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Chordal Hausdorff Convergence and Quasihyperbolic Distance

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Abstract: We study Hausdorff convergence (and related topics) in the chordalization of a metric space to better understand pointed Gromov-Hausdorff convergence of quasihyperbolic distances (and other conformal distances).

Keywords: quasihyperbolic distance, kernel convergence, Gromov-Hausdorff distance, pointed Gromov-Hausdorff distance, conformal metrics

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1 Introduction

Here we examine assorted types of convergence of sets with an eye towards securing the convergence of an associated sequence of metric spaces defined via conformal metrics. The starting point for our work is the following folklore fact, which holds because the Euclidean distances $\text{dist}(x, \partial\Omega_i)$ converge uniformly in $\mathbb{R}^n$ to $\text{dist}(x, \partial\Omega)$.

Let $\Omega, \Omega_i \subsetneq \mathbb{R}^n$ be domains. Suppose the closed sets $\mathbb{R}^n \setminus \Omega_i$ converge, with respect to Hausdorff distance in $\mathbb{R}^n$, to $\mathbb{R}^n \setminus \Omega$. Then the quasihyperbolic distances in Ω_i converge locally uniformly in Ω to quasihyperbolic distance in Ω .

We present various and sundry generalizations of this result. In particular, we believe that the correct notion of convergence (for the associated distances) is that of pointed Gromov-Hausdorff convergence, a concept that we review in §2.3. To a certain extent, this work is about the pointed Gromov-Hausdorff convergence of quasihyperbolic metric spaces. In this setting compactness can be weakened to just boundedness, and we introduce the notion of *bounded uniform convergence* which replaces local uniform convergence; see §3.1. This is especially useful when local compactness is not available, e.g., in the Banach space setting.

A non-complete locally complete rectifiably connected metric space Ω is dubbed a *quasihyperbolic space*. Each such Ω carries a *quasihyperbolic metric* $\delta^{-1}ds = \delta_{\partial\Omega}^{-1}ds$, whose length distance $k = k_\Omega$ is called *quasihyperbolic distance* in Ω ; here $\delta(x) = \delta_{\partial\Omega}(x) := \text{dist}(x, \partial\Omega)$ is the distance from x to the metric boundary $\partial\Omega$ of Ω . See §2.4.1 for more details. Using terminology coined by Bonk-Heinonen-Koskela, the *quasihyperbolization* of Ω is the metric space $\Omega_k := (\Omega, k)$. A simple, but nonetheless important, special case of a quasihyperbolic space is any open connected proper subset of Euclidean space $\mathbb{R}^n$ (with its induced Euclidean distance), and we call such an $\Omega \subsetneq \mathbb{R}^n$ an *Euclidean quasihyperbolic space*.

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Since its introduction in the 1970's by Gehring and Palka, see [9], the quasihyperbolic metric has proven to be an especially useful and important tool in many areas of geometric analysis including classical geometric function theory, quasiconformal mapping theory, potential theory, complex dynamics, and even geometric group theory. Its importance, especially with regards to the program of 'metric space analysis', cannot be overstated.

Thus it is worthwhile to know when we can approximate the quasihyperbolization of a space by simpler spaces. That is, given a quasihyperbolic space Ω , when can we exhibit a sequence (Ω_i) of 'simple' quasihyperbolic spaces whose quasihyperbolizations (Ω_i, k_i) converge, somehow, to (Ω, k) ? For example, a quasihyperbolic space with a finite boundary is 'simple'. Such a program is at the heart of recent work by Väisälä and Luiro (see [26] and [19]) who approximate quasihyperbolic plane domains by punctured planes with finite boundaries.

Here is a special case of our Theorem 4.11, which in turn relies heavily on Theorem 4.4.

Theorem. *Suppose a sequence (A_i) of closed subsets of $\hat{\mathbb{R}}^n$ converges, with respect to chordal Hausdorff distance, to a closed set $A \neq \{\infty\}$. Then, with respect to pointed Gromov-Hausdorff distance, the quasihyperbolizations of $(\mathbb{R}^n \setminus A_i)$ converge to the quasihyperbolization of $\mathbb{R}^n \setminus A$.*

We also establish analogs of the above for other conformal metrics including the Ferrand and Kulkarni-Pinkhall-Thurston metrics (on quasihyperbolic domains in $\hat{\mathbb{R}}^n$) as well as the hyperbolic metric (on hyperbolic domains in $\hat{\mathbb{C}}$). See §4.2 and §4.3.

In Theorem 4.11, we generalize the above replacing $\mathbb{R}^n$ by an arbitrary complete length space. Here Theorem 3.1 plays a crucial role; roughly speaking, when closed sets chordal Hausdorff converge, their associated distance functions converge uniformly on bounded sets.

In the pointed Gromov Hausdorff setting, our notion of *uniform convergence on bounded sets* is the natural analog of *uniform convergence on compact sets*. Just as in the Euclidean setting where the chordal distance induces the one-point compactification, the chordalization of a metric space (see §2.2) produces a bounded distance whose topology agrees with the one-point extension topology. In general, there need be no compactness nor even local compactness, and we expect these ideas to prove useful in other situations.

In Theorems 3.1, 3.5 and Corollary 3.6 we show that chordal Hausdorff convergence implies bounded uniform convergence of distances, investigate the consequences of such, and characterize chordal Hausdorff convergence in terms of bounded uniform convergence. One consequence of Corollary 3.6 is the *finite approximation property* described in Lemma 4.13; additional effort gives the *quasihyperbolic distance finite approximation property* stated in Proposition 4.14.

Many readers will anticipate a connection between the above and the classical notion of Carathéodory kernel convergence, and we explore this in Section 3. Roughly speaking, we examine three basic types of convergence: chordal Hausdorff convergence for closed sets, Carathéodory core convergence for open sets, and Kuratowski convergence for closed sets; see §§3.1, 3.2, 3.3. The latter two are always equivalent (indeed, in any topological space) and all three agree in proper spaces; see Lemma 3.14 and Corollaries 3.16, 3.17.

We also investigate the consequences of replacing chordal Hausdorff convergence with either Gromov-Hausdorff convergence or chordal Gromov-Hausdorff convergence, e.g., in the above Theorem. Our results here are recorded as Theorem 5.2 and Corollary 5.7. We conclude with Theorem 5.8, which provides a partial converse to Theorems 4.4 and 4.11, and especially with Corollary 5.9 which characterizes pointed Gromov-Hausdorff convergence for Euclidean quasihyperbolic spaces.

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2 Preliminaries

2.1 General Information

Everywhere, $(X, |\cdot|)$ denotes a generic metric space possessing no additional presumed properties; here $|x - y|$ is the distance between the points x and y in X . (We are not assuming the existence of an underlying norm nor any sort of group structure.) In the spirit of metric space analysis, we make no *a priori* assumptions concerning the metric space X preferring to give these explicitly as needed.

We write $C = C(D, \dots)$ to indicate a constant C that depends only on the data $D, \dots$. We write $K_1 \lesssim K_2$ to indicate that $K_1 \leq C K_2$ for some computable constant C that depends only on the relevant data, and $K_1 \simeq K_2$ means $K_1 \lesssim K_2 \lesssim K_1$.

For any metric spaces X, Y and maps $X \supset A \xrightarrow{f, g} Y$, we define

$$\|f - g\|_\infty^A := \sup_{x \in A} |f(x) - g(x)|.$$

When $A = X$ is compact, $Y = \mathbb{R}$, and we restrict attention to maps f, g in the space $\mathcal{C}(X, \mathbb{R})$ of all continuous real-valued functions on X , $\|\cdot\|_\infty^X$ is the usual L^∞ -norm on $\mathcal{C}(X, \mathbb{R})$.

The *open ball* and *sphere* of radius r centered at a point a in X are

$$B(a; r) := \{x : |x - a| < r\} \quad \text{and} \quad S(a; r) := \{x : |x - a| = r\}$$

and the corresponding *closed ball* is $B[a; r] := B(a; r) \cup S(a; r)$. The *open t -neighborhood* about $A \subset X$ is

$$N(A; t) := \{x \in X \mid \text{dist}(x, A) < t\} = \bigcup_{a \in A} B(a; t).$$

Given non-empty $A, B \subset X$, we write $\text{dist}(x, A)$, $\text{dist}(A, B)$, $\text{diam}(A)$, and $\text{cdiam}(A)$ for the distance from $x \in X$ to A , distance between A and B , diameter of A , and circumdiameter of A ; here the *circumdiameter* of A is

$$\text{cdiam}(A) := \inf\{\text{diam } B(a; r) \mid B(a; r) \supset A\}.$$

It is convenient to adopt the notation $\delta_A(x) := \text{dist}(x, A)$.

A metric space is *proper* if it has the Heine-Borel property that every closed bounded subset of X is compact; equivalently, every closed ball is compact. A *continuum* is a non-degenerate (so, more than a single point) compact connected space.

Recall that every metric space can be isometrically embedded into a complete metric space. See [21, Theorem 43.7, p.269] or [16, Theorem 2-72, p.82]. We let $\bar{X}$ denote the *metric completion* of the metric space X ; thus $\bar{X}$ is the closure of the image of X under such an isometric embedding. The *metric boundary* of X is $\partial X := \bar{X} \setminus X$. We write $\bar{B}(\xi; r)$ to denote the *open ball* in $\bar{X}$ centered at $\xi \in \bar{X}$ with radius $r > 0$.

For a subset A of X we write $\text{int}(A)$, $\text{cl}(A)$, $\text{bd}(A)$ to denote the *topological interior*, *closure*, *boundary* (respectively) of A . We note that when U is an open subspace of X , $\text{bd}(U) \subset \partial U$ and equality holds if X is complete¹ (but may not hold in general). In fact we always have

$$\partial U = \text{bd}(U) \cup (\partial X \cap \bar{U}).$$

Also, when $A \subsetneq X$ is closed and Ω is a component of $X \setminus A$ (and X is locally connected), $\text{bd}(\Omega) \subset A \subset X \setminus \Omega$, and so $\delta_{\partial\Omega} \geq \delta_A \geq \delta_{X \setminus \Omega}$; therefore, for all $x \in X$,

$$\text{dist}(x, \text{bd}(\Omega)) \geq \text{dist}(x, A) \geq \text{dist}(x, X \setminus \Omega).$$

¹ Here, and directly below, there is an isometric embedding $\text{bd}(U) \xrightarrow{\iota} \partial U$ that is surjective when X is complete, and we are identifying $\text{bd}(U)$ with $\iota(\text{bd}(U))$. Similarly, there is an isometric embedding $\bar{U} \hookrightarrow \bar{X}$.

Both inequalities above can be strict, but if X is a complete length space, then $\delta_{\partial\Omega} = \delta_{X \setminus \Omega}$.

A metric space X is *locally complete* provided each point is an interior point of some complete subspace; that is, for each $x \in X$ there is a complete $C \subset X$ with $x \in \text{int}(C)$. Since closed subspaces of complete spaces are complete, it is not hard to check that this is the same as requiring that for all $x \in X$, $\text{dist}(x, \partial X) > 0$. (Equivalently, ∂X is closed in $\bar{X}$, or each point has an open neighborhood whose closure is complete.) For example, this holds when X is locally compact.

When d is some other distance on X , $\bar{X}_d$ and $\partial_d X := \bar{X}_d \setminus X$ denote the metric completion and metric boundary, respectively, of $X_d := (X, d)$. Also, $B_d(x; r)$ and $S_d(x; r)$ are the open ball and sphere (of radius r centered at the point x) in X_d , and $\text{int}_d(A)$, $\text{cl}_d(A)$, $\text{bd}_d(A)$ are the interior, closure, boundary (respectively) of A in X_d . For example, when X is rectifiably connected, d could be the intrinsic length distance $l = l_X$ on X .

A *path* in X is a continuous map $\mathbb{R} \supset I \xrightarrow{\gamma} X$ where $I = I_\gamma$ is an interval (called the *parameter interval* for γ) that may be closed or open or neither and finite or infinite. The *trajectory* of such a path γ is $|\gamma| := \gamma(I)$ which we call a *curve*. When I is closed and $I \neq \mathbb{R}$, $\partial\gamma := \gamma(\partial I)$ denotes the set of *endpoints* of γ which consists of one or two points depending on whether or not I is compact. For example, if $I_\gamma = [u, v] \subset \mathbb{R}$, then $\partial\gamma = \{\gamma(u), \gamma(v)\}$. We call γ a *compact path* if its parameter interval I is compact (which we often assume to be $[0, 1]$).

When $\partial\gamma = \{a, b\}$, we write $\gamma : a \curvearrowright b$ (in X) to indicate that γ is a path (in X) with *initial point* a and *terminal point* b ; this notation is also meant to imply an orientation— a precedes b on γ .

An *arc* α is an injective compact path. Given points $a, b \in |\alpha|$, there are unique $u, v \in I$ with $\alpha(u) = a$, $\alpha(v) = b$ and we write $\alpha[a, b] := \alpha|_{[u, v]}$.

When $\alpha : a \curvearrowright b$ and $\beta : b \curvearrowright c$ are paths that join a to b and b to c respectively, $\alpha * \beta$ denotes the concatenation of α and β ; so $\alpha * \beta : a \curvearrowright c$. Of course, $|\alpha * \beta| = |\alpha| \cup |\beta|$. Also, the *reverse* of γ is the path $\tilde{\gamma}$ defined by $\tilde{\gamma}(t) := \gamma(1 - t)$ (when $I_\gamma = [0, 1]$) and going from $\gamma(1)$ to $\gamma(0)$. Of course, $|\tilde{\gamma}| = |\gamma|$.

We note that every compact path contains an arc with the same endpoints; see [23].

Euclidean n -dimensional space is denoted $\mathbb{R}^n$ and its *one point extension* is $\hat{\mathbb{R}}^n := \mathbb{R}^n \cup \{\infty\}$ together with the *Euclidean chordal distance*

$$\chi(x, y) := \begin{cases} \frac{2|x - y|}{\sqrt{1 + |x|^2}\sqrt{1 + |y|^2}} & \text{if } x \neq \infty \neq y, \\ \frac{2}{\sqrt{1 + |x|^2}} & \text{if } x \neq \infty = y. \end{cases}$$

Always $\chi(x, y) \leq 2|x - y|$, so the “identity” inclusion $\mathbb{R}^n \xrightarrow{\text{id}} \hat{\mathbb{R}}^n$ is 2-Lip. Also,

$$\forall x, y \in \mathbb{B}^n[0; R], \quad \chi(x, y) \geq \frac{2}{1 + R^2} |x - y|.$$

Thus $\mathbb{R}^n \xrightarrow{\text{id}} \hat{\mathbb{R}}^n$ is locally bi-Lipschitz, and so an embedding.

We refer to [2] for the definition and basic properties of Möbius transformations. We introduce the following notation:

$$J(x) := x^* := \frac{x}{|x|^2}, \quad \text{and then for each } p \in \mathbb{R}^n, \quad J_p(x) := J(x - p) = (x - p)^*.$$

We remind the reader that the inversion J , of $\hat{\mathbb{R}}^n$ with respect to the origin, can be viewed as reflection across the unit sphere. However, the inversion J_p is the translation $x \mapsto x - p$ followed by reflection across the unit sphere. Thus, e.g., $J^{-1} = J$ whereas $J_p^{-1}(y) = y^* + p$.

We utilize the following information about Möbius transformations; see [2, Theorem 3.6.1, p.42].

2.1 Fact. Each Möbius transformation $\hat{\mathbb{R}}^n \xrightarrow{T} \hat{\mathbb{R}}^n$ is Lipschitz with respect to the chordal distance; in fact,

$$\forall x, y \in \hat{\mathbb{R}}^n, \quad \chi(Tx, Ty) \leq H\chi(x, y)$$

where $H = H(T) := \exp(h(e_{n+1}, T(e_{n+1})))$. Here h denotes hyperbolic distance in the upper half-space $\mathbb{R}_+^{n+1}$, $e_{n+1} := (0, 1) \in \mathbb{R}^n \times \mathbb{R}$, and we use the Poincaré extension of T to $\hat{\mathbb{R}}^{n+1}$.

For later use, we record the following.

2.2 Lemma. *The Lipschitz constant $H(x) := H(J_x)$ for J_x is given by*

$$H(x) := H(J_x) = \exp(h(e_{n+1}, \tilde{x})) \quad \text{where } \tilde{x} := \frac{(-x, 1)}{1 + |x|^2}.$$

Proof. We have $J_x = J \circ T_x$ where T is translation by $-x$. The Poincaré extensions of the maps J and T_x are J for $\hat{\mathbb{R}}^{n+1}$ and translation by $-x' := (-x, 0) \in \mathbb{R}^n \times \mathbb{R}$. Thus the Poincaré extension of J_x is $J_{x'}$, and writing $x_1 := e_{n+1} - x' = (-x, 1)$ we obtain

$$\tilde{x} := J_{x'}(e_{n+1}) = J(e_{n+1} - x') = J(x_1) = \frac{x_1}{|x_1|^2} = \frac{x_1}{1 + |x|^2}$$

and so

$$H(x) = H(J_x) = \exp(h(e_{n+1}, \tilde{x})) \quad \text{where } \tilde{x} = \frac{x_1}{1 + |x|^2} = \frac{(-x, 1)}{1 + |x|^2}.$$

□

2.2 Metric Chordal Distance

The *one-point extension* $\hat{X}$ of X is defined via

$$\hat{X} := \begin{cases} X & \text{when } X \text{ is bounded,} \\ X \cup \{\infty\} & \text{when } X \text{ is unbounded} \end{cases}$$

and a set $U \subset \hat{X}$ is open in $\hat{X}$ if and only if either U is an open subset of X or $\hat{X} \setminus U$ is a bounded closed subset of X . Thus when X is proper, $\hat{X}$ is its one-point compactification. For $A \subset X$, we write $\hat{A}$ to denote the topological closure of A in $\hat{X}$; thus $\hat{A} = \text{cl}(A)$ when A is bounded and $\hat{A} = \text{cl}(A) \cup \{\infty\}$ when A is unbounded. We make the convention that whenever we write anything involving the point ∞ , we are tacitly assuming that X is unbounded.

We recall a construction of Bonk and Kleiner [5, Lemma 2.2, p.87]. Let $(X, |\cdot|)$ be any metric space, fix a base point $o \in X$, write $|x| := |x - o|$, and consider

$$c(x, y) := \frac{|x - y|}{(1 + |x|)(1 + |y|)}$$

which is defined for all $x, y \in X$. Sometimes this is a distance function, but in general it may not satisfy the triangle inequality, so we define

$$\hat{d}(x, y) := \inf \left\{ \sum_{i=1}^k c(x_i, x_{i-1}) \mid x = x_0, \dots, x_k = y \in X \right\}.$$

Then for all $x, y \in X$,

$$\frac{1}{4} c(x, y) \leq \hat{d}(x, y) \leq c(x, y) \leq |x - y| \wedge \left(\frac{1}{1 + |x|} + \frac{1}{1 + |y|} \right). \quad (2.3)$$

In particular, $\hat{d}$ is a distance function on X and the map $(X, |\cdot|) \xrightarrow{\text{id}} (X, \hat{d})$ is: 1-Lipschitz, locally bilipschitz, and a 16 t -quasimöbius homeomorphism. Moreover, when our original space X is unbounded, there is a unique point in the completion of $(X, \hat{d})$ that corresponds to the point ∞ in $\hat{X}$ and the distance function $\hat{d}$ on X extends in the usual way to $\hat{X}$. Also, the metric topology induced by $\hat{d}$ on $\hat{X}$ is precisely the one-point extension topology.

We call $\hat{d}$ the *chordal distance* on $\hat{X}$, and $(\hat{X}, \hat{d})$ is the *chordalization* of $(X, |\cdot|)$.² The $\hat{d}$ -metric quantities in $\hat{X}$ are denoted either by attaching $\hat{d}$ or by using a hat. For example: $B_{\hat{d}}(x; r)$ is a $\hat{d}$ -ball centered at x , and

² In [7] $(\hat{X}, \hat{d})$ is termed the *sphericalization* of $(X, |\cdot|)$, but this should refer to $(\hat{X}, l_{\hat{d}})$ where $l_{\hat{d}}$ is the length distance associated with $\hat{d}$.

$\hat{\text{dist}}(x, A)$, $\hat{\text{diam}}(A)$, $\hat{\ell}(\gamma)$ are the $\hat{d}$ -distance from a point to a set, the $\hat{d}$ -diameter of a set and the $\hat{d}$ -length of a path, respectively. The inequalities in (2.3) continue to hold for all points in $\hat{X}$.

As $(X, |\cdot|) \xrightarrow{\text{id}} (\hat{X}, \hat{d})$ is an embedding, whenever (x_i) is a sequence in $\hat{X}$ and $z \in X$ we have

$$\lim_{i \rightarrow \infty} \hat{d}(x_i, z) = 0 \iff x_i \in X \text{ for all sufficiently large } i \text{ and } \lim_{i \rightarrow \infty} |x_i - z| = 0. \quad (2.4)$$

Here is a useful elementary fact concerning $\hat{X}$ and the chordal distance $\hat{d}$.

2.5 Lemma. Let $R > 0$ and put $r := R/(4(2R+1)(R+1))$. Suppose $a, b \in \hat{X}$ and $\hat{d}(a, b) < r$. Then

$$a \in B[o; R] \implies b \in B[o; 2R].$$

Proof. Suppose $a \in B[o; R]$. Then $\hat{d}(a, \infty) > r$, so $b \in X$. If $|b| > 2R$ were true, then we would get

$$\frac{|a-b|}{(|a|+1)(|b|+1)} \geq \frac{1}{2} \frac{|b|}{(R+1)(|b|+1)} \geq \frac{R}{(R+1)(2R+1)}$$

which would give

$$\hat{d}(a, b) \geq \frac{1}{4} c(a, b) \geq \frac{1}{4} \frac{R}{(R+1)(2R+1)} = r. \quad \square$$

2.3 Hausdorff and Gromov-Hausdorff Distances

Here we recall the definitions of Hausdorff distance, Gromov-Hausdorff distance, pointed Gromov-Hausdorff distance, and state other relevant information.

2.6 Fact. For any non-empty $A, B \subset X$, the following quantities are all equal:

- (a) $\inf\{r > 0 \mid \forall a \in A, \text{dist}(a, B) < r \text{ and } \forall b \in B, \text{dist}(b, A) < r\},$
- (b) $\inf\{t > 0 \mid A \subset N(B; t) \text{ and } B \subset N(A; t)\},$
- (c) $\sup_{a \in A} \text{dist}(a, B) \vee \sup_{b \in B} \text{dist}(b, A),$
- (d) $\sup_{x \in X} |\text{dist}(x, A) - \text{dist}(x, B)|.$

The *Hausdorff distance* $\text{dist}_{\mathcal{H}}(A, B) = \text{dist}_{\mathcal{H}}^X(A, B)$ between two non-empty subsets A, B of X is defined to be the common value of the quantities listed in Fact 2.6.

A distance function δ on the disjoint union $X \sqcup Y$ of two metric spaces is *admissible* if its restriction to each of X, Y agrees with the original distances on X, Y respectively. Given $t > 0$, a distance function δ is *t-admissible* on $X \sqcup Y$ if it is an admissible distance on $X \sqcup Y$ with the property that $X \subset N_\delta(Y; t)$ and $Y \subset N_\delta(X; t)$; equivalently, $\text{dist}_{\mathcal{H}}^\delta(X, Y) < t$, where $\text{dist}_{\mathcal{H}}^\delta$ denotes the Hausdorff distance defined on non-empty subsets of $(X \sqcup Y, \delta)$. The *Gromov-Hausdorff distance* between two non-empty metric spaces X and Y is

$$\text{dist}_{\mathcal{GH}}(X, Y) := \inf\{t > 0 \mid \exists \text{ a } t\text{-admissible distance on } X \sqcup Y\}.$$

A *pointed metric space* is a triple $(X, d; a)$, that we often abbreviate as $(X; a)$ when the distance function is understood, where (X, d) is a metric space and a is a fixed base-point in X . Maps between pointed spaces are assumed to preserve base-points; thus a map $f : (X; a) \rightarrow (Y; b)$ satisfies $f(a) = b$. Given $t > 0$ and points $a \in X, b \in Y$, we say that δ is $(t; a, b)$ -admissible provided $\delta : X \sqcup Y \times X \sqcup Y \rightarrow [0, \infty]$ is an admissible distance function on $X \sqcup Y$ and

$$\delta(a, b) < t, \quad B_\delta[a; t^{-1}] \subset N_\delta(Y; t), \quad B_\delta[b; t^{-1}] \subset N_\delta(X; t).$$

We note that the above conditions imply that for all $r \in (0, t^{-1}]$,

$$B_\delta[a; r] \subset N_\delta(B_\delta(b; r+2t) \cap Y; t) \text{ and } B_\delta[b; r] \subset N_\delta(B_\delta(a; r+2t) \cap X; t).$$

Following Gromov, we define the *pointed Gromov-Hausdorff distance* between two pointed metric spaces $(X; a)$ and $(Y; b)$ via

$$\text{dist}_{\mathcal{GH}_*}((X; a), (Y; b)) := (1/2) \wedge \tilde{\text{dist}}_{\mathcal{GH}_*}((X; a), (Y; b))$$

where

$$\tilde{\text{dist}}_{\mathcal{GH}_*}((X; a), (Y; b)) := \inf\{t > 0 \mid \exists \text{ a } (t; a, b)\text{-admissible distance } \delta \text{ on } X \sqcup Y\}.$$

It is not difficult to see that Hausdorff distance, Gromov-Hausdorff distance, and pointed Gromov-Hausdorff distance are all non-negative, symmetric, and satisfy the triangle inequality. While these are not true distance functions on all sets, it is well-known that:

- $\text{dist}_{\mathcal{H}}$ is a distance function on the class $\mathcal{H}(X) := \mathcal{C}(X) \cap \mathcal{B}(X)$ of all non-empty closed bounded subsets of X ,
- $\text{dist}_{\mathcal{GH}}$ is a distance function on the collection of all isometry classes of compact metric spaces,
- $\text{dist}_{\mathcal{GH}_*}$ is a distance function on the collection of all isometry classes of pointed proper metric spaces.

See [11] and the many references mentioned therein.

Both Gromov-Hausdorff distance and pointed Gromov-Hausdorff distance are quantitatively equivalent to the existence of certain so-called rough isometries between the two spaces in question. Our terminology here is adopted from [6]. A map $f : X \rightarrow Y$ between metric spaces is an ε -rough isometric embedding provided

$$\forall a, b \in X, \quad |a - b| - \varepsilon \leq |f(a) - f(b)| \leq |a - b| + \varepsilon;$$

if in addition we also have

$$Y \subset N(fX; \varepsilon) \quad (\text{or equivalently, } \text{dist}_{\mathcal{H}}(Y, fX) < \varepsilon)$$

then we call f an ε -rough isometry. When $a \in A \subset X$ and $b \in B \subset Y$, we call f an ε -rough isometry from $(A; a)$ to $(B; b)$ provided $f : A \rightarrow Y$ is an ε -rough isometric embedding and

$$|f(a) - b| < \varepsilon \quad \text{and} \quad B \subset N(fA; \varepsilon).$$

We note that f need not be continuous, $f(a)$ need not be b , and $f(A)$ need not be a subset nor a superset of B .

The following is from [8, Corollary 7.3.28] and [11, Lemma 2.7].

2.7 Facts. Let $(X; a)$ and $(Y; b)$ be pointed metric spaces.

(a) We always have $\text{dist}_{\mathcal{GH}}(X, Y) \simeq \inf\{\varepsilon > 0 \mid \exists \text{ an } \varepsilon\text{-rough isometry } X \rightarrow Y\}$, with the comparability constant 2.

(b) Similarly, if $\text{dist}_{\mathcal{GH}_*}((X; a), (Y; b)) < t < 1/2$, then there is a map $B[a; t^{-1}] \xrightarrow{f} Y$ that is a $2t$ -rough isometry from $(B[a; t^{-1}]; a) \subset (X; a)$ to $(B[b; t^{-1} - 2t]; b) \subset (Y; b)$. Conversely, if $R > \varepsilon > 0$ and f is an ε -rough isometry from $(B[a; R]; a)$ to $(B[b; R - \varepsilon]; b)$, then $\text{dist}_{\mathcal{GH}_*}((X; a), (Y; b)) < \max\{2\varepsilon, (R - \varepsilon)^{-1}\}$.

For future reference we record the following information.

2.8 Facts.

(a) From Fact 2.6(d), $\text{dist}_{\mathcal{H}}^X(A, B) = \|\delta_A - \delta_B\|_\infty^X$ where $\delta_A(x) := \text{dist}(x, A)$.

(b) If $X \supset A \xrightarrow{f, g} Y$ and $\|f - g\|_\infty^A < \varepsilon$, then $\text{dist}_{\mathcal{H}}^Y(fA, gA) < \varepsilon$.

(c) If (A_i) is a sequence of non-empty sets in X , and δ_{A_i}, δ_A denote the distances to A_i, A respectively, then $\text{dist}_{\mathcal{H}}(A_i, A) \rightarrow 0 \iff \delta_{A_i} \rightarrow \delta_A$ uniformly in X .

(d) A K -Lipschitz map $X \xrightarrow{f} Y$ induces a K -Lipschitz map $\mathcal{H}(X) \xrightarrow{A \mapsto f(A)} \mathcal{H}(Y)$. Thus,

$$\forall A, B \subset B[0; R] \subset \mathbb{R}^n, \quad \frac{2}{1 + R^2} \text{dist}_{\mathcal{H}}(A, B) \leq \hat{\text{dist}}_{\mathcal{H}}(A, B) \leq 2 \text{dist}_{\mathcal{H}}(A, B)$$

where $\text{dist}_{\mathcal{H}}$ and $\hat{\text{dist}}_{\mathcal{H}}$ denote Hausdorff distance in $\mathbb{R}^n$ and in $\hat{\mathbb{R}}^n$ respectively.

(e) For any metric space X , both maps $\mathcal{H}(X) \xrightarrow{\text{diam}} \mathbb{R}$ and $\mathcal{H}(X) \xrightarrow{\text{cdiam}} \mathbb{R}$ are 2-Lipschitz.

(f) If $Z \xrightarrow{f} X$ is a K -Lipschitz surjection, then for all $a \in X, b \in Y, c \in f^{-1}(a)$,

$$\text{dist}_{\mathcal{GH}_*}((Z; c), (Y; b)) \leq K \text{dist}_{\mathcal{GH}_*}((X; a), (Y; b)).$$

2.4 Conformal Metrics

A continuous function $X \xrightarrow{\rho} (0, \infty)$ on a rectifiably connected metric space X induces a length distance d_ρ on X defined by

$$d_\rho(a, b) := \inf_{\gamma: a \curvearrowright b} \ell_\rho(\gamma) \quad \text{where} \quad \ell_\rho(\gamma) := \int_\gamma \rho \, ds$$

and where the infimum is taken over all rectifiable paths $\gamma : a \curvearrowright b$ in X . We describe this by calling $\rho \, ds = \rho(x)|dx|$ a *conformal metric* on X ; then $\rho \, ds$ is *complete* if d_ρ is complete.

We call γ a ρ -geodesic if $d_\rho(a, b) = \ell_\rho(\gamma)$; these need not be unique. We often write $[a, b]_\rho$ to indicate a ρ -geodesic with endpoints a, b , but one must be careful with this notation since these geodesics need not be unique. When we have a point x on a given fixed geodesic $[a, b]_\rho$, we write $[a, x]_\rho$ to mean the subarc of the given geodesic from a to x .

A simple but nonetheless important example is given by $\rho = 1$; here we get the *intrinsic length distance* sometimes called the *inner length distance*. $l = l_X := d_1$ associated to the given distance on X , and then $X_l := (X, l)$. The l -geodesics are shortest paths in X , and X is a *length space* if $(X, |\cdot|) = (X, l)$.

We are primarily interested in the quasihyperbolic metric, discussed below, but also consider other conformal metrics on domains $\Omega \subset \hat{\mathbb{R}}^n$. In this setting, if Ω contains the point at infinity, we must use local coordinates and remember that we are dealing with a metric; alternatively, we can work with the chordal (or the spherical) metric on Ω .

The *metric ratio* $\rho \, ds / \sigma \, ds$ of two conformal metrics $\rho \, ds$ and $\sigma \, ds$ is a well-defined positive function. We write $\rho \leq C \sigma$ to indicate that this metric ratio is bounded above by C .

2.4.1 QuasiHyperbolic Metrics

As mentioned in the Introduction, Ω is a *quasihyperbolic metric space* if it is non-complete, locally complete, and rectifiably connected. Each such Ω supports a *quasihyperbolic metric* $\delta^{-1} \, ds$ where $\delta(x) = \delta_{\partial\Omega}(x) := \text{dist}(x, \partial\Omega)$ and $\partial\Omega := \bar{\Omega} \setminus \Omega$ is the metric boundary of Ω . The length distance, $k = k_\Omega := d_{\delta^{-1}}$, is called *quasihyperbolic distance in Ω* and the length space (Ω, k) is the *quasihyperbolization of Ω* . In fact:

The quasihyperbolization (Ω, k) of a quasihyperbolic space $(\Omega, |\cdot|)$ is a complete length space; thus (Ω, k) is proper and geodesic when it is locally compact (which holds, e.g., if (Ω, l) is locally compact).

The above seems to be not well known, so we explain why (Ω, k) is complete; we reason as in [4, Proposition 2.8] but here we do not assume local compactness nor that the identity map $(\Omega, |\cdot|) \rightarrow (\Omega, l)$ is a homeomorphism. We check that each Cauchy sequence (x_n) in (Ω, k) subconverges in (Ω, k) . As Cauchy sequences are bounded, there are a point $o \in \Omega$ and $M \in (0, \infty)$ such that $k(x_n, o) \leq M$ for all n . Then from (2.11b) we obtain

$$0 < c := e^{-M} \delta(o) \leq \delta(x_n) \leq e^M \delta(o) \quad (2.9)$$

and

$$(x_m, x_n) \leq e^M \delta(o) (e^{k(x_m, x_n)} - 1). \quad (2.10)$$

It follows that (x_n) is also Cauchy in (Ω, l) , so there exists a point ξ in the metric completion of (Ω, l) such that $l(x_n, \xi) \rightarrow 0$. As $\delta_l(x_n) \geq \delta(x_n) \geq c > 0$ for all n , and $\delta_l(x_n) \rightarrow \delta_l(\xi)$, we see that $\xi \in \partial_l \Omega$, so $\xi \in \Omega$. Finally, as (Ω, l) and (Ω, k) are homeomorphic (see [4, Lemma A.4]) we deduce that $k(x_n, \xi) \rightarrow 0$.

We wish to extend this definition to domains Ω in an ambient metric space X . With this in mind, we declare X to be a *quasihyperbolic superspace* provided X is complete, connected, and locally rectifiably connected. For example, each complete length space is a quasihyperbolic superspace, as is each complete rectifiably connected space X with the identity map $X_l \rightarrow X$ a homeomorphism.

When X is a quasihyperbolic superspace, each non-empty open connected $\Omega \subsetneq X$ is a quasihyperbolic space, $\partial\Omega$ is isometrically equivalent to $\text{bd}(\Omega) \subset X$, and $\delta_{\partial\Omega} = \delta_{\text{bd}(\Omega)}$.

2.4.2 Estimates for QuasiHyperbolic Distance

Let X be a quasihyperbolic superspace, $A \subsetneq X$ a non-empty closed subspace, and Ω a connected component of $X \setminus A$. Then

$$\text{bd}(\Omega) \subset A \subset X \setminus \Omega, \quad \text{so } \forall x \in X, \quad \text{dist}(x, \text{bd}(\Omega)) \geq \text{dist}(x, A) \geq \text{dist}(x, X \setminus \Omega).$$

Either inequality above may be strict; if X is a length space, equality holds for all $x \in \Omega$.

Let $\Omega \xrightarrow{\delta} [0, +\infty)$ denote any one of $\delta_{\partial\Omega}$, δ_A , $\delta_{X \setminus \Omega}$ and consider the conformal metric $\delta^{-1}ds$ on Ω with its associated length distance $d := d_{\delta^{-1}}$.³ It is straightforward to verify that

$$\forall \text{ rectifiable } \gamma \text{ in } \Omega, \quad \ell_d(\gamma) = \ell_{\delta^{-1}}(\gamma) \geq \log \left(1 + \frac{\ell(\gamma)}{\inf_{x \in |\gamma|} \delta(x)} \right) \quad (2.11a)$$

and thus for all $a, b \in \Omega$,

$$d(a, b) \geq \log \left(1 + \frac{l(a, b)}{\delta(a) \wedge \delta(b)} \right) \geq j(a, b) := \log \left(1 + \frac{|a - b|}{\delta(a) \wedge \delta(b)} \right) \geq \left| \log \frac{\delta(a)}{\delta(b)} \right|; \quad (2.11b)$$

here $l(a, b)$ is the (intrinsic) length distance (in Ω) between a and b . For Euclidean quasihyperbolic spaces, the above basic estimates were established by Gehring and Palka; see [9, 2.1] and also [4, (2.3), (2.4)].

2.4.3 Ferrand & Kulkarni-Pinkhall Metrics

These metrics, denoted φds and μds respectively, are defined on each quasihyperbolic domain Ω in $\hat{\mathbb{R}}^n$; thus, Ω is an open connected subspace of $\hat{\mathbb{R}}^n$ with $\Omega^c := \hat{\mathbb{R}}^n \setminus \Omega$ containing at least two points.

The *Ferrand metric* $\varphi ds = \varphi_{\Omega} ds$ on Ω can be defined, for points $x \in \Omega \cap \mathbb{R}^n$, by

$$\varphi(x) = \varphi_{\Omega}(x) := \sup_{a, b \in \mathbb{R}^n \setminus \Omega} \frac{|a - b|}{|x - a||x - b|}. \quad (2.12)$$

The *Kulkarni-Pinkall-Thurston metric* $\mu ds = \mu_{\Omega} ds$ on Ω can be defined, for $x \in \Omega \cap \mathbb{R}^n$, by

$$\mu(x) = \mu_{\Omega}(x) := \inf \{ \lambda_B(x) \mid x \in B \subset \Omega, \ B \text{ a Möbius ball} \} \quad (2.13)$$

where λ_B denotes the metric density for the hyperbolic metric $\lambda_B ds$ in the ball B . This metric was introduced by Ravi Kulkarni and Ulrich Pinkall in [17].

The Ferrand and Kulkarni-Pinkall-Thurston metrics are Möbius invariant, they are bilipschitz equivalent to each other, and on domains in $\mathbb{R}^n$ bilipschitz equivalent to the quasihyperbolic metric; see (2.14), and also, [12, Cor. 4.6], [15], [14].

There is a method for calculating these metrics that is based on Euclidean diameters and circumdiameters. Recall that for any non-empty bounded set $A \subset \mathbb{R}^n$, there is a unique smallest closed (Euclidean) ball B_A that contains A ; we call B_A the *circumball* about A , and the *circumdiameter* of A is

$$\text{cdiam}(A) := \text{diam}(B_A).$$

Then for each $x \in \Omega \cap \mathbb{R}^n$ (Ω a quasihyperbolic domain in $\hat{\mathbb{R}}^n$),

$$\varphi(x) = \text{diam } J_x(\Omega^c) \quad \text{and} \quad \mu(x) = \text{cdiam } J_x(\Omega^c); \quad (2.14)$$

see [13, 3.3(a)] and [12, Prop. 4.4].

³ If $\delta = \delta_{\partial\Omega}$, d is quasihyperbolic distance in Ω , but this may not hold for the other two metrics.

2.4.4 The Poincaré Hyperbolic

Every hyperbolic plane domain carries a unique metric $\lambda ds = \lambda_\Omega ds$ which enjoys the property that its pull-back $p^*[\lambda ds]$, with respect to any holomorphic universal covering projection $p : D \rightarrow \Omega$, is the hyperbolic metric $\lambda_D(\zeta)|d\zeta| = 2(1 - |\zeta|^2)^{-1}|d\zeta|$ on the unit disk D . In terms of such a covering p , the (Euclidean) metric-density $\lambda = \lambda_\Omega$ of the *Poincaré hyperbolic metric* $\lambda_\Omega ds$ can be determined from

$$\lambda(z) = \lambda_\Omega(z) = \lambda_\Omega(p(\zeta)) = 2(1 - |\zeta|^2)^{-1}|p'(\zeta)|^{-1}.$$

Yet another description is that λds is the unique maximal (or unique complete) metric on Ω that has constant Gaussian curvature -1 . In many cases (but certainly not all), the hyperbolic metric is bi-Lipschitz equivalent to the quasihyperbolic metric.

2.4.5 Metric Conventions

At times it is convenient to work with generalized distance functions that are allowed to take the value $+\infty$. For example, if $\emptyset \neq A \subsetneq \mathbb{R}^n$ is a closed set, then we can consider quasihyperbolic distance (or Ferrand or Kulkarni-Pinkall-Thurston distance) in $\mathbb{R}^n \setminus A$; here the distance between points from different components of $\mathbb{R}^n \setminus A$ is $+\infty$. This causes no difficulties in the definitions of Hausdorff distance, Gromov-Hausdorff distance, or pointed Gromov-Hausdorff distance, although these quantities also could be infinite. Henceforth we allow such distance functions.

To minimize endless repetition, we also tacitly assume that whenever we mention some conformal distance, all necessary conditions on the underlying space hold. As an explicit example, the quasihyperbolic distance $k = k_U$ on $U := X \setminus A$ is given by

$$k(a, b) = k_U(a, b) := \begin{cases} +\infty & \text{if } a, b \text{ lie in different components of } U, \\ k_\Omega(a, b) & \text{if } a, b \text{ lie in the component } \Omega \text{ of } U; \end{cases}$$

here X is assumed to be a quasihyperbolic superspace and $\emptyset \neq A \subsetneq X$ a closed subspace.

Furthermore, we employ the following abbreviated terminology for pointed Gromov-Hausdorff convergence. Given appropriate open subspaces U, U_i of some metric space X , we write

$$\text{dist}_{\mathcal{GH}_*}((U_i, k_i), (U, k)) \rightarrow 0$$

provided each $a \in U$ lies in U_i for all but finitely many i , and

$$\text{dist}_{\mathcal{GH}_*}((U_i, k_i; a), (U, k; a)) \rightarrow 0;$$

here, again, X is a quasihyperbolic superspace, $\emptyset \neq A \subsetneq X$ a closed subspace, k is quasihyperbolic distance in $X \setminus A$, and k_i is quasihyperbolic distance in $X \setminus A_i$ (when this is defined, which will be true for all sufficiently large i).

We adopt similar conventions for other conformal metrics.

3 Convergence of Sets

Throughout this section, X is a metric space, $o \in X$ is a fixed basepoint, and $\hat{d}$ is the chordal distance on $\hat{X}$ as described in §2.2. For the most part, our various assertions are immediate if X is bounded and often we tacitly assume that X is unbounded. Except where explicitly indicated otherwise, we make no additional hypotheses for X .

Our primary goal here is to understand chordal Hausdorff convergence. In §3.2 and §3.3 we examine the related concepts of Carathéodory core convergence (for open sets) and Kuratowski convergence (for closed sets). Numerous examples are presented in §3.4.

3.1 Chordal Hausdorff Convergence

Since the identity inclusion $X \xrightarrow{\text{id}} \hat{X}$ is 1-Lipschitz, $\hat{\text{dist}}_{\mathcal{H}} \leq \text{dist}_{\mathcal{H}}$, and so $\text{dist}_{\mathcal{H}}(A_i, A) \rightarrow 0$ always implies that $\hat{\text{dist}}_{\mathcal{H}}(\hat{A}_i, \hat{A}) \rightarrow 0$. Employing Fact 2.8(c) we can rephrase this as: (δ_{A_i}) converging uniformly to δ_A in X implies that $(\hat{\delta}_{A_i})$ converges uniformly to $\hat{\delta}_A$ in $\hat{X}$. In general, the converse is false.

Using the distance estimates given in (2.3), it is straightforward to see that when $A \subset B[o; R]$ and $B \subset B[o; S]$, $\text{dist}_{\mathcal{H}}(A, B) \leq 4(R+1)(S+1) \hat{\text{dist}}_{\mathcal{H}}(A, B)$. This in turn, with Lemma 2.5, reveals that if $\hat{\text{dist}}_{\mathcal{H}}(\hat{A}_i, \hat{A}) \rightarrow 0$ with A bounded, then $\text{dist}_{\mathcal{H}}(A_i, A) \rightarrow 0$.

Everywhere, in this section, E bounded means that E is a bounded subset of X . Also, in general, A, A_i are non-empty closed sets in X with chordal closures $\hat{A}, \hat{A}_i$ in $\hat{X}$, and then δ, δ_i and $\hat{\delta}, \hat{\delta}_i$ denote the distances to A, A_i in X and to $\hat{A}, \hat{A}_i$ in $\hat{X}$ (respectively).

The notion of local uniform convergence, aka uniform convergence on compact subsets, is too restrictive for our purposes. We say that a sequence (f_i) of functions $f_i : X \rightarrow \mathbb{R}$ converges *boundedly uniformly* in X if and only if it converges uniformly in each bounded subset of X . Evidently, (letting UC denote ‘uniform convergence’)

$$UC \Rightarrow \text{bounded UC} \Rightarrow \text{local UC} \Rightarrow \text{pointwise convergence}$$

with the converses holding (from right to left) if we have equicontinuity and local compactness⁴, properness, X bounded.

We see below in §4.1 that our notion of bounded uniform convergence is most appropriate when discussing pointed Gromov Hausdorff convergence. Now we show that it is necessary for chordal Hausdorff convergence.

3.1 Theorem. *Let $E, E_i \subset \hat{X}$ with $A := \hat{E} \cap X \neq \emptyset \neq A_i := \hat{E}_i \cap X$. Suppose $\hat{\text{dist}}_{\mathcal{H}}(E_i, E) \rightarrow 0$. Then $\delta_{A_i} \rightarrow \delta_A$ boundedly uniformly in X .*

Proof. Let $\delta := \delta_A$ and $\delta_i := \delta_{A_i}$. First we explain why $\delta_i(o) \rightarrow \delta(o)$. Let $\varepsilon > 0$ be given. Select points $a \in A$ and $a_i \in A_i$ with

$$\delta(o) \leq |a| \leq \delta(o) + \frac{\varepsilon}{10} \quad \text{and} \quad \delta_i(o) \leq |a_i| \leq \delta_i(o) + \frac{\varepsilon}{10}.$$

Choose $b_i \in E_i$ with $\hat{d}(b_i, a) \rightarrow 0$. According to (2.4), $b_i \in A_i$ for all sufficiently large i and $|b_i - a| \rightarrow 0$. Picking i_0 so that for all $i \geq i_0$, $|b_i - a| < \varepsilon/10$ we thus obtain

$$\delta_i(o) \leq |b_i| \leq |b_i - a| + |a| \leq \frac{\varepsilon}{10} + \delta(o) + \frac{\varepsilon}{10}$$

and therefore

$$\forall i \geq i_0, \quad \delta_i(o) - \delta(o) < \varepsilon.$$

In particular, this implies that (a_i) is a bounded sequence in X .

Suppose that $\delta(o) - \delta_i(o) < \varepsilon$ for all sufficiently large i were *false*. Then we could find arbitrarily large i such $\delta(o) - \delta_i(o) \geq \varepsilon$, so $r := \delta(o) - \varepsilon/2 > 0$, and for such i we would obtain

$$|a_i| \leq \delta_i(o) + \frac{\varepsilon}{10} \leq \delta(o) - \varepsilon + \frac{\varepsilon}{10} = \delta(o) - \frac{9\varepsilon}{10} < r.$$

For such an i large enough we would also have

$$\hat{\text{dist}}_{\mathcal{H}}(E_i, E) < \hat{\varepsilon} := \frac{r \wedge (\varepsilon/2)}{4(2r+1)(r+1)}.$$

⁴ More correctly, all we need is equicontinuity and separability; a locally compact metric space is separable.

Pick $b \in E$ with $\hat{d}(a_i, b) < \hat{\varepsilon}$. Since $|a_i| \leq r$, Lemma 2.5 tells us that $b \in A$ with $|b| \leq 2r$, whence

$$\hat{\varepsilon} > \hat{d}(a_i, b) \geq \frac{1}{4} \frac{|a_i - b|}{(|a_i| + 1)(|b| + 1)} \geq \frac{1}{4} \frac{|a_i - b|}{(2r + 1)(r + 1)}.$$

It would follow that $|a_i - b| < \varepsilon/2$, which would yield the contradiction

$$\delta(o) \leq |b| \leq |a_i| + |a_i - b| < \delta(o) - \frac{9\varepsilon}{10} + \frac{\varepsilon}{2} < \delta(o).$$

Now we demonstrate that (δ_i) converges uniformly, in each bounded $B \subset X$, to δ . Let $\varepsilon > 0$ be given. Fix i_0 so that for all $i \geq i_0$, $|\delta_i(o) - \delta(o)| < 1$. Since B is bounded, there is an $r > 0$ with $B \subset B[o; r]$; e.g., $r := \text{dist}(o, B) + \text{diam}(B)$ has this property. Put $R := 2r + \delta(o) + 1 + \varepsilon$. Select i_1 so that

$$\forall i \geq i_1, \quad \hat{\text{dist}}_{\mathcal{H}}(E_i, E) < \hat{\varepsilon} := \frac{\varepsilon}{4(2R + 1)^2} < \frac{R}{4(2R + 1)(R + 1)}.$$

Everywhere below we take $i \geq i_0 \vee i_1$.

Let $x \in B$. Pick $a \in A$ and $a_i \in A_i$ with

$$\delta(x) \leq |x - a| \leq \delta(x) + \varepsilon \quad \text{and} \quad \delta_i(x) \leq |x - a_i| \leq \delta_i(x) + \varepsilon.$$

Then

$$\delta(x) \leq |x| + \delta(o) \leq r + \delta(o) \quad \text{and} \quad \delta_i(x) \leq |x| + \delta_i(o) \leq r + \delta(o) + 1$$

and therefore

$$|a| \leq |a - x| + |x| \leq \delta(x) + \varepsilon + r \leq 2r + \delta(o) + \varepsilon \leq R$$

and also

$$|a_i| \leq |a_i - x| + |x| \leq \delta_i(x) + \varepsilon + r \leq 2r + \delta(o) + 1 + \varepsilon = R.$$

Since $a \in A \cap B[o; R] \subset E$ and $a_i \in A_i \cap B[o; R] \subset E_i$, there exist $b_i \in E_i$ and $b \in E$ with $\hat{d}(b_i, a) < \hat{\varepsilon}$ and $\hat{d}(a_i, b) < \hat{\varepsilon}$. From Lemma 2.5 we deduce that $b, b_i \in B[o; 2R]$. Thus $b \in A$, $b_i \in A_i$ and so

$$\begin{aligned} \delta_i(x) - \delta(x) &\leq |x - b_i| - \delta(x) \leq |x - b_i| - |x - a| + \varepsilon \leq |b_i - a| + \varepsilon \\ &\leq 4(2R + 1)^2 \hat{d}(b_i, a) + \varepsilon < 2\varepsilon \end{aligned}$$

and similarly

$$\begin{aligned} \delta(x) - \delta_i(x) &\leq |x - b| - \delta_i(x) \leq |x - b| - |x - a_i| + \varepsilon \leq |a_i - b| + \varepsilon \\ &\leq 4(2R + 1)^2 \hat{d}(a_i, b) + \varepsilon < 2\varepsilon. \end{aligned}$$

□

3.2 Corollary. *Let $\emptyset \neq A, A_i \subset X$. If $\hat{\text{dist}}_{\mathcal{H}}(A_i, A) \rightarrow 0$, then $\delta_{A_i} \rightarrow \delta_A$ boundedly uniformly in X .⁵*

An easy compactness argument, in conjunction with Corollary 3.2, reveals that when $\hat{\text{dist}}_{\mathcal{H}}(\hat{A}_i, \hat{A}) \rightarrow 0$, each compact set in $X \setminus A$ lies in all but finitely many $X \setminus A_i$. However, no such result holds if we replace ‘compact’ with ‘closed and bounded’.

Our next goal is to understand the consequences of (δ_{A_i}) converging boundedly uniformly to δ_A . The following preliminary result is useful.

⁵ Remember that $\hat{\text{dist}}_{\mathcal{H}}(A_i, A) = \hat{\text{dist}}_{\mathcal{H}}(\hat{A}_i, \hat{A})$.

3.3 Lemma. Let $\emptyset \neq A, A_i \subset X$ be closed. Assume (δ_{A_i}) converges boundedly uniformly in X to δ_A . If A is unbounded, then

$$\forall R > 0, \exists i_1 \text{ such that } \forall i \geq i_1, A_i \setminus B[o; R] \neq \emptyset. \quad (3.3a)$$

Regardless of whether A is bounded or unbounded,

$$\forall \varepsilon > 0, \exists i_2 \text{ such that } \forall i \geq i_2, A \subset \begin{cases} N(A_i; \varepsilon) & \text{when } A \text{ is bounded,} \\ N_d(A_i; \varepsilon) & \text{when } A \text{ is unbounded.} \end{cases} \quad (3.3b)$$

Also,

$$\begin{aligned} \forall \varepsilon > 0, \exists i_3 \text{ such that } \forall i \geq i_3, \hat{A}_i \subset N_d(A \cup \{\infty\}; \varepsilon); \text{ in fact} \\ A_i \cap B[o; \varepsilon^{-1}] \subset N(A; \varepsilon) \quad \text{and} \quad \hat{A}_i \setminus B[o; \varepsilon^{-1}] \subset B_d(\infty; \varepsilon). \end{aligned} \quad (3.3c)$$

Proof. Let $\delta := \delta_A$ and $\delta_i := \delta_{A_i}$. To establish (3.3a), assume A is unbounded and let $R > 0$ be given. Pick $a \in A$ with $|a| \geq 10R$. As $\delta_i(a) \rightarrow \delta(a) = 0$, there exists an i_1 such that for all $i \geq i_1$, $\delta_i(a) < R$. Thus for all $i \geq i_1$, there exists an $a_i \in A_i$ with $|a_i - a| < R$, so,

$$|a_i| \geq |a| - |a_i - a| \geq 9R, \quad \text{whence } A_i \setminus B[o; R] \neq \emptyset.$$

To establish (3.3b), let $\varepsilon > 0$ be given. Suppose A is bounded. Then $\delta_i \rightarrow 0$ uniformly in A , so $A \subset N(A_i; \varepsilon)$ for all sufficiently large i . Indeed, if i_0 is such that for all $i \geq i_0$, $|\delta_i - \delta| < \varepsilon$ uniformly in A , then for all $i \geq i_0$ and all $a \in A$

$$\delta_i(a) < \varepsilon \text{ which implies } \exists a_i \in A_i \text{ such that } |a_i - a| < \varepsilon, \quad \text{whence } a \in B(a_i; \varepsilon).$$

Suppose A is unbounded. Put $R := 10/\varepsilon$. Choose i_0 so that for all $i \geq i_0$, $|\delta_i - \delta| < \varepsilon/10$ uniformly in $B[o; R]$. Then for all $i \geq i_0$ and all $a \in A \cap B[o; R]$,

$$\delta_i(a) < \varepsilon \text{ which implies } \exists a_i \in A_i \text{ such that } |a_i - a| < \varepsilon, \quad \text{whence } a \in B(a_i; \varepsilon) \subset B_d(a_i; \varepsilon).$$

Let i_1 be as promised in (3.3a). Suppose $a \in A \setminus B[o; R]$; then $\hat{d}(a, \infty) < \varepsilon/2$. By (3.3a), for each $i \geq i_1$, there is a point $a_i \in A_i \setminus B[o; R]$, so $\hat{d}(a_i, \infty) < \varepsilon/2$ and $a \in B_d(a_i; \varepsilon)$.

Thus, with $i_2 := i_0 \vee i_1$ we see that (3.3b) holds whether A is bounded or unbounded.

To establish (3.3c), let $\varepsilon > 0$ be given. Put $R := 1/\varepsilon$. Choose i_3 so that for all $i \geq i_3$, $|\delta_i - \delta| < \varepsilon$ uniformly in $B[o; R]$. Fix $i \geq i_3$. Then for all $x \in A_i \cap B[o; R]$,

$$\delta(x) < \varepsilon \text{ which implies } \exists a \in A \text{ such that } |x - a| < \varepsilon, \quad \text{whence } x \in B(a; \varepsilon).$$

Thus $A_i \cap B[o; R] \subset N(A; \varepsilon)$. If $x \in A_i \setminus B[o; R]$, then

$$\hat{d}(x, \infty) \leq \frac{1}{|x| + 1} \leq \frac{1}{R + 1} < \varepsilon, \quad \text{so } x \in B_d(\infty; \varepsilon). \quad \square$$

3.4 Corollary. Let $\emptyset \neq A, A_i \subset X$ be closed. Put

$$F := \hat{X} \setminus (X \setminus A) = \begin{cases} A & \text{when } X \text{ is bounded,} \\ A \cup \{\infty\} & \text{when } X \text{ is unbounded,} \end{cases} \quad \text{and } F_i := \hat{X} \setminus (X \setminus A_i).$$

Then $\delta_{A_i} \rightarrow \delta_A$ boundedly uniformly in X if and only if $\hat{\text{dist}}_{\mathcal{H}}(F_i, F) \rightarrow 0$.

Below we describe exactly what it means for (δ_{A_i}) to converge boundedly uniformly to δ_A (when X is unbounded). As motivation for the following technical result, we mention (see Corollary 3.6 below) that

$$\hat{\text{dist}}_{\mathcal{H}}(\hat{A}_i, \hat{A}) \rightarrow 0 \iff \delta_{A_i} \rightarrow \delta_A \text{ boundedly uniformly in } X \text{ and } \hat{\delta}_{A_i}(\infty) \rightarrow \hat{\delta}_A(\infty).$$

We now demonstrate that when (δ_{A_i}) converges boundedly uniformly in X to δ_A :

If A is unbounded, then $\hat{\text{dist}}_{\mathcal{H}}(\hat{A}_i, \hat{A}) \rightarrow 0$. If A is bounded, then either $\hat{\text{dist}}_{\mathcal{H}}(\hat{A}_i, A \cup \{\infty\}) \rightarrow 0$, or $\text{dist}_{\mathcal{H}}(A_i, A) \rightarrow 0$, or A and $A \cup \{\infty\}$ are both (and the *only*) subsequential limits of $(\hat{A}_i)$ in $(\mathcal{H}(\hat{X}), \hat{\text{dist}}_{\mathcal{H}})$.

Here is a more precise statement. See Example 3.22 for the importance of (δ_{A_i}) being boundedly uniformly convergent. Also, recall Lemma 3.3.

3.5 Theorem. Let $\emptyset \neq A, A_i \subset X$ be closed with X unbounded. Assume (δ_{A_i}) converges boundedly uniformly in X to δ_A .

(a) If A is unbounded, then $\hat{\text{dist}}_{\mathcal{H}}(\hat{A}_i, \hat{A}) \rightarrow 0$.

(b) If A is bounded, then:

(b.1) If there exist $R > 0$ and i_1 such that for all $i \geq i_1$, $A_i \subset B[0; R]$, then $\text{dist}_{\mathcal{H}}(A_i, A) \rightarrow 0$.

(b.2) Otherwise:

(b.2.i) $A \cup \{\infty\}$ is a $\hat{\text{dist}}_{\mathcal{H}}$ -subsequential limit point of $(\hat{A}_i)$ ⁶;

(b.2.ii) A and $A \cup \{\infty\}$ are the only possible $\hat{\text{dist}}_{\mathcal{H}}$ -subsequential limits point of $(\hat{A}_i)$; and

(b.2.iii) either $\hat{\text{dist}}_{\mathcal{H}}(\hat{A}_i, A \cup \{\infty\}) \rightarrow 0$ or A is a $\text{dist}_{\mathcal{H}}$ -subsequential limit of (A_i) ⁷.

Proof. Let $\delta := \delta_A$ and $\delta_i := \delta_{A_i}$. Suppose A is unbounded. We prove that $\hat{\text{dist}}_{\mathcal{H}}(\hat{A}_i, \hat{A}) \rightarrow 0$. Let $\varepsilon > 0$ be given. According to Lemma 3.3(b), there exists an i_2 so that for all $i \geq i_2$, $A \subset N_{\hat{A}}(A_i; \varepsilon)$. We must establish a similar containment valid for A_i for all large i .

Put $R := 5/\varepsilon$. Pick i_0 so that for all $i \geq i_0$, $|\delta_i - \delta| < \varepsilon/10$ uniformly in $B[0; R]$. Fix $i \geq i_0$. Let $b \in A_i \cap B[0; R]$. Then $\delta_i(b) = 0$, so $\delta(b) < \varepsilon/10$ and there exists a $c \in A$ with $|b - c| < \varepsilon$; therefore $b \in B(c; \varepsilon) \subset B_{\hat{A}}(c; \varepsilon)$. Let $b \in A_i \setminus B[0; R]$. Then $\hat{d}(b, \infty) \leq 1/(|b|+1) \leq 1/R < \varepsilon/2$. As A is unbounded, there exists a point $a \in A \setminus B[0; R]$, so $\hat{d}(a, \infty) < \varepsilon/2$ and $b \in B_{\hat{A}}(a; \varepsilon)$.

Thus for all $i \geq i_0 \vee i_2$, $\hat{\text{dist}}_{\mathcal{H}}(\hat{A}_i, \hat{A}) < \varepsilon$.

Assume A is bounded. Suppose there are $R > 0$ and i_1 so that for all $i \geq i_1$, $A_i \subset B[0; R]$. That $\text{dist}_{\mathcal{H}}(A_i, A) \rightarrow 0$ now follows from (3.3b) and (3.3c). Now suppose no such R exists. Then there exist a subsequence (A_{i_j}) and points $a_{i_j} \in A_{i_j}$ with $|a_{i_j}| \geq j$ for all $j \in \mathbb{N}$ (so, $\hat{d}(a_{i_j}, \infty) \rightarrow 0$ as $j \rightarrow \infty$). We claim that $\hat{\text{dist}}_{\mathcal{H}}(\hat{A}_{i_j}, A \cup \{\infty\}) \rightarrow 0$. Let $\varepsilon > 0$ be given. Put $R := 5/\varepsilon$. Pick j_0 so that for all $i \geq j_0$, $|\delta_i - \delta| \leq \varepsilon/10$ uniformly in $B[0; R]$.

Fix $j \geq j_0$. Let $b \in \hat{A}_{i_j}$. If $b \in B[0; R]$, then $\delta(b) < \varepsilon$ so there is an $a \in A$ with $b \in B(a; \varepsilon) \subset B_{\hat{A}}(a; \varepsilon)$. If $b \notin B[0; R]$, then $\hat{d}(b, \infty) < \varepsilon$ so $b \in B_{\hat{A}}(\infty; \varepsilon)$. Thus $\hat{A}_{i_j} \subset N_{\hat{A}}(A \cup \{\infty\}; \varepsilon)$.

Since $\hat{d}(a_{i_j}, \infty) \rightarrow 0$, there is a j_1 so that for all $j \geq j_1$, $\infty \in B_{\hat{A}}(\hat{A}_{i_j}; \varepsilon)$. By Lemma 3.3(b), there is a j_2 so that for all $i \geq j_2$, $A \subset N_{\hat{A}}(\hat{A}_i; \varepsilon)$. Thus for $j \geq j_1 \vee j_2$, $A \cup \{\infty\} \subset N_{\hat{A}}(\hat{A}_{i_j}; \varepsilon)$.

We conclude that for all $j \geq j_0 \vee j_1 \vee j_2$, $\hat{\text{dist}}_{\mathcal{H}}(\hat{A}_{i_j}, A \cup \{\infty\}) < \varepsilon$.

Next we explain why A and $A \cup \{\infty\}$ are the *only possible* subsequential limits for $(\hat{A}_i)$ (as a sequence in $(\mathcal{H}(\hat{X}), \hat{\text{dist}}_{\mathcal{H}})$). To this end, suppose $B \in \mathcal{H}(\hat{X})$ (i.e., B is a non-empty closed subset of $\hat{X}$) and there exists a subsequence $(\hat{A}_{h_k})_{k=1}^{\infty}$ of $(\hat{A}_h)_{h=1}^{\infty}$ with $\hat{\text{dist}}_{\mathcal{H}}(\hat{A}_{h_k}, B) \rightarrow 0$. Given $a \in A$, there are $a_h \in A_h$ with $|a_h - a| \rightarrow 0$ (since $\delta(a) = 0$). As $\hat{\text{dist}}_{\mathcal{H}}(\hat{A}_{h_k}, B) \rightarrow 0$, it follows that $a \in B$; so $A \subset B$.

Let $b \in B$. Assume $b \neq \infty$. Pick $b_k \in \hat{A}_{h_k}$ with $\hat{d}(b_k, b) \rightarrow 0$. Since $b \in X$, (2.4) tells us that for all sufficiently large k , $b_k \in A_{h_k}$ and $|b_k - b| \rightarrow 0$. It follows that

$$\delta(b) = \lim_{h \rightarrow \infty} \delta_h(b) = \lim_{k \rightarrow \infty} \delta_{h_k}(b) = 0$$

and therefore $b \in A$. Thus $A \subset B \subset A \cup \{\infty\}$.

⁶ Here we are regarding $(\hat{A}_i)$ as a sequence in the metric space $(\mathcal{H}(\hat{X}), \hat{\text{dist}}_{\mathcal{H}})$.

⁷ Here we are regarding (A_i) as a sequence in the metric space $(\mathcal{H}(X), \text{dist}_{\mathcal{H}})$.

Finally, suppose $\hat{\text{dist}}_{\mathcal{H}}(\hat{A}_i, A \cup \{\infty\}) \rightarrow 0$. We explain why (A_i) has a subsequence that $\text{dist}_{\mathcal{H}}$ -converges to A . There exist an $\varepsilon > 0$ and a subsequence (A_{i_j}) such that for all j , $\hat{\text{dist}}_{\mathcal{H}}(\hat{A}_{i_j}, A \cup \{\infty\}) \geq \varepsilon$. Put $R := 1/\varepsilon$. Since $\delta_i \rightarrow \delta$ uniformly in $B[o; R]$, there is a j_0 such that for all $i \geq j_0$, $|\delta_i - \delta| < \varepsilon$ uniformly in $B[o; R]$.

Since A is bounded, we can appeal to (3.3b) and (3.3c) to obtain a j_1 so that for all $i \geq j_1$,

$$A \subset N(A_i; \varepsilon) \quad \text{and} \quad \hat{A}_i \subset N_{\hat{d}}(A \cup \{\infty\}; \varepsilon).$$

Fix $j \geq j_0 \vee j_1$. Then $\hat{\text{dist}}_{\mathcal{H}}(\hat{A}_{i_j}, A \cup \{\infty\}) \geq \varepsilon$, so by the above we must have $\infty \in N_{\hat{d}}(A_{i_j}; \varepsilon)$. Therefore, for all $x \in A_{i_j}$,

$$\varepsilon \leq \hat{d}(x, \infty) \leq \frac{1}{|x| + 1} \quad \text{so, } |x| \leq \varepsilon^{-1} - 1 \leq R.$$

Thus $A_{i_j} \subset B[o; R]$, and so by a previous case we know that $\lim_{j \rightarrow \infty} \text{dist}_{\mathcal{H}}(A_{i_j}, A) = 0$. $\square$

In light of Theorem 3.5, it is convenient to say that (A_i) has the uniform bounded property (UBP) with respect to A if

$$A \text{ bounded} \Rightarrow \exists R, \exists i_0 \text{ so that } \forall i \geq i_0, A_i \subset B[o; R].$$

Here is a summary of parts of the above. Again, recall Lemma 3.3.

3.6 Corollary. For $\emptyset \neq A, A_i \subset X$ with X unbounded, $\delta := \delta_A, \delta_i := \delta_{A_i}$, the following are equivalent.

- (a) $\hat{\text{dist}}_{\mathcal{H}}(\hat{A}_i, \hat{A}) \rightarrow 0$.
- (b) $\hat{\delta}_i \rightarrow \hat{\delta}$ uniformly in $\hat{X}$.
- (c) $\delta_i \rightarrow \delta$ boundedly uniformly in X and $\hat{\delta}_i(\infty) \rightarrow \hat{\delta}(\infty)$.
- (d) $\delta_i \rightarrow \delta$ boundedly uniformly in X and (A_i) has UBP with respect to A .

Proof. That (a) $\iff$ (b) $\implies$ (c) follows from Fact 2.8(c) and Corollary 3.2. That (d) $\implies$ (a) follows from Theorem 3.5. To see that (c) $\implies$ (d), suppose $\hat{\delta}_i(\infty) \rightarrow \hat{\delta}(\infty)$ and A is bounded. Then $r := \hat{\delta}(\infty) > 0$. Pick i_0 so that for all $i \geq i_0$, $|\hat{\delta}_i(\infty) - \hat{\delta}(\infty)| < r/2$. Then for such i , and all $x \in A_i$,

$$\frac{r}{2} \leq \hat{\delta}_i(\infty) \leq \hat{d}(x, \infty) \leq \frac{1}{|x| + 1}, \quad \text{so } |x| \leq \frac{2}{r} - 1. \quad \square$$

3.7 Remarks. (a) Notice that 3.6(d) above characterizes chordal Hausdorff convergence solely in terms of the distance on X . (b) Taking $X := \mathbb{R}, A := \{0\}$, and $A_i := \{0, i\}$ we see that each A_i is a bounded set and $\delta_{A_i} \rightarrow \delta_A$ boundedly uniformly in X , but clearly (A_i) does not converge to A with respect to chordal Hausdorff distance on X . (c) We mention that for any $\emptyset \neq A_i \subset X$:

$$\hat{\text{dist}}_{\mathcal{H}}(A_i, \{\infty\}) \rightarrow 0 \iff \delta_i(o) \rightarrow +\infty \iff \delta_i \rightarrow +\infty \text{ boundedly uniformly in } X.$$

In proper spaces we only require pointwise convergence (plus UBP) to obtain chordal Hausdorff convergence.

3.8 Proposition. Assume X is a proper metric space. The following are equivalent:

- (a) $\hat{\text{dist}}_{\mathcal{H}}(\hat{A}_i, \hat{A}) \rightarrow 0$.
- (b) $\delta_i \rightarrow \delta$ pointwise in X and (A_i) has UBP with respect to A .
- (c) $\delta_i \rightarrow \rho$ pointwise in X for some $X \xrightarrow{p} \mathbb{R}$ with $\rho^{-1}(0) = A$, and (A_i) has UBP with respect to A .

Proof. That (a) $\implies$ (b) follows from Corollary 3.6, and clearly (b) $\implies$ (c) (with $\rho = \delta$). Assume (c) holds. Since distance to a set is a 1-Lipschitz function, each δ_i is 1-Lipschitz, so (δ_i) is equicontinuous and therefore we can apply Arzela-Ascoli (see for example [21]) to (δ_i) and each of its subsequences. In particular, by uniqueness of limits, (δ_i) converges locally uniformly—and hence as X is proper, boundedly uniformly—in X to ρ .

To demonstrate that $\hat{\text{dist}}_{\mathcal{H}}(\hat{A}_i, \hat{A}) \rightarrow 0$, we show that every subsequence of $(\hat{A}_i)$ has a further subsequence that converges to $\hat{A}$ with respect to $\hat{\text{dist}}_{\mathcal{H}}$. Let $(\hat{A}_{i_j})$ be such a subsequence. Since $\mathcal{K}(\hat{X})$ is compact, there is a subsubsequence $(\hat{A}_{i_{j_h}})$ and a non-empty compact set $\hat{E}$ in $\hat{X}$ such that $\hat{\text{dist}}_{\mathcal{H}}(\hat{A}_{i_{j_h}}, \hat{E}) \rightarrow 0$.

Appealing to Remark 3.7(c) we see that $E := \hat{E} \cap X \neq \emptyset$. Then by Theorem 3.1 we see that $\delta_{i_k} \rightarrow \delta_E$ locally uniformly in X . Again by uniqueness of limits, $\rho = \delta_E$ and so $E = \delta_E^{-1}(0) = \rho^{-1}(0) = A$. That $\hat{\text{dist}}_{\mathcal{H}}(\hat{A}_{i_k}, \hat{A}) \rightarrow 0$ holds now follows from Corollary 3.6. $\square$

3.9 Remarks. **(a)** In the above, properness of X is primarily used to promote pointwise convergence to boundedly uniform convergence. **(b)** There are locally compact metric spaces where neither (b) nor (c) implies (a). See Example 3.19. **(c)** Below, in §3.2, we present a topological characterization for the pointwise convergence $\delta_{A_i} \rightarrow \delta_A$; see Proposition 3.10(b).

3.2 Carathéodory Convergence

Now we shift our focus to open sets (and initially X need only be a topological space). Let (U_i) be a sequence of open sets $U_i \subset X$. Following [3], we define the *core* of (U_i) via

$$\text{core}(U_i) := \bigcup_{i \geq 1} \text{int} \left(\bigcap_{j \geq i} U_j \right);$$

this is always an open set but may be empty. We say that (U_i) *Carathéodory converges to its core* if and only if every subsequence (U_{i_j}) of (U_i) has the same core. Beardon and Minda introduced the notion of Carathéodory core convergence and examined its relation to Carathéodory kernel convergence; see [3].

Since $\text{core}(U_{i_j}) \supset \text{core}(U_i)$ always holds, it is trivial that $\text{core}(U_i) = X$ implies (U_i) converges to its core. Therefore if infinitely many U_i equal X , then (U_i) converges to its core if and only if $\text{core}(U_i) = X$. Thus it is natural to assume that for all i , $U_i \neq X$, and that $\text{core}(U_i) \neq X$.

Now suppose that $|\cdot|$ is a compatible distance function on X , so the metric topology agrees with the original topology. Here is some information relating $\text{core}(U_i)$ and the convergence of (U_i) to the behavior of the sequence of distances to $X \setminus U_i$.

3.10 Proposition. *Let $U, U_i \subsetneq X$ be open sets; $A := X \setminus U$, $A_i := X \setminus U_i$, $\delta := \delta_A$, $\delta_i := \delta_{A_i}$.*

(a) *If $\liminf \delta_i = \delta$ pointwise in X , then $U = \text{core}(U_i)$. If $\text{core}(U_i) = U$, then $\liminf \delta_i \leq \delta$ with equality in A ; when X is proper, equality holds in X , but in general it need not hold.*

(b) *If $\lim \delta_i = \delta$ pointwise in X , then $U = \text{core}(U_i)$ and (U_i) converges to its core. If (U_i) converges to its core and $\text{core}(U_i) = U$, then $\lim \delta_i = 0$ pointwise in A ; when X is proper, $\lim \delta_i = \delta$ pointwise (so, locally uniformly) in X , but in general it need not hold even if $\lim \delta_i$ exists.*

Proof. **(a)** Suppose $\liminf \delta_i = \delta$ pointwise in X . Let $x \in \text{core}(U_i)$. Then there are $i \geq 1$ and $r > 0$ so that $B(x; r) \subset \bigcap_{j \geq i} U_j$, so $\delta(x) = \liminf \delta_i(x) \geq r$ and therefore $x \in U$.

Let $x \in U$. Fix $r \in (0, \delta(x))$. Then $\liminf \delta_i(x) = \delta(x) > r$ means there is an i such that for all $j \geq i$, $\delta_j(x) > r$, so $B(x; r) \subset \bigcap_{j \geq i} U_j$ and $x \in \text{core}(U_i)$. Thus $U \subset \text{core}(U_i)$.

Now suppose $\text{core}(U_i) = U$. Let $x \in X$. We may assume $L := \liminf \delta_i(x) > 0$. Fix $r \in (0, L)$. Pick i so that for all $j \geq i$, $\delta_j(x) > r$. Then $B(x; r) \subset \bigcap_{j \geq i} U_j \subset \text{core}(U_i) = U$. Therefore, $\delta(x) \geq r$. As this holds for all $r \in (0, L)$, $\delta(x) \geq L$.

Evidently, equality holds at points in A . Assume X is proper, let $x \in X \setminus A = U$, and let $r \in (0, \delta(x))$. Since $B[x; r]$ is compact, it lies in all but finitely many U_i , so $B[x; r] \cap A_i = \emptyset$ for all but finitely many i , whence $\liminf \delta_i(x) \geq r$.

(b) Suppose $\lim \delta_i = \delta$ pointwise in X . By part (a), $\text{core}(U_i) = U$. Also, for any subsequence (U_{i_j}) , $\delta_{i_j} \rightarrow \delta$, so again by part (a), $\text{core}(U_{i_j}) = U = \text{core}(U_i)$.

Now suppose (U_i) converges to its core, $\text{core}(U_i) = U$, and let $a \in A$. For each i , pick $a_i \in A_i$ so that $\delta_i(a) \leq |a_i - a| \leq \delta_i(a) + 1/i$. Next, select (a_{i_j}) so that

$$\limsup_{i \rightarrow \infty} |a_i - a| = \lim_{j \rightarrow \infty} |a_{i_j} - a|.$$

Since $\text{core}(U_{i_j}) = \text{core}(U_i) = U$, the second part of (a) now lets us assert that

$$\begin{aligned} \limsup_{i \rightarrow \infty} \delta_i(a) &= \limsup_{i \rightarrow \infty} |a_i - a| = \lim_{j \rightarrow \infty} |a_{i_j} - a| = \liminf_{j \rightarrow \infty} |a_{i_j} - a| \\ &\leq \liminf_{j \rightarrow \infty} (\delta_{i_j}(a) + 1/i_j) = \liminf_{j \rightarrow \infty} \delta_{i_j}(a) = \delta(a) = 0 \end{aligned}$$

which shows that $\lim \delta_i(a) = 0$.

Finally, assume X is proper. Let $x \in X$. Let $(\delta_{i_j}(x))$ be an arbitrary subsequence of $(\delta_i(x))$. Since $U = \text{core}(U_i) = \text{core}(U_{i_j})$, part (a) tells us that $\liminf_{j \rightarrow \infty} \delta_{i_j}(x) = \delta(x)$, so there is a subsubsequence $(\delta_{i_{j_k}}(x))$ that converges to $\delta(x)$. $\square$

Examples 3.20 and 3.21 illustrate how the asserted equalities in the second parts of both Proposition 3.10(a,b) may fail. Notice that in Example 3.20 X is not a length space (and not proper) while in Example 3.21 X is not locally compact (and not proper).

It is useful to examine Proposition 3.10 in the setting when X is a length space; here we know that for any component Ω of an open $U \subsetneq X$, $\delta_{\partial\Omega} = \delta_{\partial U} = \delta_{X \setminus U}$ in Ω . As in [3], a component Ω of $U := \text{core}(U_i)$ is a *limit region* if and only if Ω is a component of $\text{core}(U_{i_j})$ for every subsequence (U_{i_j}) of (U_i) . Adapting the ideas in the proof of Proposition 3.10 we deduce the following, which is part of the *Quasihyperbolic Kernel Theorem* in [3].

3.11 Corollary. *Let X be a complete locally compact length space. Let $U_i \subsetneq X$ be open. Suppose $\emptyset \neq U := \text{core}(U_i) \subsetneq X$. Let Ω be a component of U . Then Ω is a limit region for (U_i) if and only if $\delta_{\partial U_i} \rightarrow \delta_{\partial\Omega}$ pointwise in Ω .*

Here is a consequence of Theorem 3.1 in conjunction with Proposition 3.10(b).

3.12 Corollary. *Let $A, A_i \subsetneq X$ be closed. Put $U := X \setminus A$, $U_i := X \setminus A_i$,*

$$F := \hat{X} \setminus U = \begin{cases} A & \text{when } X \text{ is bounded,} \\ A \cup \{\infty\} & \text{when } X \text{ is unbounded,} \end{cases} \quad \text{and } F_i := \hat{X} \setminus U_i.$$

If $\hat{\text{dist}}_{\mathcal{H}}(A_i, A) \rightarrow 0$, then (U_i) converges to its core $U = \text{core}(U_i)$. Conversely, if (U_i) converges to its core $U = \text{core}(U_i)$. Then:

*F is the only possible $\hat{\text{dist}}_{\mathcal{H}}$ -subsequential limit of (F_i)
 $\hat{A}, F$ are the only possible $\hat{\text{dist}}_{\mathcal{H}}$ -subsequential limits of $(\hat{A}_i)^g$.*

Proof. Suppose there are $E \in \mathcal{H}(\hat{X})$ and a subsequence (F_{i_j}) of (F_i) such that $\hat{\text{dist}}_{\mathcal{H}}(F_{i_j}, E) \rightarrow 0$. According to Theorem 3.1, $\delta_{A_{i_j}} \rightarrow \delta_B$ boundedly uniformly in X , where $B := E \cap X$. Therefore Proposition 3.10(b) tells us that

$$X \setminus B = \text{core}(U_{i_j}) = \text{core}(U_i) = U = X \setminus A, \quad \text{so } B = A.$$

If X is bounded, then $E = B = A = F$. If X is unbounded, then ∞ belongs to F and to each F_i , so also $\infty \in E$ and hence $E = B \cup \{\infty\} = A \cup \{\infty\} = F$.

If $E \in \mathcal{H}(\hat{X})$ and $\hat{\text{dist}}_{\mathcal{H}}(A_i, E) \rightarrow 0$, we argue similarly but now it is possible that $\infty \notin E$ and then $E = B = A = \hat{A}$ (because E is bounded in X). $\square$

Next we provide information relating $\text{core}(U_i)$ and the core convergence of (U_i) to the existence of certain sequences (a_i) with $a_i \in A_i$.

3.13 Lemma. *Let $U_i \subset X$ be open, $U := \text{core}(U_i)$, $A := X \setminus U$, $A_i := X \setminus U_i$. Suppose that $A \neq \emptyset \neq A_i$.*

⁸ Here we are regarding $(\hat{A}_i)$ and (F_i) as sequences in the metric space $(\mathcal{H}(\hat{X}), \text{dist}_{\mathcal{H}})$.

- (a) For each $a \in A$ there is a sequence (a_i) , with $a_i \in A_i$ for all i , that subconverges to a . If (U_i) converges to $U = \text{core}(U_i)$, then (a_i) converges to a .
- (b) All subsequential limits of any sequence (a_i) , with $a_i \in A_i$ for all i , lie in A .

3.3 Kuratowski Convergence

Let $\mathcal{C}$ be an infinite collection of non-empty subsets of a topological space X . Following Kuratowski and Whyburn (see [18], [28], [27]), we define

$$\bar{\mathcal{C}} = \limsup \mathcal{C} := \{x \in X \mid \forall \text{ open } U \ni x, U \cap S \neq \emptyset \text{ for infinitely many } S \in \mathcal{C}\}$$

and

$$\underline{\mathcal{C}} = \liminf \mathcal{C} := \{x \in X \mid \forall \text{ open } U \ni x, U \cap S \neq \emptyset \text{ for all but finitely many } S \in \mathcal{C}\}.$$

It is not difficult to check that both $\bar{\mathcal{C}}$ and $\underline{\mathcal{C}}$ are closed sets, and when $\mathcal{B} \subset \mathcal{C}$ is infinite, $\underline{\mathcal{C}} \subset \underline{\mathcal{B}} \subset \bar{\mathcal{B}} \subset \bar{\mathcal{C}}$. When $\underline{\mathcal{C}} = S = \bar{\mathcal{C}}$, we say that $\mathcal{C}$ converges to S . We use similar notation and terminology for a sequence of non-empty subsets of X .

In particular, a sequence (A_i) of non-empty closed sets in X Kuratowski converges to a closed set A provided $\liminf A_i = A = \limsup A_i$.

There is a direct connection with this notion of set convergence and that studied in §3.2.

3.14 Lemma. Let (U_i) be a sequence of open sets $U_i \subsetneq X$. Put $A_i := X \setminus U_i$. Then:

- (a) We always have $\text{core}(U_i) = X \setminus \limsup A_i$, and
- (b) (U_i) Carathéodory converges to its core if and only if (A_i) Kuratowski converges to $A := X \setminus \text{core}(U_i)$.

Proof. It is straightforward, using just definitions, to verify (a). That Kuratowski convergence implies Carathéodory core convergence, as stated in (b), now follows. To complete (b), assume that (A_i) does not converge; so, there is a point $x \in \limsup A_i \setminus \liminf A_i$. This means that there is an open neighborhood V of x and $i_1 < i_2 < \dots$ such that for all j , $V \cap A_{i_j} = \emptyset$, or equivalently, $V \subset U_{i_j}$. Thus $x \in \text{core}(U_{i_j}) \setminus \text{core}(U_i)$. $\square$

Now suppose that $|\cdot|$ is a compatible distance function on X , so the metric topology agrees with the original topology. Here are some alternative descriptions for $\liminf A_i$, $\limsup A_i$, and Kuratowski convergence. First,

$$\limsup A_i = \{x \in X \mid \liminf_{i \rightarrow \infty} \text{dist}(x, A_i) = 0\}$$

and

$$\liminf A_i = \{x \in X \mid \limsup_{i \rightarrow \infty} \text{dist}(x, A_i) = 0\}.$$

Also, we see that a sequence (A_i) of non-empty closed sets Kuratowski converges to a closed set A if and only if

$$\forall a \in A, \exists (a_i) \text{ with } a_i \in A_i \text{ for all } i \text{ and } \lim_{i \rightarrow \infty} a_i = a, \text{ and,} \quad (3.15a)$$

$$\forall (a_i) \text{ with } a_i \in A_i \text{ for all } i \text{ and some } (a_{i_j}) \text{ convergent, } \lim_{j \rightarrow \infty} a_{i_j} \in A. \quad (3.15b)$$

Thus we have a simple metric characterization for Kuratowski convergence (and so for Carathéodory core convergence). The conditions (3.15a) and (3.15b) should be compared with those in Lemma 3.13. This makes many examples rather transparent.

From Corollary 3.12 and Lemma 3.14(b) we obtain the following for proper spaces.

3.16 Corollary. Let $\emptyset \neq A, A_i \subseteq X$ be closed. Put $U := X \setminus A, U_i := X \setminus A_i$,

$$F := \hat{X} \setminus U = \begin{cases} A & \text{when } X \text{ is bounded,} \\ A \cup \{\infty\} & \text{when } X \text{ is unbounded,} \end{cases} \text{ and } F_i := \hat{X} \setminus U_i.$$

When X is proper, the following are equivalent:

- (a) $\hat{\text{dist}}_{\mathcal{H}}(F_i, F) \rightarrow 0$.
- (b) (U_i) converges to its core $U = \text{core}(U_i)$.
- (c) (A_i) Kuratowski converges to A .

Proof. To see that (b) implies (a), note that as X is proper, every subsequence of (F_i) must have a further subsequence that $\hat{\text{dist}}_{\mathcal{H}}$ -converges, and according to Corollary 3.12 its limit must be F . $\square$

From Corollary 3.6, Proposition 3.8, Proposition 3.10(b), and Lemma 3.14 we obtain the following for proper spaces.

3.17 Corollary. For $\emptyset \neq A, A_i \subseteq X$ closed with X proper, the following are equivalent.

- (a) $\hat{\text{dist}}_{\mathcal{H}}(\hat{A}_i, \hat{A}) \rightarrow 0$.
- (b) $(X \setminus A_i)$ converges to its core $(X \setminus A) = X \setminus A$ and (A_i) has UBP with respect to A .
- (c) (A_i) Kuratowski converges to A and $\hat{\delta}_i(\infty) \rightarrow \hat{\delta}(\infty)$.
- (d) $(\hat{A}_i)$ Kuratowski converges to $\hat{A}$ in $\hat{X}$.

3.4 Convergence Examples

Here we collect some examples that illustrate the need for various hypotheses (such as properness, locally compact, (A_i) having UBP, etc.) required in many of our results. For the most part, proofs are left to the reader.

First, we present a proper length space example.

3.18 Example. Let $X := \mathbb{R}, A := \{0\}$, and for each $i \geq 1, A_i := \{0, i\}$ (or, $A_i := \{1/i, i\}$). Then (A_i) Kuratowski converges to A , (A_i) chordal Hausdorff converges to $A \cup \{\infty\}$, and $\delta_{A_i} \rightarrow \delta_A$ boundedly uniformly in X ; however, (A_i) does not have UBP with respect to A . Taking, instead,

$$A_i := \begin{cases} \{i^{-1}, i\} & \text{if } i \text{ is odd,} \\ \{i^{-1}\} & \text{if } i \text{ is even} \end{cases}$$

we see that $\hat{\text{dist}}_{\mathcal{H}}(A_i, A \cup \{\infty\}) \rightarrow 0$, but now A and $A \cup \{\infty\}$ are both subsequential limits of $(\hat{A}_i)$ (and all the other properties continue to hold).

Next we offer two examples where X is a bounded, complete, locally compact, rectifiably connected space with $X_\ell \xrightarrow{\text{id}} X$ a homeomorphism.

3.19 Example. Let X be the real line $\mathbb{R}_b$ with its standard bounded distance $|x - y|_b := 1 \wedge |x - y|$. Taking $A_i := [-i, i]$ and $A = \mathbb{R}_b$ we have $\delta_{A_i} \rightarrow \delta_A$ locally uniformly in $\mathbb{R}_b$ with (A_i) having UBP with respect to A , but as the chordal distance in $\mathbb{R}_b = \widehat{\mathbb{R}}_b$ is bi-Lipschitz equivalent to $|\cdot|_b$, we easily see that $\hat{\text{dist}}_{\mathcal{H}}(\hat{A}_i, \hat{A}) \not\rightarrow 0$. The point is that (δ_{A_i}) fails to converge to $\delta_A = 0$ boundedly uniformly. (Similar comments apply if we use the sets A, A_i defined as in Example 3.18.)

3.20 Example. Let X be the wedge sum $X := \mathbb{R}_b \vee [0, 2]$ of the pointed spaces $(\mathbb{R}_b, 0)$ and $([0, 2], 0)$. Put $A := \{2\} \subset [0, 2]$ and $A_i := A \cup \{i\}$ with $i \in \mathbb{N} \subset \mathbb{R}_b$. Then (A_i) Kuratowski converges to A , but (δ_{A_i}) does not even converge pointwise to δ_A . (For $x \in \mathbb{R}_b, \delta_A(x) = 2 + (1 \wedge |x|)$ but if $i > 2 \vee 2|x|, \delta_{A_i}(x) = 1$. For $x \in [0, 2], \delta_A(x) = 2 - |x|$ and $\delta_{A_i}(x) = (1 + |x|) \wedge (2 - |x|)$.) For example, $\delta_A(0) = 2$ but for all $i, \delta_{A_i}(0) = 1$.

The above gives a locally compact space that illustrates how the asserted equalities in the second parts of both Proposition 3.10(a,b) may fail. Next we exhibit a bounded complete (but non-locally compact) length space with similar properties.

3.21 Example. Let $X := \bigvee_{i \geq 0} J_i$ be the wedge sum of the pointed spaces $(J_i, 0) := ([0, 2], 0)$ with its natural length distance; so, X is a bounded complete metric tree. Put $A := \{2\} \subset J_0$ and for $i \geq 1$, $A_i := A \cup \{1\}$ with $\{1\} \subset J_i$. Then (A_i) Kuratowski converges to A , but $\delta_A(0) = 2$ while $\delta_{A_i}(0) = 1$ for all $i \geq 1$. (We can realize X as a subset of the plane via $X := \bigcup_{n \geq 0} [0, z_n] \subset \mathbb{C}$ where we use complex variables notation with $z_0 := 2$ and for each $n \geq 1$, $z_n := 2 \exp(\pi i/n)$; here $A = \{z_0\}$ and $A_n = \{z_0, z_n/2\}$.)

Here is an example of an unbounded complete length space that illustrates the need for (δ_{A_i}) to converge boundedly uniformly in Theorem 3.5; it also reveals that in non-proper spaces, chordal Hausdorff convergence and Kuratowski convergence are quite different.

3.22 Example. Let $X := \bigvee_{i \geq 0} R_i$ be the wedge sum of the pointed spaces $(R_i, 0) := ([0, \infty), 0)$ with its natural length distance; so, X is an unbounded complete metric tree. Put $A := \{0\}$ and for $i \geq 1$, let $S_i := \{1, i\} \subset R_i$ and $A_i := A \cup S_i$. Then (A_i) Kuratowski converges to A , (δ_{A_i}) converges locally uniformly in X to δ_A but not boundedly uniformly, (A_i) has no subsequential limits as a sequence in $(\mathcal{H}(\hat{X}), \hat{\text{dist}}_{\mathcal{H}})$, and (A_i) fails to have UBP with respect to A .

Here is an example showing that, without properness, Kuratowski convergence is not sufficient to guarantee the pointed Gromov-Hausdorff convergence of the associated quasihyperbolizations. In contrast, in Theorem 4.11 we prove that chordal Hausdorff convergence “always” gives pointed Gromov-Hausdorff convergence of the associated quasihyperbolizations.

3.23 Example. Let $X := \bigvee_{i \geq 0} J_i$ be the wedge sum of the pointed spaces $(J_i, 0) := ([0, 2], 0)$ with its natural length distance; X is a bounded complete metric tree. Let $J_i[a, b]$ denote the interval $[a, b] \subset J_i$. Let $A := J_0[1, 2]$ and for each $i \geq 1$, define

$$A_i := A \cup \bigcup_{n \geq i} J_n[1, 2], \quad U_i := X \setminus A_i, \quad U := X \setminus A = \bigvee_{i \geq 1} J_i.$$

Let (U, k) and (U_i, k_i) be the quasihyperbolizations of U and U_i respectively. Then:

- (a) (A_i) Kuratowski converges to A ,
- (b) (δ_{A_i}) converges locally uniformly in U to δ_A ,
- (c) (δ_{A_i}) does not converge boundedly uniformly in U to δ_A ,
- (d) (U_i, k_i) does not pointed Gromov-Hausdorff converge to (U, k) .

Finally, here is a Banach space example, where again Kuratowski convergence fails to give pointed Gromov-Hausdorff convergence. (Again, see Theorem 4.11 for comparison with chordal Hausdorff convergence.)

3.24 Example. Let $X := \ell^2(\mathbb{R})$ be the usual “little ℓ^2 space” of sequences $\underline{x} = (x_n)$ with $x_n \in \mathbb{R}$ and with norm given by $\|\underline{x}\|_2^2 := \sum_{n \in \mathbb{N}} x_n^2 < \infty$. For each $i \in \mathbb{N}$, let $\underline{e}_i \in X$ be the sequence with all zero terms except for a single 1 as the i^{th} term. Put $A := \{\underline{0}\}$ and $A_i := \{\underline{0}, 2\underline{e}_i\}$. Then (A_i) Kuratowski converges to A , but the quasihyperbolizations $(X \setminus A_i, k_i)$ do not pointed Gromov-Hausdorff converge to $(X \setminus A, k)$.

Proof. First, $\Omega := X \setminus \{\underline{0}\} = X \setminus A$ and $\Omega' := X \setminus \{\underline{0}, 2\underline{e}_1\}$ are uniform spaces, hence their quasihyperbolizations (Ω, k) and (Ω', k') are Gromov hyperbolic. Also, their Gromov boundaries consist of exactly two and three points, respectively. (Note that (Ω, k) is roughly quasiisometric to $(\mathbb{R}, |\cdot|)$ while (Ω', k') is roughly quasiisometric to a metric triod with infinite legs. Alternatively, we can appeal to [25, Proposition 2.25].)

Evidently, the spaces $\Omega_i := X \setminus A_i$ are all isometric to Ω' . Suppose it were true that the quasihyperbolizations $(\Omega_i, k_i; \underline{e})$ pointed Gromov-Hausdorff converge to $(\Omega, k; \underline{e})$. Then by Fact 2.7(b) we could find $\varepsilon_i \rightarrow 0$ and maps $f_i : B_i \rightarrow \Omega$ ($B_i := B_{k'}[\underline{e}; R_i]$, $R_i := 2/\varepsilon_i$) that are ε_i -rough isometries from $(B_i; \underline{e}) \subset (\Omega', k')$ to $(B_k[\underline{e}; R_i - \varepsilon_i]; \underline{e}) \subset (\Omega, k)$.

Let $\underline{e} := \underline{e}_1$ and $\underline{z}'_i := e^{-R_i} \underline{e}$, $\underline{y}'_i := \underline{e} + \sinh(R_i) \underline{e}_2$, $\underline{x}'_i := (2 - e^{-R_i}) \underline{e}$. Then

$$k'(\underline{z}'_i, \underline{e}) = k'(\underline{y}'_i, \underline{e}) = k'(\underline{x}'_i, \underline{e}) = R_i$$

so we can define $\underline{z}_i, \underline{y}_i, \underline{x}_i$ to be the images of $\underline{z}'_i, \underline{y}'_i, \underline{x}'_i$ under f_i .

Keeping [25, Proposition 2.25] in mind we see that $(z'_i), (y'_i), (x'_i)$ are Gromov sequences in (Ω', k') that represent the boundary points $\underline{0}, \infty, 2\underline{e}$ respectively. Since f_i is an ε_i -rough-isometry, we can readily check that the Gromov products satisfy

$$(x'_i|y'_i)_{\underline{e}} - \varepsilon_i \leq (x_i|y_i)_{\underline{e}} \leq (x'_i|y'_i)_{\underline{e}} + \varepsilon_i$$

with similar inequalities for $(y_i|z_i)_{\underline{e}}, (z_i|x_i)_{\underline{e}}, (x_i|x_j)_{\underline{e}}, (y_i|y_j)_{\underline{e}}, (z_i|z_j)_{\underline{e}}$. It would follow that $(z_i), (y_i), (x_i)$ are Gromov sequences in (Ω, k) that represent distinct boundary points. As this contradicts the fact that the Gromov boundary of (Ω, k) has cardinality two, we conclude that $(\Omega_i, k_i; \underline{e})$ does *not* pointed Gromov-Hausdorff converge to $(\Omega, k; \underline{e})$. $\square$

The *moral* here is that chordal Hausdorff convergence is a handy tool, albeit somewhat technical. For analyzing complete conformal metrics in locally compact length spaces, Kuratowski convergence (and the equivalent core convergence) is more robust, easier to use, and provides more information (in the sense that we can verify pointed Gromov-Hausdorff convergence, e.g., provided we have Kuratowski convergence, even when chordal Hausdorff convergence fails). However, in non-proper spaces, Kuratowski convergence does not give pointed Gromov-Hausdorff convergence whereas chordal Hausdorff convergence does.

There is substantial interest in quasihyperbolic geometry in Banach spaces; e.g., Väisälä's definition in [24] of a quasiconformal map between Banach space domains is based on quasihyperbolic distance. In this setting Kuratowski convergence cannot be employed, but chordal Hausdorff convergence still gives useful results.

4 Convergence of Metrics

Now we turn our attention to conformal metrics and pointed Gromov-Hausdorff convergence. We show that bounded uniform convergence of conformal metrics implies the same for their associated length distances; in particular, the associated length spaces pointed Gromov-Hausdorff converge. As applications of these ideas, we get approximation results for: quasihyperbolic distance in domains in complete length spaces, Ferrand and Kulkarni-Pinkall-Thurston distances in quasihyperbolic domains in $\hat{\mathbb{R}}^n$, and hyperbolic distance for hyperbolic domains in $\hat{\mathbb{C}}$. We conclude by examining Gromov-Hausdorff convergence for Euclidean sets; especially, Corollary 5.9 characterizes pointed Gromov-Hausdorff convergence of Euclidean quasihyperbolic spaces.

4.1 Pointed Gromov-Hausdorff Convergence

Here we prove that if ρds and $\rho_i ds$ are conformal metrics defined in open subspaces $U, U_i \subset X$ of a quasihyperbolic superspace, and if $(\rho_i ds)$ converges to ρds in the appropriate sense, then the associated sequence (U_i, d_{ρ_i}) of length spaces converges to (U, d_ρ) with respect to pointed Gromov-Hausdorff convergence.

The reader is no doubt familiar with the notion of *local uniform convergence* (aka *uniform convergence on compact subsets*) which is often a handy substitute for uniform convergence. Pointed Gromov-Hausdorff convergence is specifically designed to deal with the lack of compactness. As we saw in §3.1, compactness also plays no role in chordal Hausdorff convergence; instead, convergence in bounded sets is the crucial ingredient.

Let $\rho ds, \rho_i ds$ be conformal metrics on open $U, U_i \subset X$. The sequence of conformal metrics $(\rho_i ds)$ *converges boundedly uniformly* in (U, d_ρ) to ρds provided for all $a \in U$ and all $r > 0$,

$$B_\rho[a; r] \subset U_i \text{ for finitely many } i \text{ and } \rho_i/\rho \rightarrow 1 \text{ uniformly in } B_\rho[a; r]. \quad (4.1)$$

Of course, $(\rho_i ds)$ *converges locally uniformly* in U to ρds provided each compact $C \subset U$ lies in all but finitely many U_i and $\rho_i/\rho \rightarrow 1$ uniformly in C .

4.2 Examples. Let X be a quasihyperbolic superspace, $\emptyset \neq A, A_i \subsetneq X$ be closed, and $U := X \setminus A, U_i := X \setminus A_i$. Put $\delta := \delta_A, \delta_i := \delta_{A_i}$. Let d, d_i be the length distances in U, U_i associated respectively with the conformal metrics $\delta^{-1}ds, \delta_i^{-1}ds$ on U, U_i .

- (a) Suppose $\hat{\text{dist}}_{\mathcal{H}}(A_i, A) \rightarrow 0$. Then $(\delta_i^{-1}ds)$ converges boundedly uniformly in (U, d) to $\delta^{-1}ds$.
- (b) Suppose X is proper and (A_i) Kuratowski converges to A . Then $(\delta_i^{-1}ds)$ converges boundedly uniformly in (U, d) to $\delta^{-1}ds$.
- (c) Suppose $\hat{\text{dist}}_{\mathcal{H}}(\partial\Omega_i, \partial\Omega) \rightarrow 0$ for some domains $\Omega, \Omega_i \subsetneq X$. Then the quasihyperbolic metrics in Ω_i converge boundedly uniformly to the quasihyperbolic metric in (Ω, k) .⁹
- (d) Let Ω, Ω_i be components of U, U_i respectively. Suppose X is a proper length space and $\delta_i \rightarrow \delta$ pointwise in Ω . Then the sequence of quasihyperbolic metrics $(\delta_i^{-1}ds)$ converges boundedly uniformly in $(\Omega, d) = (\Omega, k)$ to its quasihyperbolic metric $\delta^{-1}ds$.

Proof. To verify (a), first note that Corollary 3.2 asserts that $\delta_i \rightarrow \delta$ boundedly uniformly in X . Now fix $a \in U$ and $r > 0$. Appealing to (2.11b) we deduce that for all $x \in B_d[a; r]$

$$|x - a| \leq \delta(a)(e^r - 1) \quad \text{and} \quad \delta(x) \geq \delta(a)e^{-r}.$$

Thus $B_d[a; r]$ is a bounded subset of X that lies in the a -component of U . As $\delta_i \rightarrow \delta$ uniformly in $B_d[a; r]$, $B_d[a; r]$ lies in $X \setminus A_i$ for all but finitely many i , and as δ is uniformly bounded away from 0 in $B_d[a; r]$, we also see that

$$\frac{\delta}{\delta_i} = \frac{\delta_i^{-1}ds}{\delta^{-1}ds} \rightarrow 1 \quad \text{uniformly in } B_d[a; r].$$

A similar argument gives (c) and (d). For (b), Lemma 3.14, Proposition 3.10(b) and properness give $\delta_i \rightarrow \delta$ boundedly uniformly in X , so again we can argue as for (a). □

4.3 Remark. If X is a length space, then in Examples 4.2(a,b) $\delta_i^{-1}ds, \delta^{-1}ds$ are the quasihyperbolic metrics in U, U_i .

Here is our main result concerning boundedly uniformly convergent conformal metrics.

4.4 Theorem. Assume X is a quasihyperbolic superspace. Let $\rho ds, \rho_i ds$ be complete conformal metrics that are defined, respectively, on open subspaces $U, U_i \subset X$. Suppose that $(\rho_i ds)$ converges boundedly uniformly in (U, d_ρ) to ρds . Then $d_{\rho_i} \rightarrow d_\rho$ boundedly uniformly in $(U, d_\rho)^{10}$; in particular,

$$\text{dist}_{\mathcal{H}}((U_i, d_{\rho_i}), (U, d_\rho)) \rightarrow 0.$$

Proof. To simplify notation, we write $d := d_\rho$ and $d_i := d_{\rho_i}$.

Fix $a \in U$, and assume $a \in U_i$ for all i . First, we demonstrate that for each $R > 0$ and each $L > 1$, there exists an i_0 such that for all $i \geq i_0$,

$$B_{\rho_i}[a; r/L] \subset B_\rho[a; r] \quad \text{for all } r \in (0, L^2 R] \quad (4.5a)$$

and

$$B_\rho[a; r] \subset B_{\rho_i}[a; Lr] \quad \text{for all } r \in (0, R]. \quad (4.5b)$$

Let $R > 0$ and $L > 1$ be given. Choose i_0 so that for all $i \geq i_0$, $B_\rho[a; L^2 R] \subset U_i$ and so that in $B_\rho[a; L^2 R]$ we have

$$L^{-1} \leq \frac{\rho_i}{\rho} \leq L. \quad (4.6)$$

⁹ Similarly, if $\hat{\text{dist}}_{\mathcal{H}}(X \setminus \Omega_i, X \setminus \Omega) \rightarrow 0$, then the metrics $\text{dist}(x, X \setminus \Omega_i)^{-1}ds$ converge boundedly uniformly in Ω to the metric $\text{dist}(x, X \setminus \Omega)^{-1}ds$.

¹⁰ Here we simply mean that (4.7) holds, for each $R > 0, \varepsilon > 0$ and all sufficiently large i .

To verify (4.5a), we show that points not in $B_\rho[a; r]$ are not in $B_{\rho_i}(a; r/L)$. Fix $i \geq i_0$, $r \in (0, L^2R]$, and suppose $b \in U_i \setminus B_\rho[a; r]$. Let $\gamma : a \curvearrowright b$ be a rectifiable path in U_i , say with parameter interval $[0, 1]$. We claim that there is a first point c along γ with $c \in U$ and $d(a, c) = r$. To see this, let $\gamma_* := \gamma|_{[0, t_*)}$ where

$$t_* := \inf\{t \in [0, 1] \mid \gamma(t) \in B_\rho(a; r)\}.$$

Employing (4.6) we see that γ_* is rectifiable as a path in the complete metric space (U, d) , and as such it has an endpoint $c := \gamma_*(t_*) \in U$ with $d(\gamma(t), c) \rightarrow 0$ as $t \rightarrow t_*$. Then $d(a, c) = r$.

Now $\gamma[a, c] \subset B_\rho[a; r] \subset B_\rho[a; L^2R]$, so using (4.6) we obtain

$$\ell_{\rho_i}(\gamma) \geq \ell_{\rho_i}(\gamma[a, c]) = \int_{\gamma[a, c]} \rho_i ds \geq L^{-1} \int_{\gamma[a, c]} \rho ds \geq L^{-1} d(a, c) = \frac{r}{L}.$$

Therefore, $d_i(a, b) \geq r/L$. Thus $B_{\rho_i}(a; r/L) \subset B_\rho[a; r]$, so (4.5a) holds.

Notice that (4.5a) with $r = L^2R$ gives $B_{\rho_i}[a; LR] \subset B_\rho[a; L^2R]$, so (4.6) holds in $B_{\rho_i}[a; LR]$.

To verify (4.5b), we show that points not in $B_{\rho_i}[a; Lr]$ are not in $B_\rho(a; r)$. Fix $r \in (0, R]$ and suppose $b \in U \setminus B_{\rho_i}[a; Lr]$. Let $\gamma : a \curvearrowright b$ be a rectifiable path. As above, there is a first point c along γ with $c \in U_i$ and $d_i(a, c) = Lr$. Then $\gamma[a, c] \subset B_{\rho_i}[a; Lr] \subset B_{\rho_i}[a; LR]$, so using (4.6) we obtain

$$\ell_\rho(\gamma) \geq \ell_\rho(\gamma[a, c]) = \int_{\gamma[a, c]} \rho ds \geq L^{-1} \int_{\gamma[a, c]} \rho_i ds \geq L^{-1} d_i(a, c) = r.$$

Therefore, $d(a, b) \geq r$, so $B_\rho(a; r) \subset B_{\rho_i}[a; Lr]$ and hence (4.5b) holds.

Next, we verify that for each $R > 0$ and each $\varepsilon > 0$ there is an i_1 with the property that for all $i \geq i_1$, $B_\rho[a; R] \subset U_i$ and

$$x, y \in B_\rho[a; R] \implies |d_i(x, y) - d(x, y)| < \varepsilon. \quad (4.7)$$

Let $R > 0$ and $\varepsilon > 0$ be given. Choose $L \in (1, 2)$ with $2L(L^2 - 1) < \varepsilon/R$.

Since $(\rho_i ds)$ converges boundedly uniformly in U to ρds , there exists an i_1 such that for all $i \geq i_1$, $B_\rho[a; 3L^2R] \subset U_i$ and so that (4.6) holds in $B_\rho[a; 3L^2R]$, and thanks to (4.5) (with $3R$ in place of R) so that

$$B_{\rho_i}[a; r/L] \subset B_\rho[a; r] \quad \text{for all } r \in (0, 3L^2R] \quad (4.8a)$$

and

$$B_\rho[a; r] \subset B_{\rho_i}[a; Lr] \quad \text{for all } r \in (0, 3R]. \quad (4.8b)$$

In particular, for such i we have

$$B_\rho[a; 3R] \subset B_{\rho_i}[a; 3LR] \subset B_\rho[a; 3L^2R].$$

Fix $i \geq i_1$. Let $x, y \in B_\rho[a; R] \subset B_{\rho_i}[a; LR]$. Let γ and γ_i be rectifiable paths in U and U_i , respectively, both with endpoints x, y , and satisfying

$$\ell_\rho(\gamma) \leq Ld(x, y) \leq 2LR \quad \text{and} \quad \ell_{\rho_i}(\gamma_i) \leq Ld_i(x, y) \leq 2L^2R.$$

Then for all points $z \in |\gamma|$,

$$\begin{aligned} d(z, a) &\leq \min\{d(x, a) + \ell_\rho(\gamma[x, z]), d(y, a) + \ell_\rho(\gamma[y, z])\} \\ &\leq R + \frac{1}{2} \ell_\rho(\gamma) \leq R + LR \leq 3R, \end{aligned}$$

so $|\gamma| \subset B_\rho[a; 3R]$ and similarly $|\gamma_i| \subset B_{\rho_i}[a; 3LR]$; therefore (4.6) holds on these paths. Thus

$$d_i(x, y) \leq \ell_{\rho_i}(\gamma) = \int_{\gamma} \rho_i ds \leq L \int_{\gamma} \rho ds \leq L^2 d(x, y)$$

and

$$d(x, y) \leq \ell_\rho(\gamma_i) = \int_{\gamma_i} \rho \, ds \leq L \int_{\gamma_i} \rho_i \, ds \leq L^2 d_i(x, y).$$

It now follows that

$$d_i(x, y) - d(x, y) \leq (L^2 - 1)d(x, y) \leq 2(L^2 - 1)R$$

and

$$d(x, y) - d_i(x, y) \leq (L^2 - 1)d_i(x, y) \leq 2L(L^2 - 1)R$$

whence

$$|d_i(x, y) - d(x, y)| \leq 2L(L^2 - 1)R < \varepsilon$$

and thus (4.7) holds.

Finally, we establish the asserted pointed Gromov-Hausdorff convergence. Let $R > \varepsilon > 0$ be given. We must show that for all sufficiently large i there exist maps $B_{\rho_i}[a; R] \xrightarrow{f_i} U_\rho$ that are ε -rough isometries from $(B_{\rho_i}[a; R]; a)$ to $(B_\rho[a; R - \varepsilon]; a)$, meaning that

$$d(f_i(a), a) < \varepsilon \quad \text{and} \quad B_\rho[a; R - \varepsilon] \subset N_d(f_i(B_{\rho_i}[a; R]); \varepsilon)$$

and

$$x, y \in B_{\rho_i}[a; R] \implies |d_i(x, y) - d(f_i(x), f_i(y))| < \varepsilon.$$

Employing (4.7), we obtain an i_2 such that for all $i \geq i_2$, $B_\rho[a; 2R] \subset U_i$ and so that

$$x, y \in B_\rho[a; 2R] \implies |d_i(x, y) - d(x, y)| < \varepsilon.$$

It is not difficult to check that for such i , $B_{\rho_i}[a; R] \subset U$ and

$$B_\rho[a; R - \varepsilon] \subset B_{\rho_i}[a; R] \subset B_\rho[a; R + \varepsilon].$$

It now follows that the identity inclusions $B_{\rho_i}[a; R] \xrightarrow{f_i := \text{id}} U$ are well defined, and are the desired ε -rough pointed isometries. $\square$

4.9 Remarks. The above argument consists of three parts. First, we have the “ball engulfing” property described in (4.5). Beardon and Minda illuminated the key ideas behind this fundamental property (in the complete, locally compact, length space setting); see [3]. Similar techniques have been useful in the study of BLD mappings (see, for example, [10] and [20]). Next, we have that bounded uniform convergence of the metrics gives “the same” for the associated length distances in the sense that (4.7) holds. Finally, we have the pointed Gromov-Hausdorff convergence. In fact this last part is true in general: If (U_i, d_i) are length spaces and (U, d) is a metric space such that for each $a \in U$, each $R > 0$, and each $\varepsilon > 0$ there is an i_0 with the property that for all $i \geq i_0$, $B_d[a; R] \subset U_i$ and (4.7) holds, then it is straightforward to show that $\text{dist}_{\mathcal{GH}^*}((U_i, d_i; a), (U, d; a)) \rightarrow 0$.

4.2 Ferrand and Kulkarni-Pinkall-Thurston Convergence

Here we verify that in the Euclidean setting we have local uniform convergence (which implies bounded uniform convergence) of both Ferrand and Kulkarni-Pinkhall-Thurston metrics.

4.10 Theorem. Let $A, A_i \subsetneq \hat{\mathbb{R}}^n$ be closed with $\text{card}(A) \geq 2$. Suppose $\hat{\text{dist}}_{\mathcal{H}}(A_i, A) \rightarrow 0$. Then $(\varphi_i ds)$ and $(\mu_i ds)$ converge boundedly uniformly to φds and μds (respectively) in $\hat{\mathbb{R}}^n \setminus A$.

Here $\varphi_i ds$, φds and $\mu_i ds$, μds represent the Ferrand and Kulkarni-Pinkall-Thurston metrics (respectively) in $\hat{\mathbb{R}}^n \setminus A_i$, $\hat{\mathbb{R}}^n \setminus A$ (respectively). See §2.4.3 and also §2.4.5.

Proof. Since rotations of $\hat{\mathbb{R}}^n = \mathbb{S}^n$ are chordal isometries, as well as Ferrand and Kulkarni-Pinkall-Thurston isometries, we may assume that A contains the ‘point at infinity’. Furthermore, we may assume that each A_i contains at least two points, so each component of $\hat{\mathbb{R}}^n \setminus A$, and of $\hat{\mathbb{R}}^n \setminus A_i$, is a quasihyperbolic domain.

Let $a \in \hat{\mathbb{R}}^n \setminus A$ and fix $r \in (0, \text{dist}(a, A)/2)$. Let Ω be the a -component of $\hat{\mathbb{R}}^n \setminus A$. Then $B[a; r] \subset \Omega$. As $\hat{\text{dist}}_{\mathcal{H}}(A_i, A) \rightarrow 0$, for all sufficiently large i , $B[a; r] \subset \hat{\mathbb{R}}^n \setminus A_i$; for such i there is a unique component Ω_i of $\hat{\mathbb{R}}^n \setminus A_i$ that contains $B[a; r]$. We verify that

$$\varphi_i \rightarrow \varphi \quad \text{and} \quad \mu_i \rightarrow \mu \quad \text{uniformly in } B[a; r].$$

According to (2.14), for all $x \in B[a; r]$ we have

$$\varphi(x) = \text{diam } J_x(\Omega^c) \quad \text{and} \quad \mu(x) = \text{cdiam } J_x(\Omega^c)$$

with similar expressions for $\varphi_i(x)$ and $\mu_i(x)$. Note that $\partial\Omega \subset A \subset \Omega^c = \hat{\mathbb{R}}^n \setminus \Omega$, so $J_x(\partial\Omega) \subset J_x(A) \subset J_x(\Omega^c)$ and thus $\text{diam } J_x(\partial\Omega) \leq \text{diam } J_x(A) \leq \text{diam } J_x(\Omega^c) = \text{diam } J_x(\partial\Omega)$ and $\text{cdiam } J_x(\partial\Omega) \leq \text{cdiam } J_x(A) \leq \text{cdiam } J_x(\Omega^c) = \text{cdiam } J_x(\partial\Omega)$. Therefore, for all $x \in B[a; r]$,

$$\varphi(x) = \text{diam } J_x(A) \quad \text{and} \quad \mu(x) = \text{cdiam } J_x(A)$$

with similar expressions for $\varphi_i(x)$ and $\mu_i(x)$.

Recall from Lemma 2.2 that the chordal Lipschitz constant $H(x) := H(J_x)$ for J_x is given by

$$H(x) = \exp h(e_{n+1}, \tilde{x}) \quad \text{where } \tilde{x} = \frac{x_1}{1 + |x|^2} = \frac{(-x, 1)}{1 + |x|^2}$$

and $x_1 := e_{n+1} - x' = (-x, 1)$. When $x \in B[a; r]$ we have

$$\begin{aligned} |h(e_{n+1}, \tilde{x}) - h(e_{n+1}, \tilde{a})| &\leq h(\tilde{x}, \tilde{a}) = h(J(x_1), J(a_1)) = h(x_1, a_1) \\ &\leq 2j(x_1, a_1) = 2 \log(1 + |x - a|) \\ &\leq 2 \log(1 + r). \end{aligned}$$

The middle equality above holds because J is an isometry of $(\mathbb{R}_+^{n+1}, h)$, and we have also used the well known estimate¹¹ $h \leq 2j$ that is valid in $\mathbb{R}_+^{n+1}$; here

$$j(x, y) := \log \left(1 + \frac{|x - y|}{d(x) \wedge d(y)} \right) \quad \text{where } d((x_1, \dots, x_{n+1})) := x_{n+1}.$$

We thus have

$$\forall x \in B[a; r], \quad (1 + r)^{-2} \leq \frac{H(x)}{H(a)} \leq (1 + r)^2.$$

Finally, for all $x \in B[a; r]$,

$$\begin{aligned} |\varphi_i(x) - \varphi(x)| &= |\text{diam } J_x(A_i) - \text{diam } J_x(A)| \leq 2 \text{dist}_{\mathcal{H}}(J_x(A_i), J_x(A)) \\ &\leq (1 + r^{-2}) \hat{\text{dist}}_{\mathcal{H}}(J_x(A_i), J_x(A)) \leq (1 + r^{-2}) H(x) \hat{\text{dist}}_{\mathcal{H}}(A_i, A) \\ &\leq (1 + r^{-2})(1 + r)^2 H(a) \hat{\text{dist}}_{\mathcal{H}}(A_i, A). \end{aligned}$$

Since $\hat{\text{dist}}_{\mathcal{H}}(A_i, A) \rightarrow 0$, it follows from the above that $\varphi_i \rightarrow \varphi$ uniformly in $B[a; r]$. By replacing diam with cdiam in the above we also see that $\mu_i \rightarrow \mu$ uniformly in $B[a; r]$. $\square$

¹¹ It is trivial that $h((0, t), (0, s)) = j((0, t), (0, s))$ for $(0, t), (0, s) \in \mathbb{R}^n \times (0, +\infty)$ and then $h \leq 2j$ follows by Möbius quasiinvariance of the relative distance j .

4.3 Approximation Properties

Here we explore some consequences of Theorem 4.4. First we give an example of how we can use chordal Hausdorff convergence to get pointed Gromov-Hausdorff convergence of various conformal metrics; here there is no assumption of local compactness. Recall §2.4.5.

4.11 Theorem. *Let X be a complete length space. Let $\emptyset \neq U, U_i \subsetneq X$ be open with quasihyperbolic distances $k := k_U, k_i := k_{U_i}$ respectively. Suppose $\hat{\text{dist}}_{\mathcal{H}}(X \setminus U_i, X \setminus U) \rightarrow 0$. Then $\text{dist}_{\mathcal{GH}}((U_i, k_i), (U, k)) \rightarrow 0$.*

When $X = \hat{\mathbb{R}}^n$, similar results hold for both the Ferrand and Kulkarni-Pinkall-Thurston metrics. When $X = \hat{\mathbb{C}}$, similar results hold for the Poincaré hyperbolic metric.

Proof. As X is a length space, $\delta := \delta_{X \setminus U} = \delta_{\partial U}$ in U and $\delta_i := \delta_{X \setminus U_i} = \delta_{\partial U_i}$ in U_i . By Corollary 3.2, $\delta_i \rightarrow \delta$ boundedly uniformly in X , and thus we can reason as in Examples 4.2 to see that the quasihyperbolic metrics $\delta_i^{-1}ds$ for U_i converge boundedly uniformly in (U, k) to its quasihyperbolic metric $\delta^{-1}ds$; an appeal to Theorem 4.4 completes the argument.

When $X = \hat{\mathbb{R}}^n$, we appeal to Theorem 4.10 to obtain bounded uniform convergence of the Ferrand and/or Kulkarni-Pinkall-Thurston metrics; if $X = \hat{\mathbb{C}}$ we use [3]. $\square$

4.12 Remark. Let $\Omega, \Omega_i \subsetneq X$ be non-empty domains in a quasihyperbolic superspace X . Suppose $\hat{\text{dist}}_{\mathcal{H}}(\partial\Omega_i, \partial\Omega) \rightarrow 0$. Reasoning as above, using Example 4.2(c), we see that the quasihyperbolic metrics in Ω_i converge boundedly uniformly in (Ω, k) to its quasihyperbolic metric, and therefore $\text{dist}_{\mathcal{GH}}((\Omega_i, k_i), (\Omega, k)) \rightarrow 0$.

It is easy to establish the following.

4.13 Lemma. *A locally compact length space X has the finite approximation property that for each closed $\emptyset \neq A \subseteq X$ there exist finite sets $A_i \subset X$ such that $\hat{\text{dist}}_{\mathcal{H}}(A_i, A) \rightarrow 0$.*

Proof. Note that X is proper. Fix a point $o \in X$. Then for each i , let A_i be a maximal $(1/i)$ -separated set in $B_i := A \cap B[o; i]$. Since B_i is compact, A_i is finite; also, $A_i \subset B_i \subset N(A_i; 1/i)$ and (A_i) has UBP with respect to A . It is easy to check that $\delta_{A_i} \rightarrow \delta_A$ pointwise, and hence by properness, boundedly uniformly, in X . Now apply 3.8. $\square$

The above provides a powerful tool to understand quasihyperbolic geometry as it allows one to approximate a general quasihyperbolization (U, k) by, say, (U', k') where $\partial U' = X \setminus U'$ is finite; here $\emptyset \neq U \subsetneq X$ is open and X is any locally compact complete length space.

It is easy to see that this “chordal Hausdorff distance finite approximation” fails without both the local compactness and the length space hypotheses. If $X = \mathbb{R}_b^2$ (see Example 3.19), $A := \mathbb{Z}^2 \subset X$, and $B \subset X$ is finite, then $\hat{\text{dist}}_{\mathcal{H}}(A, B) \simeq \text{dist}_{\mathcal{H}}(A, B) = 1$. If $X = \bigvee_{n \in \mathbb{N}} [0, e^{\pi i/n}] \subset \mathbb{C}$ with its induced length metric, and we set $A := \{e^{\pi i/n} \mid n \in \mathbb{N}\}$, and $B \subset X$ is finite, then $\text{dist}_{\mathcal{H}}(A, B) \simeq \text{dist}_{\mathcal{H}}(A, B) \geq 1$. In fact, the related space $\bigvee_{t \in [0, 2\pi)} [0, e^{\pi i t}] \subset \mathbb{C}$ with its induced length metric, does not even have the countable approximation property.

It is worth noting the following “quasihyperbolic distance finite approximation” result.

4.14 Proposition. *Let X be a quasihyperbolic superspace. Suppose $X_l := (X, l)$ is locally compact. Then for each domain $\Omega \subsetneq X$ there is a sequence (Ω_i) of domains $\Omega_i \subsetneq X$ such that each $\partial\Omega_i$ is a finite set and $\text{dist}_{\mathcal{GH}}((\Omega_i, k_i), (\Omega, k)) \rightarrow 0$.*

Proof. Let $\Omega \subsetneq X$ be a domain. Put $A := \partial\Omega$, fix a point $o \in A$, and consider $B_i := A \cap B_1[o; i]$ (for $i \in \mathbb{N}$). Since B_i is compact, there is a finite maximal $(1/i)$ -separated set $A_i \subset B_i$; so for all $a, b \in A_i$, $|a - b| \geq 1/i$ and for all $x \in B_i$, there is an $a \in A_i$ such that $|x - a| < 1/i$.

Let $\delta := \delta_A = \delta_{\partial\Omega}$ and $\delta_i := \delta_{A_i}$. It is not difficult to check that $\delta_i \rightarrow \delta$ pointwise in X . Let $x \in X$ and $\varepsilon > 0$ be given. Pick $a \in A$ with $\delta(x) \leq |x - a| < \delta(x) + \varepsilon/10$. Fix $i_o \geq l(a, o)$. Then for all $i \geq i_o$, $a \in B_i$ so there exist $a_i \in A_i$ with $|a - a_i| < 1/i$. Thus for $i \geq i_o \vee (2/\varepsilon)$

$$\delta(x) \leq \delta_i(x) \leq |x - a_i| \leq |x - a| + |a - a_i| < \delta(x) + \frac{\varepsilon}{10} + \frac{1}{i} < \delta(x) + \varepsilon.$$

Since X_l is proper, (δ_i) converges locally (even boundedly) uniformly in X_l to δ .

Let Ω_i be the component of $X \setminus A_i$ that contains Ω . Then $\partial\Omega_i = A_i$, so $\delta_i = \delta_{A_i} = \delta_{\partial\Omega_i}$ and therefore in Ω_i , $\delta_i^{-1}ds$ is the quasihyperbolic metric for Ω_i .

Fix $z \in \Omega$ and $r > 0$. Then $B_k[z; r] \subset \Omega \subset \Omega_i$ for all i . As $B_k[z; r]$ is bounded in X_l , it is compact and so $\delta_i \rightarrow \delta$ uniformly in $B_k[z; r]$. Also, δ is bounded away from 0 in $B_k[z; r]$, so $\delta_i/\delta \rightarrow 1$ uniformly in $B_k[z; r]$. Thus we can appeal to Theorem 4.4 to assert that $\text{dist}_{\mathcal{GH}}((\Omega_i, k_i), (\Omega, k)) \rightarrow 0$. Also, see Remark 4.12. $\square$

Again, local compactness is a crucial hypothesis: If $X = \bigvee_{n \in \mathbb{N}} [0, e^{\pi i/n}] \subset \mathbb{C}$ with its induced Euclidean length metric (so $X = X_l$ is a locally compact quasihyperbolic superspace, but is not locally compact), and $\Omega := X \setminus \{e^{\pi i/n} \mid n \in \mathbb{N}\}$, then (Ω, k) cannot be approximated by any (Ω', k') with $\Omega' \subsetneq X$ having a finite boundary.

5 Gromov Hausdorff Convergence

We conclude by examining Gromov-Hausdorff convergence for Euclidean sets; especially, Corollary 5.9 below characterizes pointed Gromov-Hausdorff convergence of Euclidean quasihyperbolic spaces. More precisely, we return to the Theorem in our Introduction and explore the consequences of replacing chordal Hausdorff convergence with either Gromov-Hausdorff convergence or chordal Gromov-Hausdorff convergence.

We employ the following from [1, Theorem 2.2]; see §2.3 for the notion of a rough isometry.

5.1 Fact. Suppose $\mathbb{R}^n \supset K \xrightarrow{f} \mathbb{R}^n$ is a $d\varepsilon$ -rough isometric embedding with K compact, $d := \text{diam}(K)$, $0 < \varepsilon \leq 1$. Then there is an Euclidean isometry $\mathbb{R}^n \xrightarrow{\varphi} \mathbb{R}^n$ such that

$$\|f - \varphi\|_{\infty}^K \leq cd\sqrt{\varepsilon} \quad \text{where } c = c(n).$$

5.2 Theorem. Let $A, A_i \subset \mathbb{R}^n$ be closed. Suppose $\text{dist}_{\mathcal{GH}}(A_i, A) \rightarrow 0$. Then there are Euclidean isometries $\mathbb{R}^n \xrightarrow{\varphi_i} \mathbb{R}^n$ such that:

- (a) When A is compact, $\text{dist}_{\mathcal{GH}}(\varphi_i A_i, A) \rightarrow 0$.
- (b) When $A, A_i \subset S := S^{n-1}(c; R)$ for some $c \in \mathbb{R}^n$ and some $R > 0$, $\varphi_i(S) = S$ and $\text{dist}_{\mathcal{GH}}(\varphi_i A_i, A) \rightarrow 0$.
- (c) In general, $\hat{\text{dist}}_{\mathcal{GH}}(\varphi_i A_i, A) \rightarrow 0$.

Proof. Since $\text{dist}_{\mathcal{GH}}(A_i, A) \rightarrow 0$, an appeal to Fact 2.7(a) produces a sequence (ε_i) with $\varepsilon_i \rightarrow 0$ and such that for each i there exists an ε_i -rough isometry $A_i \xrightarrow{f_i} A$; thus,

$$\forall i, \quad f_i(A_i) \subset A \subset N(f_i A_i; \varepsilon_i) \quad (\text{so, } \text{dist}_{\mathcal{GH}}(f_i A_i, A) < \varepsilon_i) \quad (5.3a)$$

and

$$\forall i, \forall x, y \in A_i, \quad ||f_i(x) - f_i(y)| - |x - y|| < \varepsilon_i. \quad (5.3b)$$

We may, and do, assume that each $\varepsilon_i \in (0, 1)$.

(a) Suppose A is compact. Since $d_i := \text{diam } f_i(A_i) \rightarrow \text{diam}(A) =: d$, we may assume each A_i is compact. By Fact 5.1, there are Euclidean isometries $\mathbb{R}^n \xrightarrow{\varphi_i} \mathbb{R}^n$ with

$$\|f_i - \varphi_i\|_{\infty}^{A_i} \leq c\sqrt{d_i \varepsilon_i}$$

where $c = c(n)$. Thus Fact 2.8(b) asserts that for all sufficiently large i ,

$$\text{dist}_{\mathcal{H}}(f_i A_i, \varphi_i A_i) \leq c \sqrt{d_i \varepsilon_i} \leq 2c \sqrt{d \varepsilon_i},$$

and therefore

$$\text{dist}_{\mathcal{H}}(\varphi_i A_i, A) \leq \text{dist}_{\mathcal{H}}(\varphi_i A_i, f_i A_i) + \text{dist}_{\mathcal{H}}(f_i A_i, A) \leq 2c \sqrt{d \varepsilon_i} + \varepsilon_i \rightarrow 0.$$

(b) Suppose $A, A_i \subset S := S^{n-1}(c; R)$ for some $c \in \mathbb{R}^n$ and some $R > 0$. We may assume that $c = 0$. Let $B := A \cup \{0\}$, $B_i := A_i \cup \{0\}$ and define $B_i \xrightarrow{g_i} B$ by

$$g_i(x) := \begin{cases} f_i(x) & \text{if } x \in A_i, \\ 0 & \text{if } x = 0. \end{cases}$$

It is easy to check that each g_i is an ε_i -rough isometry.

As in part (a), there are Euclidean isometries $\mathbb{R}^n \xrightarrow{\varphi_i} \mathbb{R}^n$ with

$$\|g_i - \varphi_i\|_{\infty}^{B_i} \leq c \sqrt{d_i \varepsilon_i}$$

where $c = c(n)$ and $d_i := \text{diam}(B_i) \rightarrow \text{diam}(B) =: d$. Note that

$$|\varphi_i(0)| = |\varphi_i(0) - g_i(0)| \leq \|g_i - \varphi_i\|_{\infty}^{B_i} \leq c \sqrt{d_i \varepsilon_i}.$$

Define $\mathbb{R}^n \xrightarrow{\psi_i} \mathbb{R}^n$ by $\psi_i := \varphi_i - \varphi_i(0)$; ψ_i are Euclidean isometries with $\psi_i(S) = S$. Also, for each $x \in B_i$,

$$|\psi_i(x) - g_i(x)| \leq |\varphi_i(x) - g_i(x)| + |\varphi_i(0)| \leq 2\|g_i - \varphi_i\|_{\infty}^{B_i} \leq 2c \sqrt{d_i \varepsilon_i}.$$

Arguing as above we deduce that $\text{dist}_{\mathcal{H}}(\psi_i B_i, B) \rightarrow 0$, and therefore $\text{dist}_{\mathcal{H}}(\psi_i A_i, A) \rightarrow 0$.

(c) Here we consider the general case which requires more effort. We may assume that A is unbounded and that $0 \in A$. By (5.3a), there are points $a_i \in A_i$ such that $b_i := f_i(a_i)$ satisfy $|b_i| < \varepsilon_i$. Put $r_i := \varepsilon_i^{-1/2}$ and $B_i := A_i \cap B[a_i; r_i]$.

It is not difficult to check that

$$A \cap B[0; r_i - 3\varepsilon_i] \subset N(f_i B_i; \varepsilon_i) \quad (5.4a)$$

and

$$f_i(B_i) \subset A \cap B[0; r_i + 2\varepsilon_i]. \quad (5.4b)$$

Put $g_i := f_i|_{B_i}$. It is easy to check that each g_i is a ε_i -rough isometric embedding, and so by Fact 5.1 there are Euclidean isometries $\mathbb{R}^n \xrightarrow{\vartheta_i} \mathbb{R}^n$ with

$$\|g_i - \vartheta_i\|_{\infty}^{B_i} \leq t_i := c \sqrt{d_i \varepsilon_i} \leq c \sqrt{2r_i \varepsilon_i} \leq 2c \varepsilon_i^{\frac{1}{4}}$$

where $c = c(n)$ and $d_i := \text{diam}(B_i)$. Hence by Facts 2.8(a,b,c)

$$\text{dist}_{\mathcal{H}}(g_i B_i, \vartheta_i B_i) \leq t_i \rightarrow 0$$

and therefore

$$\delta_{\vartheta_i B_i} - \delta_{g_i B_i} \rightarrow 0 \text{ uniformly in } \mathbb{R}^n. \quad (5.5)$$

We demonstrate that $\hat{\text{dist}}_{\mathcal{H}}(\vartheta_i A_i, A) \rightarrow 0$. Since A is unbounded, Theorem 3.5 tells us that it suffices to show that $\delta_{\vartheta_i A_i} \rightarrow \delta_A$ boundedly uniformly in $\mathbb{R}^n$

Let $\delta := \delta_A$, $\delta_i := \delta_{g_i A_i}$, $\tilde{\delta}_i(x) := \delta_{f_i A_i}$ (i.e., the distances to A , $\mathcal{G}_i(A_i)$, $f_i(A_i)$). We claim that for each $R > 0$ and each $\varepsilon > 0$ there is an i_o such that for all $i \geq i_o$ and all $x \in B[0; R]$

$$|\tilde{\delta}_i(x) - \delta(x)| < \varepsilon, \quad (5.6a)$$

$$\tilde{\delta}_i(x) \leq \delta_{g_i B_i}(x) \leq \tilde{\delta}_i(x) + \varepsilon_i, \quad (5.6b)$$

$$\delta_{g_i B_i}(x) = \delta_i(x). \quad (5.6c)$$

We establish these inequalities below. Employing (5.6b) and (5.6c) in conjunction with (5.5) we see that

$$|\delta_i - \tilde{\delta}_i| \leq |\delta_{g_i B_i} - \delta_{g_i B_i}| + \varepsilon_i \rightarrow 0 \text{ uniformly in } B[0; R].$$

By (5.6a), $|\tilde{\delta}_i - \delta| \rightarrow 0$ uniformly in $B[0; R]$. Since $|\delta_i - \delta| \leq |\delta_i - \tilde{\delta}_i| + |\tilde{\delta}_i - \delta|$, it now follows that $|\delta_i - \delta| \rightarrow 0$ uniformly in $B[0; R]$, as required.

It remains to corroborate the inequalities (5.6). Let $R > 0$ and $\varepsilon > 0$ be given. As $\varepsilon_i \rightarrow 0$, $r_i \rightarrow \infty$, and $t_i \leq 2c\varepsilon_i^{1/4}$, there is an i_o such that for all $i \geq i_o$, $r_i > 10R$, $\varepsilon_i < \varepsilon$, and $2R + 4\varepsilon_i + 2t_i \leq r_i$. Pick such an i_o and fix $i \geq i_o$ and $x \in B[0; R]$.

Since $f_i(A_i) \subset A$, $\delta \leq \tilde{\delta}_i$ holds everywhere. As A is closed, there is an $a \in A$ with $\delta(x) = |x - a|$. Since $0 \in A$, $|a| \leq 2R < r_i - 3\varepsilon_i$, so by (5.4a) there are $c_i \in B_i$ with $|a - f_i(c_i)| < \varepsilon_i$. Therefore,

$$\delta(x) \leq \tilde{\delta}_i(x) \leq |x - f_i(c_i)| \leq |x - a| + |a - f_i(c_i)| < \delta(x) + \varepsilon_i$$

which establishes (5.6a)

Since $g_i(B_i) \subset f_i(A_i)$, $\tilde{\delta}_i \leq \delta_{g_i B_i}$ holds everywhere. Pick $\tilde{a}_i \in A_i$ so that $\tilde{b}_i := f_i(\tilde{a}_i)$ satisfies the inequality $\tilde{\delta}_i(x) \leq |x - \tilde{b}_i| < \tilde{\delta}_i(x) + \varepsilon_i$. Note that as f_i needn't be continuous, $f_i(A_i)$ may not be closed. Since $b_i \in f_i(A_i)$, $|x - \tilde{b}_i| < |x - b_i| + \varepsilon_i$, so by (5.3b)

$$\begin{aligned} |\tilde{a}_i - a_i| &\leq |f(\tilde{a}_i) - f_i(a_i)| + \varepsilon_i = |\tilde{b}_i - b_i| + \varepsilon_i \leq |x - \tilde{b}_i| + |x - b_i| + \varepsilon_i \\ &\leq 2(|x - b_i| + \varepsilon_i) \leq 2(|x| + |b_i| + \varepsilon_i) \leq 2R + 4\varepsilon_i < r_i. \end{aligned}$$

Thus $\tilde{a}_i \in A_i \cap B[a_i; r_i] = B_i$, so $\tilde{b}_i = f_i(\tilde{a}_i) = g_i(\tilde{a}_i)$ and therefore

$$\tilde{\delta}_i(x) \leq \delta_{g_i B_i}(x) \leq |x - \tilde{b}_i| < \tilde{\delta}_i(x) + \varepsilon_i$$

which establishes (5.6b).

Since $\mathcal{G}_i(B_i) \subset \mathcal{G}_i(A_i)$, $\delta_i \leq \delta_{g_i B_i}$ holds everywhere. Pick $v_i \in A_i$ with $\delta_i(x) = |x - \mathcal{G}_i(v_i)|$. Now

$$|\mathcal{G}_i(a_i) - b_i| = |\mathcal{G}_i(a_i) - g_i(a_i)| \leq t_i,$$

so $|\mathcal{G}_i(a_i)| \leq |b_i| + t_i \leq \varepsilon_i + t_i$. Thus, as $\mathcal{G}_i$ is an isometry,

$$\begin{aligned} |v_i - a_i| &= |\mathcal{G}_i(v_i) - \mathcal{G}_i(a_i)| \leq |\mathcal{G}_i(v_i) - x| + |x - \mathcal{G}_i(a_i)| \leq 2|x - \mathcal{G}_i(a_i)| \\ &\leq 2(|x| + |\mathcal{G}_i(a_i)|) \leq 2(R + \varepsilon_i + t_i) \leq r_i. \end{aligned}$$

Therefore, $v_i \in A_i \cap B[a_i; r_i] = B_i$ and so

$$\delta_i(x) \leq \delta_{g_i B_i}(x) \leq |x - \mathcal{G}_i(v_i)| = \delta_i(x)$$

which establishes (5.6c). □

We can now establish an improved version of Theorem 4.11 for Euclidean regions.

5.7 Corollary.

- (a) Let $\emptyset \neq A, A_i \subsetneq \mathbb{R}^n$ be closed. Suppose $\text{dist}_{\mathcal{GH}}(A_i, A) \rightarrow 0$. Then $\text{dist}_{\mathcal{GH}^*}((\mathbb{R}^n \setminus A_i, k_i), (\mathbb{R}^n \setminus A, k)) \rightarrow 0$.

(b) Let $\emptyset \neq A, A_i \subsetneq \hat{\mathbb{R}}^n$ be closed. Suppose $\hat{\text{dist}}_{\mathcal{H}}(A_i, A) \rightarrow 0$. Then $\text{dist}_{\mathcal{H}}((\hat{\mathbb{R}}^n \setminus A_i, \hat{k}_i), (\hat{\mathbb{R}}^n \setminus A, \hat{k})) \rightarrow 0$.

Similar results hold for the Ferrand, Kulkarni-Pinkall-Thurston, and hyperbolic metrics.

Proof. The interested reader can readily verify (a). In (b), we view $(\hat{\mathbb{R}}^n, \chi)$ as the Euclidean unit sphere $S^n \subset \mathbb{R}^{n+1}$ with its inherited Euclidean distance. By Theorem 5.2(b) there are spherical isometries $\hat{\mathbb{R}}^n \xrightarrow{\psi_i} \hat{\mathbb{R}}^n$ such that $\hat{\text{dist}}_{\mathcal{H}}(\psi_i A_i, A) \rightarrow 0$. Thus $\hat{\delta}_{A'_i} \rightarrow \hat{\delta}_A$ uniformly in $\hat{\mathbb{R}}^n$, where $A'_i := \psi_i(A_i)$. Then by Theorem 4.4, $\text{dist}_{\mathcal{H}}((U'_i, \hat{k}'_i), (U, \hat{k})) \rightarrow 0$ where $U := \hat{\mathbb{R}}^n \setminus A$ and $U'_i := \hat{\mathbb{R}}^n \setminus A'_i$. Finally, $\text{dist}_{\mathcal{H}}((U'_i, \hat{k}'_i), (U_i, \hat{k}_i)) = 0$ where $U_i := \hat{\mathbb{R}}^n \setminus A_i$. $\square$

We conclude our work with the following partial converse to Theorems 4.4 and 4.11; this addresses the question: “What can we say when we have pointed Gromov-Hausdorff convergence of quasihyperbolic spaces?” Note that if (Ω_i) is a sequence of any open Euclidean balls $\Omega_i := B^n(a_i; r_i) \subset \mathbb{R}^n$, then $\text{dist}_{\mathcal{H}}((\Omega_i, k_i; a_i), (B^n, k_{B^n}; 0)) \rightarrow 0$; however, neither $(\partial\Omega_i)$ nor $(\mathbb{R}^n \setminus \Omega_i)$ need Hausdorff converge nor Gromov-Hausdorff converge to S^{n-1} or $\mathbb{R}^n \setminus B^n$. In particular, some sort of normalization is necessary.

Recall that $\mathcal{O}(n)$ is the group of orthogonal automorphisms of $\mathbb{R}^n$; it is the subgroup of all Euclidean isometries of $\mathbb{R}^n$ that fix the origin. Each $\theta \in \mathcal{O}(n)$ extends to $\hat{\mathbb{R}}^n$ via $\theta(\infty) := \infty$.

5.8 Theorem. Assume $n \geq 3$. Let $\Omega, \Omega_i \subsetneq \mathbb{R}^n$ be quasihyperbolic domains, $A := \mathbb{R}^n \setminus \Omega, A_i := \mathbb{R}^n \setminus \Omega_i, \delta := \delta_A, \delta_i := \delta_{A_i}$. Suppose $0 \in \Omega, \delta_i(0) \rightarrow \delta(0)$, and $\text{dist}_{\mathcal{H}}((\Omega_i, k_i; 0), (\Omega, k; 0)) \rightarrow 0$. Then there exist $\theta_i \in \mathcal{O}(n)$ such that $(\delta_i \circ \theta_i^{-1})$ converges boundedly uniformly in Ω to δ . Consequently, there are domains $\Omega'_i := \theta_i(\Omega_i)$ such that the sequence $(\delta_i^{-1} ds)$ of quasihyperbolic metrics (so $\delta'_i := \delta_{\partial\Omega'_i} = \delta_i|_{\Omega_i} \circ \theta_i^{-1}$) converges boundedly uniformly in (Ω, k) to $\delta^{-1} ds$.

Note that $\delta_i \circ \theta_i^{-1} = \delta_{A_i} \circ \theta_i^{-1} = \delta_{\theta_i A_i}$.

Proof. Since $\delta_i(0) > \frac{1}{2}\delta(0) > 0$ for all sufficiently large i , we may assume $0 \in \Omega_i$ for all i . Define

$$\Delta_i := \inf \left\{ \hat{\text{dist}}_{\mathcal{H}}(\theta(\hat{A}_i), F) \mid \theta \in \mathcal{O}(n), \emptyset \neq F \subset \hat{\mathbb{R}}^n \text{ closed with } \partial\Omega \subset F \cap \mathbb{R}^n \subset \mathbb{R}^n \setminus \Omega \right\}.$$

Below we demonstrate that $\Delta_i \rightarrow 0$, so there exists $\theta_i \in \mathcal{O}(n)$ and closed $\emptyset \neq F_i \subset \hat{\mathbb{R}}^n$ with $\partial\Omega \subset F_i \cap \mathbb{R}^n \subset \mathbb{R}^n \setminus \Omega$ such that $\hat{\text{dist}}_{\mathcal{H}}(\theta_i(\hat{A}_i), F_i) \rightarrow 0$.

To verify that $(\delta_i \circ \theta_i^{-1})$ converges boundedly uniformly in Ω to δ , we start with any subsequence $(\delta_h \circ \theta_h^{-1})$ of $(\delta_i \circ \theta_i^{-1})$. As (F_h) is a sequence in the compact metric space $(\mathcal{H}(\hat{\mathbb{R}}^n), \hat{\text{dist}}_{\mathcal{H}})$, there exist a closed nonempty set $F \subset \hat{\mathbb{R}}^n$ and a subsequence (F_j) of (F_h) (so a subsubsequence of (F_i)) such that $\hat{\text{dist}}_{\mathcal{H}}(F_j, F) \rightarrow 0$. The conditions on the sets F_j ensure that $\partial\Omega \subset B := F \cap \mathbb{R}^n \subset \mathbb{R}^n \setminus \Omega$. In particular, $B \neq \emptyset$ and in $\Omega, \delta_B = \delta_{\partial\Omega} = \delta$.

Evidently, $\hat{\text{dist}}_{\mathcal{H}}(\theta_j(\hat{A}_j), F) \rightarrow 0$. Therefore by Theorem 3.1,

$$\delta_j \circ \theta_j^{-1} = \delta_{A_j} \circ \theta_j^{-1} = \delta_{\theta_j A_j} \rightarrow \delta_B$$

boundedly uniformly in $\mathbb{R}^n$. As $\delta_B = \delta$ in Ω , we have established that $\delta_j \circ \theta_j^{-1} \rightarrow \delta$ boundedly uniformly in Ω . Thus every subsequence of $(\delta_i \circ \theta_i^{-1})$ has a further subsequence that converges boundedly uniformly in Ω to δ , so $(\delta_i \circ \theta_i^{-1})$ converges boundedly uniformly in Ω to δ .

It remains to explain why $\Delta_i \rightarrow 0$. Let (A_h) be any subsequence of (A_i) . As $(\hat{A}_h)$ is a sequence in the compact metric space $(\mathcal{H}(\hat{\mathbb{R}}^n), \hat{\text{dist}}_{\mathcal{H}})$, there exist a closed set $\emptyset \neq E \subset \hat{\mathbb{R}}^n$ and a subsequence (A_j) of (A_h) such that $\hat{\text{dist}}_{\mathcal{H}}(\hat{A}_j, E) \rightarrow 0$. Evidently, $0 \notin E$; also, $E \cap S^{n-1}(0; \delta(0)) \neq \emptyset$.

Thus $C := E \cap \mathbb{R}^n \neq \emptyset$ and $0 \notin C$. Appealing again to Theorem 3.1 we deduce that $\delta_j = \delta_{A_j} \rightarrow \delta_C$ boundedly uniformly in $\mathbb{R}^n$. Let D be the component of $\mathbb{R}^n \setminus C$ that contains 0; so $\partial D \subset C \subset \mathbb{R}^n \setminus D$ and in $D, \delta_{\partial D} = \delta_C$. It now follows that the sequence of metrics $(\delta_j^{-1} ds)$ converges boundedly uniformly in (D, k_D) to $\delta_D^{-1} ds$; see Example 4.2(d). Noting that in $\Omega_j, \delta_j = \delta_{\partial\Omega_j}$, we employ Theorem 4.4 to assert that

$$\text{dist}_{\mathcal{H}}((\Omega_j, k_j; 0), (D, k_D; 0)) \rightarrow 0.$$

Hence the pointed metric spaces $(\Omega, k; 0)$ and $(D, k_D; 0)$ are isometric; see [11, Proposition 3.12].

As $\delta_{\partial\Omega}(0) = \delta_{\partial D}(0)$, and $n \geq 3$, [22, Theorem 7.1.6] asserts that there exists $\theta \in \mathcal{O}(n)$ with $\theta(D) = \Omega$; so $\theta(\partial D) = \partial\Omega$. Put $F := \theta(E)$. From $\partial D \subset C = E \cap \mathbb{R}^n \subset \mathbb{R}^n \setminus \partial D$ we deduce that $\partial\Omega \subset F \cap \mathbb{R}^n \subset \mathbb{R}^n \setminus \Omega$. In particular, F is an admissible set in the definition of Δ_i . Hence

$$\Delta_j \leq \hat{\text{dist}}_{\mathcal{H}}(\theta(\hat{A}_j), F) = \hat{\text{dist}}_{\mathcal{H}}(\theta(\hat{A}_j), \theta(E)) = \hat{\text{dist}}_{\mathcal{H}}(\hat{A}_j, E) \rightarrow 0.$$

Since every subsequence of (Δ_i) has a further subsequence that converges to 0, $\Delta_i \rightarrow 0$. $\square$

5.9 Corollary. Assume $n \geq 3$. Let $\Omega, \Omega_i \subsetneq \mathbb{R}^n$ be quasihyperbolic domains, $a \in \Omega$, $a_i \in \Omega_i$. Put $\delta := \delta_{\mathbb{R}^n \setminus \Omega}$ and $\delta_i := \delta_{\mathbb{R}^n \setminus \Omega_i}$. The following are equivalent:

- (a) $\text{dist}_{\mathcal{G}\mathcal{H}}((\Omega_i, k_i; a_i), (\Omega, k; a)) \rightarrow 0$.
- (b) There exist similarities $\psi_i : (\mathbb{R}^n; a_i) \rightarrow (\mathbb{R}^n; a)$ such that the quasihyperbolic metrics in $\Omega'_i := \psi_i(\Omega_i)$ converge boundedly uniformly in (Ω, k) to its quasihyperbolic metric.
- (c) The sequence (δ'_i) converges pointwise in Ω to δ , where $\delta'_i := \delta_i \circ \psi_i^{-1}$.
- (d) For every sequence $i_1 < i_2 < \dots$ of natural numbers, Ω is a component of $\text{core}(\Omega'_{i_j})$.
- (e) The sequence (Ω'_i) kernel converges to Ω with respect to a .

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