem 3.19]:

Theorem 2.14. $T_p^g X$ is $CAT(0)$ and $\Sigma_p^g X$ is $CAT(1)$.

Note that this theorem in particular implies that $T_p^g X$ is a geodesic metric space which is not obvious from the definition. More precisely, it means that each path component of $\Sigma_p^g X$ is $CAT(1)$ (and hence geodesic) and the distance between points in different components is π . Note however, that $\Sigma_p^g X$ itself need not be path connected.

2.4 BV functions and DC calculus

Recall that a function $g : V \subset \mathbb{R}^n \rightarrow \mathbb{R}$ of bounded variation (BV) admits a derivative in the distributional sense [17, Theorem 5.1] that is a signed vector valued Radon measure $[Dg] = (\frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_n}) = [Dg]_{ac} + [Dg]_s$. Moreover, if g is BV, then it is L^1 -differentiable [17, Theorem 6.1] a.e. with L^1 -derivative $[Dg]_{ac}$, and approximately differentiable a.e. [17, Theorem 6.4] with approximate derivative $D^{ap}g = (\frac{\partial^{ap}g}{\partial x_1}, \dots, \frac{\partial^{ap}g}{\partial x_n})$ that coincides almost everywhere with $[Dg]_{ac}$. The set of BV functions $BV(V)$ on V is closed under addition and multiplication [32, Section 4]. We'll call BV functions BV_0 if they are continuous.

Remark 2.15. In [32] and [1] BV functions are called BV_0 if they are continuous away from an $\mathcal{H}^{n-1}$ -negligible set. However, for the purposes of the present paper it will be more convenient to work with the more restrictive definition above.

Then for $f, g \in BV_0(V)$ we have

$$\frac{\partial fg}{\partial x_i} = \frac{\partial f}{\partial x_i} g + f \frac{\partial g}{\partial x_i} \quad (2.3)$$

as signed Radon measures [32, Section 4, Lemma]. By taking the $\mathcal{L}^n$ -absolutely continuous part of this equality it follows that (2.3) also holds a.e. in the sense of approximate derivatives. In fact, it holds at *all* points of approximate differentiability of f and g . This easily follows by a minor variation of the standard proof that $d(fg) = fdg + gdf$ for differentiable functions.

A function $f : V \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is called a DC function if in a small neighborhood of each point $x \in V$ one can write f as a difference of two semi-convex functions. The set of DC functions on V is denoted by $DC(V)$ and contains the class $C^{1,1}(V)$. The space $DC(V)$ is closed under addition and multiplication. The first partial derivatives $\frac{\partial f}{\partial x_i}$ of a DC function $f : V \rightarrow \mathbb{R}$ are BV, and hence the second partial derivatives $\frac{\partial}{\partial x_j} \frac{\partial f}{\partial x_i}$ exist as signed Radon measure that satisfy

$$\frac{\partial}{\partial x_i} \frac{\partial f}{\partial x_j} = \frac{\partial}{\partial x_j} \frac{\partial f}{\partial x_i}$$

[17, Theorem 6.8], and hence

$$\frac{\partial^{ap}}{\partial x_i} \frac{\partial f}{\partial x_j} = \frac{\partial^{ap}}{\partial x_j} \frac{\partial f}{\partial x_i} \quad \text{a.e. on } V. \quad (2.4)$$

A map $F : V \rightarrow \mathbb{R}^l$, $l \in \mathbb{N}$, is called a DC map if each coordinate function F_i is DC. The composition of two DC maps is again DC. A function f on V is called DC_0 if it's DC and C^1 .

Let (X, d) be a geodesic metric space. A function $f : X \rightarrow \mathbb{R}$ is called a DC function if it can be locally represented as the difference of two Lipschitz semi-convex functions. A map $F : Z \rightarrow Y$ between metric spaces Z and Y that is locally Lipschitz is called a DC map if for each DC function f that is defined on an open set $U \subset Y$ the composition $f \circ F$ is DC on $F^{-1}(U)$. In particular, a map $F : Z \rightarrow \mathbb{R}^l$ is DC if and only if its coordinates are DC. If F is a bi-Lipschitz homeomorphism and its inverse is DC, we say F is a DC isomorphism.

2.5 DC coordinates in CAT spaces

The following was developed in [27] based on previous work by Perelman [32].

Assume (X, d) is a CAT space, let $p \in X$ such that there exists an open neighborhood $\hat{U}$ of p that is homeomorphic to $\mathbb{R}^n$. It is well known (see e.g. [24, Lemma 3.1]) that this implies that geodesics in $\hat{U}$ are locally extendible.

Suppose $T_p^g X \cong \mathbb{R}^n$.

Then, there exist DC coordinates near p with respect to which the distance on $\hat{U}$ is induced by a BV Riemannian metric g .

More precisely, let $a_1, \dots, a_n, b_1, \dots, b_n$ be points near p such that $d(p, a_i) = d(p, b_i) = r$, p is the midpoint of $[a_i, b_i]$ and $\angle a_i p a_j = \pi/2$ for all $i \neq j$ and all comparison angles $\tilde{\angle} a_i p a_j, \tilde{\angle} a_i p b_j, \tilde{\angle} b_i p b_j$ are sufficiently close to $\pi/2$ for all $i \neq j$.

Let $x: \hat{U} \rightarrow \mathbb{R}^n$ be given by $x = (x_1, \dots, x_n) = (d(\cdot, a_1), \dots, d(\cdot, a_n))$.

Then by [27, Corollary 11.12] for any sufficiently small $0 < \varepsilon < \pi_k/4$ the restriction $x|_{B_{2\varepsilon}(p)}$ is Bilipschitz onto an open subset of $\mathbb{R}^n$. Let $U = B_\varepsilon(p)$ and $V = x(U)$. By [27, Proposition 14.4] $x: U \rightarrow V$ is a DC equivalence in the sense that $h: U \rightarrow \mathbb{R}$ is DC iff $h \circ x^{-1}$ is DC on V .

Further, the distance on U is induced by a BV Riemannian metric g which in x coordinates is given by a 2-tensor $g^{ij}(p) = \cos \alpha_{ij}$ where α_{ij} is the angle at p between geodesics connecting p and a_i and a_j respectively. By the first variation formula g^{ij} is the derivative of $d(a_i, y(t))$ at 0 where y is the geodesic with $y(0) = p$ and $y(1) = a_j$. Since $d(a_i, \cdot)$, $i = 1, \dots, n$, are Lipschitz, g^{ij} is in L^∞ . We denote $\langle v, w \rangle_g(p) = g^{ij}(p)v_i w_j$ the inner product of $v, w \in \mathbb{R}^n$ at p . The Riemannian metric g^{ij} induces a distance function d_g on V such that x is a metric space isomorphism for $\varepsilon > 0$ sufficiently small.

If u is a Lipschitz function on U , $u \circ x^{-1}$ is a Lipschitz function on V , and therefore differentiable $\mathcal{L}^n$ -a.e. in V by Rademacher's theorem. Hence, we can define the gradient of u at points of differentiability of u in the usual way as the metric dual of its differential. Then the usual Riemannian formulas hold and $\nabla u = g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x_j}$ and $|\nabla u|_g^2 = g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}$ a.e. .

3 Structure theory of RCD+CAT spaces

In this section we study metric measure spaces (X, d, m) satisfying

$$(X, d, m) \text{ is } CAT(\kappa) \text{ and satisfies the conditions } RCD(K, N) \text{ for } 1 \leq N < \infty, K, \kappa < \infty. \quad (3.1)$$

The following result was proved in [24]

Theorem 3.1 ([24]). *Let (X, d, m) satisfy $CD(K, N)$ for $1 \leq N < \infty, K, \kappa \in \mathbb{R}$. Then X is infinitesimally Hilbertian. In particular, (X, d, m) satisfies $RCD(K, N)$.*

Remark 3.2. It was shown in [24] that the above theorem also holds if the $CD(K, N)$ assumption in (3.1) is replaced by $CD^*(K, N)$ or $CD^e(K, N)$ conditions (see [24] for the definitions). Moreover, in a recent paper [30] Di Marino, Gigli, Pasqualetto and Soultanis show that a $CAT(\kappa)$ space with any Radon measure is infinitesimally Hilbertian. For these reasons (3.1) is equivalent to assuming that X is $CAT(\kappa)$ and satisfies one of the assumptions $CD(K, N)$, $CD^*(K, N)$ or $CD^e(K, N)$ with $1 \leq N < \infty, K, \kappa < \infty$.

In [24] we also established the following property of spaces satisfying (3.1):

Proposition 3.3 ([24]). *Let X satisfy (3.1). Then X is non-branching.*

Next we prove

Proposition 3.4. *Let X satisfy (3.1). Then for almost all $p \in X$ it holds that $T_p^g X \cong \mathbb{R}^k$ for some $k \leq N$.*

Remark 3.5. Note that from the fact that X is an RCD space it follows that $T_p X$ is an Euclidean space for almost all $p \in X$ [21]. However, at this point in the proof we don't know if $T_p X \cong T_p^g X$ at all such points (we expect this to be true for all p).

Proof. First, recall that by the CAT condition, geodesics of length less than π_κ in X are unique. Moreover, since X is nonbranching and CD , for any $p \in X$ the set E_p of points q , such that the geodesic which connects p and q is not extendible, has measure zero (Remark 2.5).

Let $A = \{p_i\}_{i=1}^\infty$ be a countable dense set of points in X , and let $C = \bigcup_{i \in \mathbb{N}} E_{p_i}$. For any $q \in X \setminus C$ and any i with $d(p_i, q) < \pi_\kappa$ the geodesic $[p_i q]$ can be extended slightly past q . Since A is dense this implies that for any $q \in X \setminus C$ there is a dense subset in $T_q^g X$ consisting of directions v which have "opposites" (i.e. making angle π with v).

For every $p \in X$ and every tangent cone $T_p X$ the geodesic tangent cone $T_p^g X$ is naturally a closed convex subset of $T_p X$. Since X is RCD this means that for almost all p the geodesic tangent cone $T_p^g X$ is a convex subset of a Euclidean space. Thus, for almost all $p \in X$ it holds that $T_p^g X$ is a convex subset in $\mathbb{R}^m$ for some $m \leq N$, is a metric cone over $\Sigma_p^g X$ and contains a dense subset of points with opposites also in $T_p^g X$. In particular, $\Sigma_p^g X$ is a convex subset of $\mathbb{S}^m$. Since a closed convex subset of $\mathbb{S}^m$ is either $\mathbb{S}^k$ with $k \leq m$ or has boundary this means that for any such p $T_p^g X$ is isometric to a Euclidean space of dimension $k \leq m$. $\square$

Proposition 3.6. *Let X satisfy (3.1).*

- i) *Let $p \in X$ satisfy $T_p^g X \cong \mathbb{R}^m$ for some $m \leq N$.
Then an open neighbourhood W of p is homeomorphic to $\mathbb{R}^m$.*
- ii) *If an open neighborhood W of p is homeomorphic to $\mathbb{R}^m$ then for any $q \in W$ it holds that $T_q^g X \cong T_q X \cong \mathbb{R}^m$.
Moreover, for any compact set $C \subset W$ there is $\varepsilon = \varepsilon(C) > 0$ such that every geodesic starting in C can be extended to length at least ε .*

Proof. Let us first prove part i). Suppose $T_p^g X \cong \mathbb{R}^m$. By [26, Theorem A] there is a small $R > 0$ such that $B_R(p) \setminus \{p\}$ is homotopy equivalent to $\mathbb{S}^{m-1}$. Since $\mathbb{S}^{m-1}$ is not contractible, by [28, Theorem 1.5] there is $0 < \varepsilon < \pi_\kappa/2$ such that every geodesic starting at p extends to a geodesic of length ε . The natural "logarithm" map $\Phi: \bar{B}_\varepsilon(p) \rightarrow \bar{B}_\varepsilon(0) \subset T_p^g X$ is Lipschitz since X is $CAT(\kappa)$. By the above mentioned result of Lytchak and Schroeder [28, Theorem 1.5] Φ is onto.

We also claim that Φ is 1-1. If Φ is not 1-1 then there exist two distinct unit speed geodesics γ_1, γ_2 of the same length $\varepsilon' \leq \varepsilon$ such that $p = \gamma_1(0) = \gamma_2(0)$, $\gamma_1'(\varepsilon') = \gamma_2'(\varepsilon')$ but $\gamma_1(\varepsilon') \neq \gamma_2(\varepsilon')$.

Let $v = \gamma_1'(\varepsilon') = \gamma_2'(\varepsilon')$. Since $T_p^g X \cong \mathbb{R}^m$ the space of directions $T_p^g X$ contains the "opposite" vector $-v$. Then there is a geodesic γ_3 of length ε starting at p in the direction $-v$. Since X is $CAT(\kappa)$ and $2\varepsilon < \pi_\kappa$, the concatenation of γ_3 with γ_1 is a geodesic and the same is true for γ_2 . This contradicts the fact that X is nonbranching.

Thus, Φ is a continuous bijection and since both $\bar{B}_\varepsilon(p)$ and $\bar{B}_\varepsilon(0)$ are compact and Hausdorff it's a homeomorphism. This proves part i).

Let us now prove part ii). Suppose an open neighborhood W of p is homeomorphic to $\mathbb{R}^m$. By [24, Lemma 3.1] or by the same argument as above using [26] and [28], for any $q \in W$ all geodesics starting at q can be extended to length at least $\varepsilon(q) > 0$. Therefore $T_q^g X \cong T_q X$. By the splitting theorem $T_q X \cong \mathbb{R}^l$ where $l = l(q) \leq N$ might a priori depend on q . However, using part i) we conclude that an open neighbourhood of q is homeomorphic to $\mathbb{R}^{l(q)}$. Since W is homeomorphic to $\mathbb{R}^m$ this can only happen if $l(q) = m$.

The last part of ii) immediately follows from above and compactness of C . $\square$

3.1 DC coordinates in RCD + CAT spaces.

Let X_{reg}^g be the set of points p in X with $T_p X \cong T_p^g X \cong \mathbb{R}^n$. Then by Proposition 3.6 there is an open neighbourhood $\hat{U}$ of p homeomorphic to $\mathbb{R}^n$ such that every $q \in \hat{U}$ also lies in X_{reg}^g . In particular, X_{reg}^g is open. Further, geodesics in $\hat{U}$ are locally extendible by Proposition 3.6.

Thus the theory of Lytchak–Nagano from [27] applies, and let $x: U \rightarrow V$ with $U = B_{2\varepsilon}(p) \subset \hat{U}$ be DC coordinates as in Subsection 2.5. The pushforward of the Hausdorff measure $\mathcal{H}^n$ on U under x coordinates is given by $\sqrt{|g|}\mathcal{L}^n$ where $|g|$ is the determinant of g_{ij} . Consequently, the map $x: (U, d, \mathcal{H}^n) \rightarrow (V, d_g, \sqrt{|g|}\mathcal{L}^n)$ is a metric-measure isomorphism.

With a slight abuse of notations we will identify these metric-measure spaces as well as functions on them, i.e we will identify any function u on U with $u \circ x^{-1}$ on V .

Lemma 3.7. *Angles between geodesics in U are continuous. That is if $q_i \rightarrow q \in U$, $[q_i s_i] \rightarrow [qs]$, $[q_i t_i] \rightarrow [qt]$ are converging sequences with $q \neq s$, $q \neq t$ then $\angle s_i q_i t_i \rightarrow \angle sqt$.*

Proof. Without loss of generality we can assume that $q_i \in U$ for all i . Let $\alpha_i = \angle s_i q_i t_i$, $\alpha = \angle s q t$. Let $\{\alpha_{i_k}\}$ be a converging subsequence and let $\bar{\alpha} = \lim_{k \rightarrow \infty} \alpha_{i_k}$. Then by upper semicontinuity of angles in $CAT(\kappa)$ spaces it holds that $\alpha \geq \bar{\alpha}$. We claim that $\alpha = \bar{\alpha}$.

By Proposition 3.6 we can extend $[s_i q_i]$ past q_i as geodesics a definite amount δ to geodesics $[s_i z_i]$. Let $\beta_i = \angle z_i q_i t_i$. By possibly passing to a subsequence of $\{i_k\}$ we can assume that $[s_{i_k} z_{i_k}] \rightarrow [sz]$. Let $\beta = \angle z q t$. Then since all spaces of directions $T_{q_i}^g X$ and $T_q^g X$ are Euclidean by Proposition 3.6, we have that $\alpha_i + \beta_i = \alpha + \beta = \pi$ for all i . Again using semicontinuity of angles we get that $\beta \geq \bar{\beta}$.

We therefore have

$$\pi = \alpha + \beta \geq \bar{\alpha} + \bar{\beta} = \pi$$

Hence all the inequalities above are equalities and $\alpha = \bar{\alpha}$. Since this holds for an arbitrary converging subsequence $\{\alpha_{i_k}\}$ it follows that $\lim_{i \rightarrow \infty} \alpha_i = \alpha$. $\square$

Let $\tilde{\mathcal{A}}$ be the algebra of functions of the form $\varphi(f_1, \dots, f_m)$ where $f_i = d(\cdot, q_i)$ for some $q_1, \dots, q_m$ with $|q_i p| > \varepsilon$ and φ is smooth. Together with the first variation formula for distance functions Lemma 3.7 implies that for any $u, h \in \tilde{\mathcal{A}}$ it holds that $\langle \nabla u, \nabla h \rangle_g$ is continuous on V . In particular, $g^{ij} = \langle \nabla x_i, \nabla x_j \rangle_g$ is continuous and hence g is BV_0 and not just BV .

Furthermore, since $\frac{\partial}{\partial x_i} = \sum_j g_{ij} \nabla x_j$ where g_{ij} is the pointwise inverse of g^{ij} , Lemma 3.7 also implies that any $u \in \tilde{\mathcal{A}}$ is C^1 on V . Hence, any such u is DC_0 on V .

Recall that for a Lipschitz function u on V we have two a-priori different notions of the norm of the gradient defined m -a.e.: the "Riemannian" norm of the gradient $|\nabla u|_g^2 = g^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j}$ and the minimal weak upper gradient $|\nabla u|$ when u is viewed as a Sobolev function in $W^{1,2}(m)$. We observe that these two notions are equivalent.

Lemma 3.8. *Let $u, h : U \rightarrow \mathbb{R}$ be Lipschitz functions. Then $|\nabla u| = |\nabla u|_g$, $|\nabla h| = |\nabla h|_g$ m -a.e. and $\langle \nabla u, \nabla h \rangle = \langle \nabla u, \nabla h \rangle_g$ m -a.e..*

In particular, $g^{ij} = \langle \nabla x_i, \nabla x_j \rangle_g = \langle \nabla x_i, \nabla x_j \rangle$ m -a.e..

Proof. First note that since both $\langle \nabla u, \nabla h \rangle$ and $\langle \nabla u, \nabla h \rangle_g$ satisfy the parallelogram rule, it's enough to prove that $|\nabla u| = |\nabla u|_g$ a.e..

Recall that g^{ij} is continuous on U . Fix a point p where u is differentiable. Then

$$\begin{aligned} \text{Lip } u(p) &= \limsup_{q \rightarrow p} \frac{|u(p) - u(q)|}{d(p, q)} = \limsup_{q \rightarrow p} \frac{|u(p) - u(q)|}{|p - q|_{g(p)}} \\ &= \sup_{|v|_{g(p)}=1} D_v u = \sup_{|v|_{g(p)}=1} \langle v, \nabla u \rangle_{g(p)} = |\nabla u|_{g(p)}. \end{aligned}$$

In the second equality we used that d is induced by g^{ij} , and that g^{ij} is continuous. Since (U, d, m) admits a local 1-1 Poincaré inequality and is doubling, the claim follows from [14] where it is proved that for such spaces $\text{Lip } u = |\nabla u|$ a.e.. $\square$

In view of the above Lemma from now on we will not distinguish between $|\nabla u|$ and $|\nabla u|_g$ and between $\langle \nabla u, \nabla h \rangle$ and $\langle \nabla u, \nabla h \rangle_g$.

Proposition 3.9. *If $u \in W^{1,2}(m) \cap BV(U)$, then $|\nabla u|^2 = g^{ij} \frac{\partial^{ap} u}{\partial x_i} \frac{\partial^{ap} u}{\partial x_j}$ m -a.e..*

Proof. We choose a set $S \subset U$ of full measure such that u and $|\nabla u|$ are defined pointwise on S and u is approximately differentiable at every $x \in S$. Since u is $BV(U)$, for $\eta > 0$ there exist $\hat{u}_\eta \in C^1(U)$ such that for the set

$$B_\eta = \{x \in S : u(x) \neq \hat{u}_\eta(x), D^{ap} u(x) \neq D\hat{u}_\eta(x)\} \cap S$$

one has $m(B_\eta) \leq \eta$ [17, Theorem 6.13]. Note, since f is continuous, there exists a constant $\lambda > 0$ such that $\lambda^{-1} m \leq \mathcal{H}^n \leq \lambda m$ on U . Moreover, since g^{ij} is continuous, one can check that $\hat{u}_\eta$ is Lipschitz w.r.t. d_g , and hence $\hat{u}_\eta \in W^{1,2}(m)$.

By [3, Proposition 4.8] we know that $|\nabla u|_{A_\eta} = |\nabla \hat{u}_\eta|_{A_\eta}$ m -a.e. for $A_\eta = S \setminus B_\eta$. On the other hand, uniqueness of approximative derivatives also yields that $g^{ij} \frac{\partial^{ap} u}{\partial x_i} \frac{\partial^{ap} u}{\partial x_j} |_{A_\eta} = g^{ij} \frac{\partial^{ap} \hat{u}_\eta}{\partial x_i} \frac{\partial^{ap} \hat{u}_\eta}{\partial x_j} |_{A_\eta}$ m -a.e. . Hence, since $\hat{u}$ is Lipschitz w.r.t. d ,

$$|\nabla u|_{A_\eta} = g^{ij} \frac{\partial^{ap} u}{\partial x_i} \frac{\partial^{ap} u}{\partial x_j} |_{A_\eta} \quad m\text{-a.e. .}$$

by Lemma 3.8.

Now, we pick a sequence η_k for $k \in \mathbb{N}$ such that $\sum_{k=1}^{\infty} \eta_k < \infty$. Then, by the Borel-Cantelli Lemma the set

$$B = \{x \in S : \exists \text{ infinitely many } k \in \mathbb{N} \text{ s.t. } x \in B_{\eta_k}\}$$

is of m -measure 0. Consequently, for $x \in A = S \setminus B$ we can pick a $k \in \mathbb{N}$ such that $x \in A_{\eta_k} \subset S$. It follows

$$|\nabla u|^2(x) = g^{ij} \frac{\partial^{ap} u}{\partial x_i} \frac{\partial^{ap} u}{\partial x_j}(x) \quad \forall x \in S$$

and hence m -a.e. . □

4 Proof of the main theorem

0. Let $(X, d, f\mathcal{H}^n)$ be $RCD(K, n)$ and $CAT(\kappa)$ where $0 \leq f \in L^1_{loc}(\mathcal{H}^n)$.

Remark 4.1. If (X, d, m) is a weakly non-collapsed RCD space in the sense of [16] or a space satisfying the generalized Bishop inequality in the sense of [23] and if (X, d) is $CAT(\kappa)$, the assumptions are satisfied by [16, Theorem 1.10].

Following Gigli and De Philippis [16] for any $x \in X$ we consider the monotone quantity $\frac{m(B_r(x))}{v_{k,n}(r)}$ which is non increasing in r by the Bishop-Gromov volume comparison. Let $\theta_{n,r}(x) = \frac{m(B_r(x))}{\omega_n r^n}$. Consider the density function $\theta_n(x) = \lim_{r \rightarrow 0} \theta_{n,r}(x) = \lim_{r \rightarrow 0} \frac{m(B_r(x))}{\omega_n r^n}$.

Since n is fixed throughout the proof we will drop the subscripts n and from now on use the notations $\theta(x)$ and $\theta_r(x)$ for $\theta_n(x)$ and $\theta_{n,r}(x)$ respectively.

By Propositions 3.4, 3.6 and [16, Theorem 1.10] we have that for almost all $p \in X$ it holds that $T_p X \cong T_p^g X \cong \mathbb{R}^n$ and $\theta(x) = f(x)$.

Therefore we can and will assume from now on that $f = \theta$ everywhere.

Remark 4.2. Monotonicity of $r \mapsto \frac{m(B_r(x))}{v_{k,n}(r)}$ immediately implies that $f(x) = \theta(x) > 0$ for all x .

Let $x \in X_{reg}^g$. Then $T_p^g X \cong \mathbb{R}^l$ for some $l \leq n$. We claim that $l = n$. By Proposition 3.6 X_{reg}^g is an l -manifold near p and by section 2.5 DC coordinates near p give a *biLipschitz* homeomorphism of an open neighborhood of p onto an open set in $\mathbb{R}^m$. Since $m = f\mathcal{H}^n$ this can only happen if $l = n$.

Lemma 4.3. [24, Lemma 5.4] $\theta^{\frac{1}{n}} = f^{\frac{1}{n}}$ is semiconcave on X .

Corollary 4.4. $\theta = f$ is locally Lipschitz near any $p \in X_{reg}^g$.

Proof. First observe that semiconcavity of $\theta^{\frac{1}{n}}$, the fact that $\theta > 0$ and local extendability of geodesics on X_{reg}^g imply that θ must be locally bounded on X_{reg}^g . Now the corollary becomes an easy consequence of Lemma 4.3, the fact that geodesics are locally extendible a definite amount near p by Proposition 3.6 and the fact that a semiconcave function on $(0, 1)$ is locally Lipschitz. □

1. Since small balls in spaces with curvature bounded above are geodesically convex, we can assume that $\text{diam } X < \pi_\kappa$. Let $p \in X$, $x : U \rightarrow \mathbb{R}^n$ and $\tilde{\mathcal{A}}$ be as in the previous subsection.

By the same argument as in [32, Section 4] (cf. [33], [1]) it follows that any $u \in \tilde{\mathcal{A}}$ lies in $D(\Delta, U, \mathcal{H}^n)$ and the $\mathcal{H}^n$ -absolutely continuous part of $\Delta_0 u$ can be computed using standard Riemannian geometry formulas that

is

$$\Delta_0^g(u) = \frac{1}{\sqrt{|g|}} \frac{\partial^{ap}}{\partial x_j} (g^{jk} \sqrt{|g|} \frac{\partial u}{\partial x_k}) \quad (4.1)$$

where $|g|$ denotes the pointwise determinant of g^{ij} . Here Δ_0 denotes the measure valued Laplacian on $(U, d, \mathcal{H}^n)$. Note that g , $\sqrt{|g|}$ and $\frac{\partial u}{\partial x_i}$ are BV_0 functions, and the derivatives on the right are understood as approximate derivatives.

Indeed, w.l.o.g. let $u \in DC_0(U)$, and let v be Lipschitz with compact support in U . As before we identify u and v with their representatives in x coordinates. First, we note that, since g , $\sqrt{|g|}$ and $\frac{\partial u}{\partial x_i}$ are BV_0 , their product is also in BV_0 , as well as the product with v . Then, the Leibniz rule (2.3) for the approximate partial derivatives yields that

$$\sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial^{ap} v}{\partial x_j} = - \frac{\partial^{ap}}{\partial x_j} \left(\sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_i} \right) v + \frac{\partial^{ap}}{\partial x_j} \left(\sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_i} v \right) \mathcal{L}^n\text{-a.e.}$$

Again using (2.3) we also have that

$$\sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} = - \frac{\partial}{\partial x_j} \left(\sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_i} \right) v + \frac{\partial}{\partial x_j} \left(\sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_i} v \right) \text{ as measures} \quad (4.2)$$

and the absolutely continuous with respect to $\mathcal{L}^n$ part of this equation is given by the previous identity.

The fundamental theorem of calculus for BV functions (see [17, Theorem 5.6]) yields that

$$\int_V \frac{\partial}{\partial x_j} \left(\sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_i} v \right) = 0. \quad (4.3)$$

Moreover, by Lemma 3.8 $\langle \nabla v, \nabla u \rangle$ is given in x coordinates by $g^{ij} \frac{\partial v}{\partial x_j} \frac{\partial u}{\partial x_i} \mathcal{L}^n\text{-a.e.}$

Combining the above formulas gives that

$$\begin{aligned} - \int_V \langle \nabla u, \nabla v \rangle \sqrt{|g|} d\mathcal{L}^n &= \int_V \left[\frac{\partial^{ap}}{\partial x_j} \left(\sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_i} \right) v - \frac{\partial^{ap}}{\partial x_j} \left(\sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_i} v \right) \right] d\mathcal{L}^n \\ &= \int_V \frac{1}{\sqrt{|g|}} \frac{\partial^{ap}}{\partial x_j} \left(\sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_i} \right) v \sqrt{|g|} d\mathcal{L}^n + \int_V v d\mu \end{aligned}$$

where μ is some signed measure such that $\mu \perp \mathcal{L}^n$. This implies (4.1).

2. Since (X, d, m) is $RCD(K, n)$ for any $q \in X$, we have that d_q lies in $D(\Delta, U \setminus \{q\}, m)$ and Δd_q is locally bounded above on $U \setminus \{q\}$ by $\text{const} \cdot m$ by Theorem 2.4.

Furthermore, since by Proposition 3.6 all geodesics in U are locally extendible we have $\Delta d_q = [\Delta d_q]^{reg} \cdot m$ on $U \setminus \{q\}$ and $[\Delta d_q]^{reg}$ is locally bounded below on $U \setminus \{q\}$ again by Theorem 2.4. Therefore $[\Delta d_q]^{reg}$ is in $L_{loc}^\infty(U \setminus \{q\})$ with respect to m (and also $\mathcal{H}^n$), and in particular, Δd_q is locally L^2 in $U \setminus \{q\}$.

By the chain rule for Δ [19] the same holds for any $u, h \in \tilde{\mathcal{A}}$ on all of U as by construction u and h only involve distance functions to points outside U .

Recall the following lemma from [7, Lemma 6.7] (see also [31]).

Lemma 4.5. *Let (X, d, m) be a metric measure space satisfying a RCD condition. Then for all $E \subset X$ compact and all $G \subset X$ open such that $E \subset G$ there exists a Lipschitz function $\chi : X \rightarrow [0, 1]$ with*

(i) $\chi = 1$ on $E_v = \{x \in X : \exists y \in E : d(x, y) < v\}$ and $\text{supp } \chi \subset G$,

(ii) $\Delta \chi \in L^\infty(m)$ and $|\nabla \chi|^2 \in W^{1,2}(X)$.

Let us choose a cut-off function $\chi : X \rightarrow [0, 1]$ as in the previous lemma for G with $\tilde{G} \subset U$ and $E = \tilde{B}_\delta(p) \subset G$ for some $\delta \in (0, \varepsilon)$.

Let $u, h \in \tilde{\mathcal{A}}$. By the chain rule for Δ it again follows that

$$\Delta(\chi u) = [\Delta(\chi u)]^{reg} m \text{ \& } [\Delta(\chi u)]^{reg} \in L^2(m).$$

Moreover, (2.1) holds for Lipschitz functions on X . Hence $\chi u \in D_{L^2(m)}(\Delta)$.

Therefore $\chi u, \chi h \in H^{2,2}(X)$ by Remark 2.8, $\langle \nabla \chi u, \nabla \chi h \rangle \in W^{1,2}(U)$ by Proposition 2.9 and the Hessian of χu can be computed by the formula (2.2). Moreover, by locality of the minimal weak upper gradient

$$\begin{aligned} 2 \text{Hess}(\chi u)(\nabla(\chi h_1), \nabla(\chi h_2))|_{B_\delta(p)} \\ = \langle \nabla h_1, \nabla \langle \nabla u, \nabla h_2 \rangle \rangle + \langle \nabla h_2, \nabla \langle \nabla u, \nabla h_1 \rangle \rangle - \langle \nabla u, \nabla \langle \nabla h_1, \nabla h_2 \rangle \rangle \text{ m-a.e. in } \bar{B}_\delta(p). \end{aligned} \quad (4.4)$$

Note that, for instance,

$$W^{1,2}(\bar{B}_\delta(p)) \ni \langle \nabla \chi u, \nabla \chi h_2 \rangle|_{\bar{B}_\delta(p)} = \langle \nabla \chi u|_{\bar{B}_\delta(p)}, \nabla \chi h_2|_{\bar{B}_\delta(p)} \rangle = \langle \nabla u|_{\bar{B}_\delta(p)}, \nabla h_2|_{\bar{B}_\delta(p)} \rangle.$$

Remark 4.6. It is not clear that u itself is in the domain of Gigli's Hessian since u is not contained $D_{L^2(m)}(\Delta)$ (integration by parts for u would involve boundary terms). Nevertheless, the equality and the RHS in (4.4) are well-defined on $B_\delta(p)$. We denote the RHS in (4.4) with $Hu(h_1, h_2)$.

3. The aim of this paragraph is to compute $Hu(x_i, x_j)g_{ij}$ on $B_\delta(p)$ in the DC_0 coordinate chart x . In the following we assume w.l.o.g. that $B_\varepsilon(p) = B_\delta(p)$ for δ like in the previous paragraph.

Since u, h_1, h_2 are DC_0 in x coordinates we have that $\langle \nabla h_1, \nabla h_2 \rangle = g^{ij} \frac{\partial h_1}{\partial x_i} \frac{\partial h_2}{\partial x_j}$ is BV and the same holds for $\langle \nabla u, \nabla h_1 \rangle, \langle \nabla u, \nabla h_2 \rangle$. Moreover, $\langle \nabla h_i, \nabla h_j \rangle, \langle \nabla u, \nabla h_i \rangle \in W^{1,2}(\bar{B}_\delta(p))$ as we saw before.

Hence, with the help of Proposition 3.9 the RHS of (4.4) can be computed pointwise in x coordinates at points of approximate differentiability of $\frac{\partial u}{\partial x_i}, \frac{\partial h_1}{\partial x_i}$ and $\frac{\partial h_2}{\partial x_i}, i = 1, \dots, n$, and (4.4) can be understood to hold a.e. in the sense of approximate derivatives. That is, we can write

$$\langle \nabla u, \nabla \langle \nabla h_1, \nabla h_2 \rangle \rangle = g^{ij} \frac{\partial^{ap} u}{\partial x_i} \frac{\partial^{ap}}{\partial x_j} \langle \nabla h_1, \nabla h_2 \rangle = g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial^{ap}}{\partial x_j} (g^{kl} \frac{\partial h_1}{\partial x_k} \frac{\partial h_2}{\partial x_l}) \quad (4.5)$$

and do the same for the other two terms in the RHS of (4.4).

Using that $g^{ij} = \langle \nabla x_i, \nabla x_j \rangle$ and $\frac{\partial}{\partial x_i} = \sum_j g_{ij} \nabla x_j$ a standard computation shows that for any $u \in \tilde{\mathcal{A}}$ it holds that

$$\frac{1}{\sqrt{|g|}} \frac{\partial^{ap}}{\partial x_j} (g^{jk} \sqrt{|g|} \frac{\partial u}{\partial x_k}) = Hu(x_i, x_j)g_{ij} \quad (4.6)$$

on $B_\delta(p)$.

Let S be the set of points in V where $\nabla u, g_{ij}$ have approximate derivatives and $\frac{\partial^{ap}}{\partial x_i}(\frac{\partial u}{\partial x_j}) = \frac{\partial^{ap}}{\partial x_j}(\frac{\partial u}{\partial x_i})$. Then by (2.4) S has full measure in V , and hence it's enough to verify (4.6) pointwise on S .

Let $q \in S$. Let $\hat{g}$ be a smooth metric on a neighborhood of q which such that $\hat{g}(q) = g(q)$ and $D\hat{g}(q) = D^{ap}g(q)$. Likewise let $\hat{u}$ be a smooth function on a neighborhood of q such that $\hat{u}(q) = u(q)$, $D\hat{u}(q) = Du(q)$ and $D\frac{\partial \hat{u}}{\partial x_i}(q) = D^{ap}\frac{\partial u}{\partial x_i}(q)$ for all i . Such $\hat{u}$ exists (we can take it to be quadratic in x) since $\frac{\partial^{ap}}{\partial x_i}(\frac{\partial u}{\partial x_j})(q) = \frac{\partial^{ap}}{\partial x_j}(\frac{\partial u}{\partial x_i})(q)$. Then

$$\frac{1}{\sqrt{|g|}} \frac{\partial^{ap}}{\partial x_j} (g^{jk} \sqrt{|g|} \frac{\partial u}{\partial x_k})(q) = \frac{1}{\sqrt{|\hat{g}|}} \frac{\partial}{\partial x_j} (\hat{g}^{jk} \sqrt{|\hat{g}|} \frac{\partial \hat{u}}{\partial x_k})(q)$$

where all the derivatives are approximate derivatives.

Similarly

$$Hu(x_i, x_j)(q)g_{ij}(q) = H\hat{u}(x_i, x_j)(q)\hat{g}_{ij}(q)$$

where again all the derivatives in (4.4) and (4.5) are approximate derivatives.

But

$$\frac{1}{\sqrt{|\hat{g}|}} \frac{\partial}{\partial x_j} (\hat{g}^{jk} \sqrt{|\hat{g}|} \frac{\partial \hat{u}}{\partial x_k})(q) = \text{Hess}_{\hat{g}} \hat{u}(\nabla_{\hat{g}} x_i, \nabla_{\hat{g}} x_j) \hat{g}_{ij}(q)$$

by standard Riemannian geometry since all functions involved are smooth. Since $q \in S$ was arbitrary this proves that (4.6) holds a.e. in the sense of approximate derivatives as claimed.

4. It follows that

$$\begin{aligned}\mathrm{Tr} \, \mathrm{Hess}(\chi u)|_{B_\delta(p)} &= \mathrm{Hess}(\chi u)(\nabla x_i, \nabla x_j)g_{ij}|_{B_\delta(p)} \\ &= \mathrm{Hess}(\chi u)(\nabla \chi x_i, \nabla \chi x_j)g_{ij}|_{B_\delta(p)} = Hu(x_i, x_j)g_{ij} = \Delta_0 u|_{B_\delta(p)}.\end{aligned}\quad (4.7)$$

for every $u \in \tilde{\mathcal{A}}$ where Hess is the Hessian in the sense of Gigli, and $H(u)$ denotes the RHS of (4.4). The first equality in (4.7) is the definition of Tr , the second equality is the L^∞ -homogeneity of the tensor $\mathrm{Hess}(\chi u)$, and the third equality is the identity (4.4).

Since f is locally Lipschitz and positive on $B_\delta(p)$, we can perform the following integration by parts in DC_0 coordinates. Let $u \in \tilde{\mathcal{A}}$ and let g be a Lipschitz function with compact support in $B_\delta(p)$. $\chi u \in D_{L^2(m)}(\Delta)$ implies $u|_{B_\delta(p)} \in D(\Delta, B_\delta(p))$. Then

$$\begin{aligned}\int_{B_\delta(p)} g \Delta u dm &= - \int_{B_\delta(p)} \langle \nabla u, \nabla g \rangle dm = - \int_{B_\delta(p)} \langle \nabla u, \nabla g \rangle f d\mathcal{H}^n \\ &= - \int_{B_\delta(p)} \langle \nabla u, \nabla(gf) \rangle d\mathcal{H}^n + \int_{B_\delta(p)} \langle \nabla u, \nabla \log f \rangle gf d\mathcal{H}^n \\ &= \int_{B_\delta(p)} (\Delta_0 u + \langle \nabla u, \nabla \log f \rangle) g dm\end{aligned}$$

yields

$$\Delta u = \Delta_0 u + \langle \nabla u, \nabla \log f \rangle$$

on $B_\delta(p)$ for any $u \in \tilde{\mathcal{A}}$. Note again that only χu is in $D_{L^2(m)}(\Delta)$.

On the other hand, by Corollary 2.12 it holds that $\Delta(\chi u) = \mathrm{Tr} \, \mathrm{Hess}(\chi u)$ m -a.e.. Thus

$$0 = \mathrm{Tr} \, \mathrm{Hess}(\chi u)|_{B_\delta(p)} - \mathrm{Tr} \, \mathrm{Hess}(\chi u)|_{B_\delta(p)} = \Delta u|_{B_\delta(p)} - \Delta_0(\chi u)|_{B_\delta(p)} = \langle \nabla u, \nabla \log f \rangle|_{B_\delta(p)}$$

a.e. for any $u \in \tilde{\mathcal{A}}$.

5. Therefore $f \nabla \log f|_{B_\delta(p)} = \nabla f|_{B_\delta(p)} = 0$. Indeed, since f is semiconcave, $f \circ x^{-1}$ is DC by [27]. Hence $\nabla f = g^{ij} \frac{\partial f}{\partial x_i}$ is continuous on a set of full measure Z in $B_\delta(p)$ since this is true for convex functions on $\mathbb{R}^n$. Let $q \in Z$ be a point of continuity of $\nabla f|_Z$ and $v = \nabla f(q)$. Assume $v \neq 0$. Then due to extendability of geodesics there exists $z \notin U$ such that $\nabla d_z(q) = \frac{v}{|v|}$. Since ∇d_z is continuous near q and ∇f is continuous on Z it follows $\langle \nabla f, \nabla d_z \rangle \neq 0$ on a set of positive measure. Hence $\nabla f|_{B_\delta(p)} = 0$ and $f|_{B_\delta(p)} = \text{const}$.

6. We claim that this implies that f is constant on X_{reg}^g . (This is not immediate since we don't know yet that X_{reg}^g is connected.) Indeed, since X is essentially nonbranching, radial disintegration of m centered at p (Theorem 2.4) implies that for almost all $q \in X$ the set $[pq] \cap X_{reg}^g$ has full measure in $[p, q]$. It is also open in $[p, q]$ since X_{reg}^g is open.

Suppose $q \in X_{reg}^g$ is as above.

Since θ is semiconcave on X and locally constant on X_{reg}^g it is locally Lipschitz (and hence Lipschitz) on the geodesic segment $[p, q]$. A Lipschitz function on $[0, 1]$ which is locally constant on an open set of full $\mathcal{L}^1$ measure is constant. Therefore θ is constant on $[p, q]$ and hence θ is constant on X_{reg}^g which has full measure. Therefore $f = \theta = \text{const}$ a.e. globally. $\square$

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