Research Article

Ky Ho and Inbo Sim*

A-priori bounds and existence for solutions of weighted elliptic equations with a convection term

DOI: 10.1515/anona-2015-0177

Received December 22, 2015; revised March 3, 2016; accepted March 4, 2016

Abstract: We investigate weighted elliptic equations containing a convection term with variable exponents that are subject to Dirichlet or Neumann boundary condition. By employing the De Giorgi iteration and a localization method, we give a-priori bounds for solutions to these problems. The existence of solutions is also established using Brezis' theorem for pseudomonotone operators.

Keywords: $p(x)$ -Laplacian, weighted variable exponent Lebesgue–Sobolev spaces, a-priori bound, De Giorgi iteration, localization method, pseudomonotone operators

MSC 2010: 35J20, 35J60, 35J70, 47J10, 46E35

1 Introduction

Recently, PDE problems containing $p(x)$ -Laplacian, $\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, have been studied intensively (see for example [2, 4, 6, 8, 14–18, 20] and the references therein). One reason is that various real fields require PDE problems with variable exponents; for example, electrorheological fluids and image processing. The other reason is that such problems are mathematically challengeable since $p(x)$ -Laplacian is not homogeneous and do not guarantee the positivity and simplicity of the first eigenvalue subject to the Dirichlet boundary condition. However, very few works have addressed $p(x)$ -Laplacian-type problems with either degeneracy (a weight w may neither bounded nor away from 0) or the convection term.

In this article, first we study a-priori bounds of solutions for the following types of problems:

$$\begin{cases} -\operatorname{div} A(x, u, \nabla u) = B(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \quad (1.1)$$

and

$$\begin{cases} -\operatorname{div} A(x, u, \nabla u) = B(x, u, \nabla u) & \text{in } \Omega, \\ A(x, u, \nabla u) \cdot \nu = C(x, u) & \text{on } \Gamma, \end{cases} \quad (1.2)$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) with a Lipschitz boundary $\Gamma = \partial\Omega$, $\nu(x)$ denotes the outer unit normal of Ω at $x \in \Gamma$, and the three maps $A : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$, $B : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ and $C : \Gamma \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy the Carathéodory condition. On Ω and Γ , we use the N -dimensional Lebesgue measure and the usual $(N-1)$ -dimensional surface measure on $\mathbb{R}^N$, respectively. Throughout the paper, w is a weight function that is measurable and positive a.e. in Ω , $p \in C(\overline{\Omega}, (1, \infty)) =: C_+(\overline{\Omega})$, and for each $q \in C(\overline{\Omega})$, $q^- := \min_{x \in \overline{\Omega}} q(x)$, $q^+ := \max_{x \in \overline{\Omega}} q(x)$, and q' is the conjugate function of q , i.e., $\frac{1}{q(x)} + \frac{1}{q'(x)} = 1$ for all $x \in \overline{\Omega}$. Furthermore, we

Ky Ho: NTIS, University of West Bohemia, Technická 8, 306 14 Plzeň, Czech Republic, e-mail: ngockyh@ntis.zcu.cz

***Corresponding author: Inbo Sim:** Department of Mathematics, University of Ulsan, Ulsan 680-749, Republic of Korea, e-mail: ibsim@ulsan.ac.kr. <http://orcid.org/0000-0002-1618-054X>

assume that

$$|A(x, t, \xi)| \leq \alpha_1 \left[k(x) + w(x)^{\frac{1}{p(x)}} |t|^{\frac{q(x)}{p'(x)}} + w(x) |\xi|^{p(x)-1} \right] \quad \text{for a.e. } x \in \Omega \text{ and all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N, \quad (\text{A1})$$

$$A(x, t, \xi) \cdot \xi \geq \alpha_2 w(x) |\xi|^{p(x)} - \alpha_3 (|t|^{q(x)} + 1) \quad \text{for a.e. } x \in \Omega \text{ and all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N, \quad (\text{A2})$$

$$|B(x, t, \xi)| \leq \beta \left[1 + |t|^{q(x)-1} + w(x)^{\frac{1}{q'(x)}} |\xi|^{\frac{p(x)}{q'(x)}} \right] \quad \text{for a.e. } x \in \Omega \text{ and all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N, \quad (\text{B1})$$

$$|C(x, t)| \leq \gamma [1 + |t|^{r(x)-1}] \quad \text{for a.e. } x \in \Gamma \text{ and all } t \in \mathbb{R}, \quad (\text{C1})$$

where $\alpha_1, \alpha_2, \alpha_3, \beta, \gamma$ are positive constants, $q \in C_+(\overline{\Omega})$ and $r \in C(\Gamma, (1, \infty))$ are subcritical exponents specified later, and k is a measurable function such that $kw^{-\frac{1}{p}} \in L^{p'(x)}(\Omega)$. Fan–Zhao [7] studied the boundedness of solutions to the equation

$$-\operatorname{div} A(x, u, \nabla u) = B(x, u, \nabla u) \quad \text{in } \Omega. \quad (1.3)$$

They showed that, under conditions (A1), (A2), (B1) in which w, k are constants, q is a subcritical exponent, a solution for (1.3) is in $L_{\text{loc}}^\infty(\Omega)$ and is in $L^\infty(\Omega)$ if it is bounded on Γ . Gasiński–Papageorgiou [9] showed the boundedness of a solution of (1.2) when $w \equiv 1$, $A(x, t, \xi) = |\xi|^{p(x)-2} \xi$, B is independent of ξ , and $C(x, t) \equiv 0$. Winkert–Zacher [19] also gave an a-priori bound for solutions of (1.2) under conditions (A1), (A2), (B1), (C1) in which w, k are constants and q, r are subcritical exponents. A weighted Dirichlet problem (1.1) when $A(x, t, \xi) = w(x) |\xi|^{p(x)-2} \xi$ and B is independent of ξ was addressed by us in [10]. The main idea of the proofs is based on the De Giorgi iteration technique (in [7, 10, 19]) or the Moser iteration technique (in [9]).

Next, we study the existence of solutions for the following types of problems:

$$\begin{cases} -\operatorname{div} A(x, u, \nabla u) + b(x) |u|^{p(x)-2} u = B(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \quad (1.4)$$

and

$$\begin{cases} -\operatorname{div} A(x, u, \nabla u) + b(x) |u|^{p(x)-2} u = B(x, u, \nabla u) & \text{in } \Omega, \\ A(x, u, \nabla u) \cdot \nu = C(x, u) & \text{on } \Gamma, \end{cases} \quad (1.5)$$

where $b \in L^\infty(\Omega)$. Note that, under conditions (A1), (A2), (B1), (C1) with $p(x) \leq q(x)$ for all $x \in \overline{\Omega}$, problems (1.4) and (1.5) are equivalent to (1.1) and (1.2), respectively. To verify the existence of solutions, we assume the monotonicity of A with respect to the variable corresponding to the gradient term and modify conditions (A2), (B1) and (C1) as follows:

$$A(x, t, \xi) \cdot \xi \geq \alpha_2 w(x) |\xi|^{p(x)} - \alpha_3 (|t|^{\alpha(x)} + 1) \quad \text{for a.e. } x \in \Omega \text{ and all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N, \quad (\tilde{\text{A2}})$$

where $\alpha \in C_+(\overline{\Omega})$ such that $\alpha^+ < p^-$,

$$0 < [A(x, t, \xi_1) - A(x, t, \xi_2)] \cdot (\xi_1 - \xi_2) \quad \text{for a.e. } x \in \Omega, \text{ all } t \in \mathbb{R} \text{ and all } \xi_1, \xi_2 \in \mathbb{R}^N, \xi_1 \neq \xi_2, \quad (\text{A3})$$

$$|B(x, t, \xi)| \leq \beta \left[h(x) + |t|^{\eta(x)} + w(x)^{\frac{1}{q'(x)}} |\xi|^{\frac{\delta(x)}{q'(x)}} \right] \quad \text{for a.e. } x \in \Omega \text{ and all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N, \quad (\tilde{\text{B1}})$$

where $h \in L^{q'(x)}(\Omega)$, $\eta \in C(\overline{\Omega})$, $\delta \in C_+(\overline{\Omega})$ such that $0 < \eta^- \leq \eta^+ < p^- - 1$, $(\frac{\delta}{q})^+ < p^- - 1$,

$$|C(x, t)| \leq \gamma [m(x) + |t|^{r(x)-1}] \quad \text{for a.e. } x \in \Gamma \text{ and all } t \in \mathbb{R}, \quad (\tilde{\text{C1}})$$

where $m \in L^{r'(x)}(\Gamma)$ and $r^+ < p^-$. Benboubker–Azroul–Barbara [1] showed the existence of a solution for (1.4) with $b \equiv 0$ under (A1), ($\tilde{\text{A2}}$), (A3) in which $w \equiv 1$, $q = p$, $\alpha_3 = 0$ and a stronger condition than ($\tilde{\text{B1}}$). Boureanu–Rădulescu [3] studied (1.5) with $A = A(x, \xi) = (a_1(x, \xi_1), \dots, a_N(x, \xi_N))$, where $\xi = (\xi_1, \dots, \xi_N)$, $B = B(x, t)$ under (A1), ($\tilde{\text{A2}}$), (A3), ($\tilde{\text{B1}}$), ($\tilde{\text{C1}}$) in which $\operatorname{ess\,inf}_\Omega b > 0$, $w = h \equiv 1$, $m \equiv 1$, and $\alpha_3 = 0$. In fact, the condition corresponding to ($\tilde{\text{A2}}$) in their paper was in the anisotropic form $a_i(x, \xi_i) \xi_i \geq |\xi_i|^{p_i(x)}$, $p_i \in C_+(\overline{\Omega})$ for all $i \in \{1, \dots, N\}$, which is a special one of our case when $p_1 = \dots = p_N$. The proofs for (1.4) and (1.5) in these works come from the Leray–Lions Theorem or direct methods in Calculus of Variations.

The novelty of this paper is that we consider simultaneously the degeneracy and the convection term in A and B , which is the first attempt as far as the authors know. In addition, we found some flaws in the proofs in [19] and noticed the authors in [19] a corresponding correct argument in the proof of Theorem 3.3 in the present paper. Note that, in [1], although they only assumed that $\eta(x) < p(x) - 1$ and $\tilde{\delta}(x) = \frac{\delta(x)}{q'(x)} < \frac{p(x)-1}{p'(x)}$

for all $x \in \bar{\Omega}$, they indeed used the facts $\eta^+ < p^- - 1$ and $\bar{\delta}^+ < (\frac{p^- - 1}{p^-})^-$ in their proof. Thus, our assumption $\bar{\delta}^+ < p^- - 1$ is weaker than theirs. Lastly, Theorem 4.2 shows the existence of solutions for a problem that is more general than that of [3] in the case $p_1 = \dots = p_N$.

The paper is organized as follows. In Section 2, we review the weighted variable exponent Lebesgue–Sobolev spaces and list the properties of these spaces. In Section 3, we give a-priori bounds for solutions of problems (1.1) and (1.2) using the De Giorgi iteration and a localization method. In Section 4, we establish the existence of solutions for (1.4) and (1.5) using Brezis’ theorem for pseudomonotone operators.

2 Abstract framework and preliminary results

We review the weighted variable exponent Lebesgue–Sobolev spaces $L^{p(x)}(w, \Omega)$ and $W^{1,p(x)}(w, \Omega)$, which were studied in [10, 13]. For the variable exponent Lebesgue–Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, we refer to [6, 8, 14] and the references therein.

Let Ω be a bounded domain in $\mathbb{R}^N$ with a Lipschitz boundary Γ . For an arbitrary weight w that is measurable and positive a.e. in Ω and $p \in C_+(\bar{\Omega})$, we define the weighted variable exponent Lebesgue space by

$$L^{p(x)}(w, \Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} \text{ is measurable, } \int_{\Omega} w(x)|u(x)|^{p(x)} dx < \infty \right\}.$$

Then $L^{p(x)}(w, \Omega)$ endowed with the norm

$$|u|_{p,w} = \inf \left\{ \lambda > 0 : \int_{\Omega} w(x) \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}$$

becomes a normed space. When $w(x) \equiv 1$, we have $L^{p(x)}(w, \Omega) \equiv L^{p(x)}(\Omega)$ and we use the notation $|u|_p$ instead of $|u|_{p,w}$.

Proposition 2.1 ([6, 14]). *The space $L^{p(x)}(\Omega)$ is a separable and uniformly convex Banach space, and its conjugate space is $L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$\left| \int_{\Omega} uv dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) |u|_p |v|_{p'} \leq 2 |u|_p |v|_{p'}.$$

Define $\rho : L^{p(x)}(w, \Omega) \rightarrow \mathbb{R}$ by

$$\rho(u) = \int_{\Omega} w(x)|u(x)|^{p(x)} dx \quad \text{for all } u \in L^{p(x)}(w, \Omega).$$

Proposition 2.2 ([13]). *For all $u \in L^{p(x)}(w, \Omega)$, we have:*

- (i) $|u|_{p,w} < 1$ ($= 1, > 1$) if and only if $\rho(u) < 1$ ($= 1, > 1$), respectively,
- (ii) if $|u|_{p,w} > 1$, then $|u|_{p,w}^{p^-} \leq \rho(u) \leq |u|_{p,w}^{p^+}$,
- (iii) if $|u|_{p,w} < 1$, then $|u|_{p,w}^{p^+} \leq \rho(u) \leq |u|_{p,w}^{p^-}$.

Consequently, we have

$$|u|_{p,w}^{p^-} - 1 \leq \rho(u) \leq |u|_{p,w}^{p^+} + 1.$$

Proposition 2.3 ([10]). *If $u, u_n \in L^{p(x)}(w, \Omega)$ ($n = 1, 2, \dots$), then the following statements are equivalent:*

- (i) $\lim_{n \rightarrow \infty} |u_n - u|_{p,w} = 0$,
- (ii) $\lim_{n \rightarrow \infty} \rho(u_n - u) = 0$.

Proposition 2.4 ([12]). *Let $p \in C_+(\bar{\Omega})$ and $q \in C(\bar{\Omega})$ such that $pq \in C_+(\bar{\Omega})$. For all $u \in L^{p(x)q(x)}(w, \Omega)$, we have:*

- (i) if $|u|_{pq,w} \geq 1$, then $|u|_{pq,w}^{q^-} \leq ||u|^q|_{p,w} \leq |u|_{pq,w}^{q^+}$,
- (ii) if $|u|_{pq,w} < 1$, then $|u|_{pq,w}^{q^+} \leq ||u|^q|_{p,w} \leq |u|_{pq,w}^{q^-}$.

Consequently, we have

$$|u|_{pq,w}^{q^-} - 1 \leq ||u|^q|_{p,w} \leq |u|_{pq,w}^{q^+} + 1.$$

The weighted variable exponent Sobolev space $W^{1,p(x)}(w, \Omega)$ is defined by

$$W^{1,p(x)}(w, \Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(w, \Omega)\},$$

with the norm

$$\|u\| = |u|_p + \|\nabla u\|_{p,w}.$$

This norm is equivalent to the norm

$$\|u\|_1 = \inf \left\{ \lambda > 0 : \rho_1\left(\frac{u}{\lambda}\right) \leq 1 \right\}$$

corresponding to the modular

$$\rho_1(u) := \int_{\Omega} (|u(x)|^{p(x)} + w(x)|\nabla u(x)|^{p(x)}) dx$$

with the following estimate:

$$\frac{1}{2}\|u\| \leq \|u\|_1 \leq 2\|u\| \quad \text{for all } u \in W^{1,p(x)}(w, \Omega). \quad (2.1)$$

The space $W_0^{1,p(x)}(w, \Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(w, \Omega)$ with respect to the norm $\|\cdot\|$. When $w(x) \equiv 1$, we have $W^{1,p(x)}(w, \Omega) \equiv W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(w, \Omega) \equiv W_0^{1,p(x)}(\Omega)$.

Throughout this paper, we assume that the weight w satisfies

$$w \in L^1(\Omega) \text{ and } w^{-s} \in L^1(\Omega) \text{ for some } s \in C(\overline{\Omega}) \text{ such that } s(x) \in \left(\frac{N}{p(x)}, \infty\right) \cap \left(\frac{1}{p(x)-1}, \infty\right) \text{ for all } x \in \overline{\Omega}. \quad (w1)$$

Proposition 2.5 ([11, 13]). Assume that (w1) holds. Then $W^{1,p(x)}(w, \Omega)$ is a separable reflexive Banach space.

Let us denote

$$p_s(x) := \frac{p(x)s(x)}{1+s(x)} < p(x),$$

where $s(x)$ is given in (w1), and

$$p_s^*(x) := \begin{cases} \frac{Np_s(x)}{N-p_s(x)} & \text{if } p_s(x) < N, \\ +\infty & \text{if } p_s(x) \geq N, \end{cases} \quad p_s^\partial(x) := \begin{cases} \frac{(N-1)p_s(x)}{N-p_s(x)} & \text{if } p_s(x) < N, \\ +\infty & \text{if } p_s(x) \geq N, \end{cases}$$

for all $x \in \overline{\Omega}$. The next imbedding results are crucial in our argument.

Proposition 2.6 ([13]). Assume that (w1) holds. If $q \in C_+(\overline{\Omega})$ satisfies $q(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$, then we obtain the continuous compact imbedding

$$W^{1,p(x)}(w, \Omega) \hookrightarrow W^{1,p_s(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega).$$

By this result, we have $W^{1,p(x)}(w, \Omega) \subset W^{1,p_s(x)}(\Omega) \subset W^{1,1}(\Omega)$. Since $W^{1,1}(\Omega) \hookrightarrow L^1(\Gamma)$, the trace $u|_\Gamma$ has definite meaning for any $u \in W^{1,p(x)}(w, \Omega)$. The classical boundary trace imbedding theorem was extended to variable exponent cases when the weight $w \equiv 1$.

Proposition 2.7 ([5]). Assume $q \in C_+(\overline{\Omega})$ and $r \in C(\Gamma)$ satisfy $1 \leq r(x) < q^\partial(x)$ for all $x \in \Gamma$, where

$$q^\partial(x) := \begin{cases} \frac{(N-1)q(x)}{N-q(x)} & \text{if } q(x) < N, \\ +\infty & \text{if } q(x) \geq N. \end{cases}$$

Then

$$W^{1,q(x)}(\Omega) \hookrightarrow L^{r(x)}(\Gamma).$$

Thanks to the last two propositions we obtain the following boundary trace imbedding for the weighted variable exponent Sobolev spaces.

Proposition 2.8. Assume that (w1) holds. If $r \in C(\Gamma)$ and $1 \leq r(x) < p_s^\partial(x)$ for all $x \in \Gamma$, then we obtain the continuous compact imbedding

$$W^{1,p(x)}(w, \Omega) \hookrightarrow L^{r(x)}(\Gamma).$$

Proof. By Proposition 2.6, $W^{1,p(x)}(w, \Omega) \hookrightarrow W^{1,p_s(x)}(\Omega)$. Meanwhile, we have $W^{1,p_s(x)}(\Omega) \hookrightarrow L^{r(x)}(\Gamma)$ due to Proposition 2.7. Combining these facts, we deduce $W^{1,p(x)}(w, \Omega) \hookrightarrow L^{r(x)}(\Gamma)$. $\square$

In fact, in Propositions 2.5, 2.6 and 2.8, the condition $w \in L^1(\Omega)$ in (w1) can be replaced by $w \in L_{\text{loc}}^1(\Omega)$. In spite of this, we assume $w \in L^1(\Omega)$ in (w1) because we need this restriction for all our results in what follows.

Definition 2.9. We say that $u \in W_0^{1,p(x)}(w, \Omega)$ is a weak solution (resp. subsolution, supersolution) of (1.1) if

$$\int_{\Omega} A(x, u, \nabla u) \cdot \nabla \varphi \, dx = (\text{resp. } \leq, \geq) \int_{\Omega} B(x, u, \nabla u) \varphi \, dx$$

holds for all nonnegative test functions $\varphi \in W_0^{1,p(x)}(w, \Omega)$.

We say that $u \in W^{1,p(x)}(w, \Omega)$ is a weak solution (resp. subsolution, supersolution) of (1.2) if

$$\int_{\Omega} A(x, u, \nabla u) \cdot \nabla \varphi \, dx = (\text{resp. } \leq, \geq) \int_{\Omega} B(x, u, \nabla u) \varphi \, dx + \int_{\Gamma} C(x, u) \varphi \, d\sigma \quad (2.2)$$

holds for all nonnegative test functions $\varphi \in W^{1,p(x)}(w, \Omega)$.

Note that, by Propositions 2.6 and 2.8, the definitions above make sense under conditions (w1), (A1), (B1), (C1), in which $q(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$ and $r(x) < p_s^\partial(x)$ for all $x \in \Gamma$. Similarly, we can define a weak solution of (1.4) and (1.5) with $b \in L^\infty(\Omega)$, (w1), (A1), ($\bar{B}$ 1), ($\bar{C}$ 1), in which $p(x) \leq q(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$.

3 A-priori bounds

In this section, we show the L^∞ -bound of a weak solution for (1.1) and (1.2) by modifying the De Giorgi iteration and localization method, which were used in [10, 19]. To do this, we need to restrict the weight w as below:

$$w \in L^\infty(\Omega) \text{ and } w^{-s^+} \in L^1(\Omega) \text{ for some } s \in C(\overline{\Omega}) \text{ such that } s(x) \in \left(\frac{N}{p(x)}, \infty\right) \cap \left(\frac{1}{p(x)-1}, \infty\right) \text{ for all } x \in \overline{\Omega}. \quad (w2)$$

Denote $u_+ = \max(u, 0)$. The next theorem gives us an a-priori bound of solutions for (1.1).

Theorem 3.1. Assume that (w2), (A1), (A2), (B1) hold, in which $q(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$. Then there exist positive constants τ, C such that

$$\text{ess sup}_{\Omega} u \leq C \left[1 + \left(\int_{\Omega} u_+^{\tilde{q}(x)} \, dx \right)^\tau \right]$$

and

$$-C \left[1 + \left(\int_{\Omega} (-v)_+^{\tilde{q}(x)} \, dx \right)^\tau \right] \leq \text{ess inf}_{\Omega} v,$$

for any weak subsolution u of (1.1) and for any weak supersolution v of (1.1), respectively, where $\tilde{q}(x) = \max(p(x), q(x))$ for all $x \in \overline{\Omega}$.

Corollary 3.2. Using the assumptions in Theorem 3.1, there exist positive constants τ, C such that, if u is a weak solution of (1.1), then $u \in L^\infty(\Omega)$, and

$$\|u\|_{L^\infty(\Omega)} \leq C \left[1 + \left(\int_{\Omega} |u|^{\tilde{q}(x)} \, dx \right)^\tau \right],$$

where $\tilde{q}(x) = \max(p(x), q(x))$ for all $x \in \overline{\Omega}$.

To obtain an a-priori bound of solutions for (1.2), we need to additionally impose $p(x) < p_s^\partial(x)$ for all $x \in \Gamma$, that is, $s(x) > \frac{N}{p(x)-1}$ for all $x \in \Gamma$. The next theorem gives us an a-priori bound of solutions for (1.2) up to boundary (remind that on the boundary we are in fact considering the trace of solutions). Such an estimate is meaningful since we have no information about the continuity of solutions for (1.2) up to boundary.

Theorem 3.3. Assume that (w2), (A1), (A2), (B1), (C1) hold, in which $s(x) > \frac{N}{p(x)-1}$ and $r(x) < p_s^\partial(x)$ for all $x \in \Gamma$ and $q(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$. Then there exist positive constants τ, C such that

$$\max\left(\operatorname{ess\,sup}_{\Omega} u, \operatorname{ess\,sup}_{\Gamma} u\right) \leq C \left[1 + \left(\int_{\Omega} u_+^{\tilde{q}(x)} dx + \int_{\Gamma} u_+^{\tilde{r}(x)} d\sigma \right)^{\tau} \right],$$

and

$$-C \left[1 + \left(\int_{\Omega} (-v)_+^{\tilde{q}(x)} dx + \int_{\Gamma} (-v)_+^{\tilde{r}(x)} d\sigma \right)^{\tau} \right] \leq \min\left(\operatorname{ess\,inf}_{\Omega} v, \operatorname{ess\,inf}_{\Gamma} v\right),$$

for any weak subsolution u of (1.2) and for any weak supersolution v of (1.2), respectively, where $\tilde{q}(x) = \max(p(x), q(x))$ for all $x \in \overline{\Omega}$ and $\tilde{r}(x) = \max(p(x), r(x))$ for all $x \in \Gamma$.

Corollary 3.4. Using the assumptions in Theorem 3.3, there exist positive constants τ, C such that, if u is a weak solution of (1.2), then $u \in L^\infty(\Omega) \cap L^\infty(\Gamma)$, and

$$\max(|u|_{L^\infty(\Omega)}, |u|_{L^\infty(\Gamma)}) \leq C \left[1 + \left(\int_{\Omega} |u|^{\tilde{q}(x)} dx + \int_{\Gamma} |u|^{\tilde{r}(x)} d\sigma \right)^{\tau} \right],$$

where $\tilde{q}(x) = \max(p(x), q(x))$ for all $x \in \overline{\Omega}$ and $\tilde{r}(x) = \max(p(x), r(x))$ for all $x \in \Gamma$.

Example 3.5. It is easy to verify that

$$w(x) = |x|^a \quad (0 \leq a < \min(p^-, N(p^- - 1)))$$

satisfies (w2) with $s(x) \equiv \frac{N-\epsilon}{a}$, where $0 < \epsilon < \min(\frac{N(p^- - a)}{p^-}, \frac{N(p^- - 1) - a}{p^- - 1})$. Similarly, we can verify that

$$w(x) = |x|^a \quad (0 \leq a < p^- - 1)$$

satisfies (w2) with some $s(x) > \frac{N}{p(x)-1}$ on Γ (taking $s(x) \equiv \frac{N-\epsilon}{a}$, where $0 < \epsilon < \frac{N(p^- - 1) - a}{p^- - 1}$).

To employ the De Giorgi iteration, we need the following lemma, that was obtained by us in [10].

Lemma 3.6 ([10]). Let $\{Y_n\}_{n=0}^\infty$ be a sequence of positive numbers satisfying the recursion inequality

$$Y_{n+1} \leq Kb^n (Y_n^{1+\delta_1} + Y_n^{1+\delta_2}), \quad n = 0, 1, 2, \dots, \quad (3.1)$$

for some $b > 1, K > 0$ and $\delta_2 \geq \delta_1 > 0$. If $Y_0 \leq \min(1, (2K)^{\frac{1}{\delta_1}} b^{\frac{1}{\delta_1}})$ or

$$Y_0 \leq \min\left((2K)^{\frac{1}{\delta_1}} b^{\frac{1}{\delta_1}}, (2K)^{\frac{1}{\delta_2}} b^{-\frac{1}{\delta_1\delta_2} - \frac{\delta_2 - \delta_1}{\delta_2^2}}\right),$$

then $Y_n \leq 1$ for some $n \in \mathbb{N} \cup \{0\}$. Moreover,

$$Y_n \leq \min\left(1, (2K)^{\frac{1}{\delta_1}} b^{\frac{1}{\delta_1}} b^{-\frac{n}{\delta_1}}\right) \quad \text{for all } n \geq n_0,$$

where n_0 is the smallest $n \in \mathbb{N} \cup \{0\}$ satisfying $Y_n \leq 1$. In particular, $Y_n \rightarrow 0$ as $n \rightarrow \infty$.

Remark 3.7. It is worth noting that Lemma 3.6 is a correction of [19, Lemma 2.1] where it was incorrectly stated that

$$Y_n \leq (2K)^{\frac{1}{\delta_1}} b^{\frac{1}{\delta_1}} b^{-\frac{n}{\delta_1}} \quad \text{for all } n \in \mathbb{N}$$

if $Y_0 \leq (2K)^{\frac{1}{\delta_1}} b^{\frac{1}{\delta_1}}$. It is easy to see that if we choose a sequence $\{Y_n\}_{n=0}^\infty$ with $Y_0 = \frac{5}{4}, Y_1 = \frac{45}{64}$ satisfying the recursion inequality (3.1) with $\delta_1 = 1, \delta_2 = 2, K = \frac{1}{5}$, and $b = 2$, then we have

$$Y_0 \leq (2K)^{\frac{1}{\delta_1}} b^{\frac{1}{\delta_1}} \quad \text{but} \quad Y_1 > (2K)^{\frac{1}{\delta_1}} b^{\frac{1}{\delta_1}} b^{-\frac{1}{\delta_1}} = \frac{5}{8}.$$

We emphasize that, to deal with the De Giorgi iteration on the weighted spaces, we need to obtain some crucial imbeddings on these spaces. Since the proof of Theorem 3.1 is similar to that of Theorem 3.3 and is simpler, we will give the proof of Theorem 3.3 in detail and sketch the proof of Theorem 3.1.

Proof of Theorem 3.3. It is easy to see that (A1), (A2), and (B1) still hold with $\tilde{q}(x)$ instead of $q(x)$ (with new k, α_3, β). Similarly, (C1) holds with $\tilde{r}(x)$ instead of $r(x)$ (with a new γ). All following estimates use $\tilde{q}(x), \tilde{r}(x)$; however, for simplicity, we write $q(x)$ (resp. $r(x)$) instead of $\tilde{q}(x)$ (resp. $\tilde{r}(x)$) and still use α_3, β, γ for corresponding new constants. By means of this and the assumption on w , we have

$$p(x) \leq q(x) < p_s^*(x) \quad \text{for all } x \in \bar{\Omega} \quad \text{and} \quad p(x) \leq r(x) < p_s^{\partial}(x) \quad \text{for all } x \in \Gamma. \quad (3.2)$$

(i) Truncated energy estimates. For a given $u \in W^{1,p(x)}(w, \Omega)$, denote

$$\begin{aligned} A_k &= \{x \in \Omega : u(x) > k\}, & \tilde{A}_k &= \{x \in \Omega : -u(x) > k\}, \\ \Gamma_k &= \{x \in \Gamma : u(x) > k\}, & \tilde{\Gamma}_k &= \{x \in \Gamma : -u(x) > k\}, \quad k \in \mathbb{N}. \end{aligned}$$

We claim that there exist positive constants d_1, d_2 independent of u such that, for all $k \in \mathbb{N}$, if u is a weak subsolution of (1.2), then

$$\int_{A_k} w(x) |\nabla u|^{p(x)} dx \leq d_1 \int_{A_k} u^{q(x)} dx + d_2 \int_{\Gamma_k} u^{r(x)} d\sigma, \quad (3.3)$$

and if u is a weak supersolution of (1.2), then

$$\int_{\tilde{A}_k} w(x) |\nabla(-u)|^{p(x)} dx \leq d_1 \int_{\tilde{A}_k} (-u)^{q(x)} dx + d_2 \int_{\tilde{\Gamma}_k} (-u)^{r(x)} d\sigma. \quad (3.4)$$

Indeed, taking $\varphi = (u - k)_+ \in W^{1,p(x)}(w, \Omega)$ in (2.2) when u is a weak subsolution of (1.2), it yields

$$\int_{\Omega} A(x, u, \nabla u) \cdot \nabla \varphi dx \leq \int_{\Omega} B(x, u, \nabla u) \varphi dx + \int_{\Gamma} C(x, u) \varphi d\sigma.$$

Equivalently,

$$\int_{A_k} A(x, u, \nabla u) \cdot \nabla u dx \leq \int_{A_k} B(x, u, \nabla u)(u - k) dx + \int_{\Gamma_k} C(x, u)(u - k) d\sigma.$$

Since $u \geq u - k > 0$ and $u \geq k \geq 1$ on A_k, Γ_k , using (A2), (B1), and (C1) and taking into account a generalized Young's inequality, we estimate each term in the preceding inequality as follows:

$$\begin{aligned} \int_{A_k} A(x, u, \nabla u) \cdot \nabla u dx &\geq \alpha_2 \int_{A_k} w(x) |\nabla u|^{p(x)} dx - \alpha_3 \int_{A_k} (1 + u^{q(x)}) dx \\ &\geq \alpha_2 \int_{A_k} w(x) |\nabla u|^{p(x)} dx - 2\alpha_3 \int_{A_k} u^{q(x)} dx, \\ \int_{A_k} B(x, u, \nabla u)(u - k) dx &\leq \beta \int_{A_k} \left(1 + u^{q(x)-1} + w^{\frac{1}{q^+(x)}} |\nabla u|^{\frac{p(x)}{q^+(x)}}\right) u dx \\ &\leq 2\beta \int_{A_k} u^{q(x)} dx + \frac{\alpha_2}{2} \int_{A_k} w(x) |\nabla u|^{p(x)} dx + C_1(\alpha_2, \beta, q^+, q^-) \int_{A_k} u^{q(x)} dx, \end{aligned}$$

and

$$\int_{\Gamma_k} C(x, u)(u - k) d\sigma \leq \gamma \int_{\Gamma_k} (1 + u^{r(x)-1}) u d\sigma \leq 2\gamma \int_{\Gamma_k} u^{r(x)} d\sigma.$$

Combining the last four estimates, we obtain (3.3). Similarly, taking $\varphi = (-u - k)_+$ as a test function in (2.2) when u is a weak supersolution and using the same argument as above, replacing u with $-u$, we easily obtain (3.4).

In the rest of the proof, we obtain a sup-bound for a weak subsolution u of (1.2) by utilizing Lemma 3.6. The corresponding result for a weak supersolution of (1.2) can be similarly obtained.

(ii) **Definition of iteration variables and basic estimates.** Define $Y_n := Z_n + \tilde{Z}_n$ by

$$Z_n := \int_{A_{k_n}} (u - k_n)^{q(x)} dx, \quad \tilde{Z}_n := \int_{\Gamma_{k_n}} (u - k_n)^{r(x)} d\sigma,$$

where $k_n := k_*(2 - \frac{1}{2^n})$, $n = 0, 1, 2, \dots$, with $k_* \geq 1$ specified later. As in [19], we easily obtain the following estimates:

$$\int_{A_{k_{n+1}}} u^{q(x)} dx \leq 2^{(n+2)q^+} Z_n, \quad \int_{\Gamma_{k_{n+1}}} u^{r(x)} d\sigma \leq 2^{(n+2)r^+} \tilde{Z}_n, \quad (3.5)$$

and

$$|A_{k_{n+1}}| \leq \frac{2^{(n+1)q^+}}{k_*^{q^-}} Z_n, \quad (3.6)$$

where $|A_{k_{n+1}}|$ is the Lebesgue measure of $A_{k_{n+1}}$. Combining (3.3) and (3.5), we get

$$\int_{A_{k_{n+1}}} w(x) |\nabla u|^{p(x)} dx \leq d_3 a^n Y_n, \quad (3.7)$$

where $d_3 = \max(d_1 2^{2q^+}, d_2 2^{2r^+})$, $a := \max(2^{q^+}, 2^{r^+}) > 1$.

(iii) **Localization.** The compactness of $\bar{\Omega}$ implies that, for any $R > 0$, there exists a finite open covering $\{B_i(R)\}_{i=1}^m$ of balls $B_i := B_i(R)$ with radius R such that $\bar{\Omega} \subset \bigcup_{i=1}^m B_i$. Denote

$$\begin{aligned} p_i^+ &= \max_{\bar{B}_i \cap \bar{\Omega}} p(x), & p_i^- &= \min_{\bar{B}_i \cap \bar{\Omega}} p(x), & p_{s,i}^- &= \min_{\bar{B}_i \cap \bar{\Omega}} p_s(x), \\ q_i^+ &= \max_{\bar{B}_i \cap \bar{\Omega}} q(x), & q_i^- &= \min_{\bar{B}_i \cap \bar{\Omega}} q(x), \\ r_i^+ &= \max_{\bar{B}_i \cap \Gamma} r(x), & r_i^- &= \min_{\bar{B}_i \cap \Gamma} r(x) \quad \text{if } \bar{B}_i \cap \Gamma \neq \emptyset. \end{aligned}$$

For any $t \in \mathbb{R}$, define

$$t^* = \begin{cases} \frac{Nt}{N-t} & \text{if } t < N, \\ +\infty & \text{if } t \geq N, \end{cases} \quad t_* = \begin{cases} \frac{(N-1)t}{N-t} & \text{if } t < N, \\ +\infty & \text{if } t \geq N. \end{cases}$$

Since $p, q, s \in C(\bar{\Omega})$, $r \in C(\Gamma)$ satisfy (3.2), we may take a sufficiently small $R > 0$ such that

$$p_i^- \leq q_i^- \leq q_i^+ < (p_{s,i}^-)^*, \quad p_j^- \leq r_j^- \leq r_j^+ < (p_{s,j}^-)^* \quad (3.8)$$

for all $i \in \{1, \dots, m\}$ and for all $j \in I$, where I is the set of all $i \in \{1, \dots, m\}$ such that $\bar{B}_i \cap \Gamma \neq \emptyset$. Next, let $\{\xi_i\}_{i=1}^m \subset C_0^\infty(\mathbb{R}^N)$ be a partition of unity associated with open covering $\{B_i\}_{i=1}^m$ of $\bar{\Omega}$; that is,

$$\text{supp } \xi_i \subset B_i, \quad 0 \leq \xi_i \leq 1 \quad (i = 1, \dots, m) \quad \text{and} \quad \sum_{i=1}^m \xi_i = 1 \quad \text{on } \bar{\Omega}.$$

Let L be a positive constant such that

$$\|\nabla \xi_i\|_{L^\infty(\mathbb{R}^N)} \leq L, \quad i = 1, \dots, m. \quad (3.9)$$

Then we have

$$\begin{aligned} \int_{A_{k_{n+1}}} w(x) |\nabla u|^{p(x)} dx &= \int_{A_{k_{n+1}}} w(x) |\nabla u|^{p(x)} \sum_{i=1}^m \xi_i dx \\ &\geq \sum_{i=1}^m \int_{A_{k_{n+1}}} w(x) (|\nabla u|^{p_i^-} - 1) \xi_i dx \\ &\geq \sum_{i=1}^m \int_{A_{k_{n+1}}} w(x) |\nabla u|^{p_i^-} \xi_i^{p_i^-} dx - |A_{k_{n+1}}| \text{ess sup}_\Omega w. \end{aligned}$$

By this, (3.7) and (3.6), we obtain

$$\sum_{i=1}^m \int_{A_{k_{n+1}}} w(x) |\xi_i \nabla u|^{p_i^-} dx \leq d_4 a^n Y_n, \quad (3.10)$$

where $d_4 = d_3 + \frac{Ma}{k_*^{q^-}}$ and $M := \text{ess sup}_\Omega w$.

(iv) Estimating Z_{n+1} by Y_n . Using the partition of unity and Jensen's inequality, we get

$$Z_{n+1} = \int_{A_{k_{n+1}}} (u - k_{n+1})^{q(x)} \left(\sum_{i=1}^m \xi_i \right)^{q(x)} dx \leq m^{q^+-1} \sum_{i=1}^m \int_{A_{k_{n+1}}} (u - k_{n+1})^{q(x)} \xi_i^{q(x)} dx.$$

Thus,

$$Z_{n+1} \leq m^{q^+-1} \sum_{i=1}^m \left[\int_{A_{k_{n+1}}} (u - k_{n+1})^{q_i^+} \xi_i^{q_i^+} dx + \int_{A_{k_{n+1}}} (u - k_{n+1})^{q_i^-} \xi_i^{q_i^-} dx \right]. \quad (3.11)$$

For each $i \in \{1, \dots, m\}$, define

$$\tilde{q}_i := \begin{cases} \frac{q_i^+ + (p_{s,i}^-)^*}{2} & \text{if } (p_{s,i}^-)^* < \infty, \\ q_i^+ + 1 & \text{if } (p_{s,i}^-)^* = \infty. \end{cases}$$

Thus, (3.8) gives

$$q_i^+ < \tilde{q}_i < (p_{s,i}^-)^*. \quad (3.12)$$

We claim that

$$W^{1,p_i^-}(w, \Omega) \hookrightarrow L^{\tilde{q}_i}(\Omega). \quad (3.13)$$

Indeed, denoting $x_i \in \overline{B_i} \cap \overline{\Omega}$ with $p(x_i) = p_i^-$, since $w^{-s^+} \in L^1(\Omega)$ and the boundedness of Ω , we infer $w^{-s(x_i)} \in L^1(\Omega)$. Applying Proposition 2.6 for $p(x) \equiv p(x_i)$, $s(x) \equiv s(x_i)$, we obtain

$$W^{1,p_i^-}(w, \Omega) = W^{1,p(x_i)}(w, \Omega) \hookrightarrow W^{1,p_s(x_i)}(\Omega).$$

Noting that $p_s(x_i) \geq p_{s,i}^-$, we deduce from the last imbedding that

$$W^{1,p_i^-}(w, \Omega) \hookrightarrow W^{1,p_{s,i}^-}(\Omega). \quad (3.14)$$

Combining (3.14) and the classical imbedding $W^{1,p_{s,i}^-}(\Omega) \hookrightarrow L^{\tilde{q}_i}(\Omega)$ (see (3.12)), we obtain (3.13).

For $\tilde{q} \in \{q_i^-, q_i^+\}$, then $p_i^- \leq \tilde{q} \leq q_i^+ < \tilde{q}_i$. Using the Hölder inequality and (3.13), we get

$$\begin{aligned} \int_{A_{k_{n+1}}} (u - k_{n+1})^{\tilde{q}} \xi_i^{\tilde{q}} dx &\leq \left(\int_{\Omega} [\xi_i(u - k_{n+1})_+]^{\tilde{q}_i} dx \right)^{\frac{\tilde{q}}{\tilde{q}_i}} |A_{k_{n+1}}|^{1-\frac{\tilde{q}}{\tilde{q}_i}} \\ &\leq c_i^{\tilde{q}} \left(\int_{\Omega} |\xi_i(u - k_{n+1})_+|^{p_i^-} dx + \int_{\Omega} w(x) |\nabla [\xi_i(u - k_{n+1})_+]|^{p_i^-} dx \right)^{\frac{\tilde{q}}{p_i^-}} |A_{k_{n+1}}|^{1-\frac{\tilde{q}}{p_i^-}}, \end{aligned} \quad (3.15)$$

where c_i is the imbedding constant in (3.13). Meanwhile, utilizing (3.5), (3.9) and (3.10), we have

$$\int_{\Omega} |\xi_i(u - k_{n+1})_+|^{p_i^-} dx \leq \int_{A_{k_{n+1}}} u^{q(x)} dx \leq a^2 a^n Y_n, \quad (3.16)$$

and

$$\begin{aligned} \int_{\Omega} w(x) |\nabla [\xi_i(u - k_{n+1})_+]|^{p_i^-} dx &\leq 2^{p_i^- - 1} \int_{A_{k_{n+1}}} w(x) |\xi_i \nabla u|^{p_i^-} dx + 2^{p_i^- - 1} \int_{A_{k_{n+1}}} w(x) (u - k_{n+1})^{p_i^-} |\nabla \xi_i|^{p_i^-} dx \\ &\leq 2^{p_i^- - 1} \int_{A_{k_{n+1}}} w(x) |\xi_i \nabla u|^{p_i^-} dx + 2^{p_i^- - 1} M L^{p_i^-} \int_{A_{k_{n+1}}} u^{q(x)} dx \\ &\leq 2^{p_i^- - 1} d_4 a^n Y_n + 2^{p_i^- - 1} M L^{p_i^-} a^2 a^n Y_n. \end{aligned} \quad (3.17)$$

Combining (3.15)–(3.17) and (3.6), we obtain

$$\begin{aligned} \int_{A_{k_{n+1}}} (u - k_{n+1})^{\bar{q}} \xi_i^{\bar{q}} dx &\leq \frac{C_2(c_i, p_i^-, \bar{q}_i, \bar{q})}{k_*^{q^-(1-\frac{\bar{q}}{q_i})}} \left(a^{\frac{\bar{q}}{p_i^-}+1-\frac{\bar{q}}{q_i}}\right)^n Y_n^{\frac{\bar{q}}{p_i^-}} Y_n^{1-\frac{\bar{q}}{q_i}} \\ &\leq \frac{C_2(c_i, p_i^-, \bar{q}_i, \bar{q})}{k_*^{q^-(1-\eta)}} \left(a^{1+\frac{q^+}{p^-}}\right)^n (Y_n + Y_n^{\frac{q^+}{p^-}})(Y_n + Y_n^{1-\eta}), \end{aligned}$$

where $C_2(c_i, p_i^-, \bar{q}_i, \bar{q}) = c_i^{\bar{q}}(a^2 + 2p_i^{-1}d_4 + 2p_i^{-1}ML^{p_i^-}a^2)^{\frac{\bar{q}}{p_i^-}}a^{1-\frac{\bar{q}}{q_i}}$ and $0 < \eta := \max_{1 \leq i \leq m} \frac{q_i^+}{q_i} < 1$. By this and estimate (3.11), we deduce

$$Z_{n+1} \leq \frac{d_5}{k_*^{q^-(1-\eta)}} \left(a^{1+\frac{q^+}{p^-}}\right)^n (Y_n^2 + Y_n^{2-\eta} + Y_n^{1+\frac{q^+}{p^-}} + Y_n^{1+\frac{q^+}{p^-}-\eta}), \quad (3.18)$$

where $d_5 = m^{q^+-1} \sum_{i=1}^m [C_2(c_i, p_i^-, \bar{q}_i, q_i^-) + C_2(c_i, p_i^-, \bar{q}_i, q_i^+)]$ is a positive constant independent of n and u .

(v) Estimating $\tilde{Z}_{n+1}$ by Y_n . Using the similar arguments as in the previous step, we have

$$\tilde{Z}_{n+1} \leq m^{r^+-1} \sum_{i \in I} \left[\int_{\Gamma_{k_{n+1}}} (u - k_{n+1})^{r_i^+} \xi_i^{r_i^+} d\sigma + \int_{\Gamma_{k_{n+1}}} (u - k_{n+1})^{r_i^-} \xi_i^{r_i^-} d\sigma \right]. \quad (3.19)$$

Fix $i \in I$ and take $t \in \{r_i^-, r_i^+\}$, then by (3.8), we have $p_i^- \leq t \leq r_i^+ < (p_{s,i}^-)_*$. Let $\delta = \delta(t) \in (1, N)$ be such that

$$\delta_* = \begin{cases} \frac{t+(p_{s,i}^-)_*}{2} & \text{if } (p_{s,i}^-)_* < \infty, \\ t+1 & \text{if } (p_{s,i}^-)_* = \infty. \end{cases}$$

This implies that $1 < \delta < p_{s,i}^- < p_i^- \leq t < \delta_*$. Therefore, we have

$$W^{1,\delta}(\Omega) \hookrightarrow L^t(\Gamma). \quad (3.20)$$

Now, let us estimate the integrals in (3.19). Using (3.20), we have

$$\begin{aligned} \int_{\Gamma_{k_{n+1}}} (u - k_{n+1})^t \xi_i^t d\sigma &= \int_{\Gamma} |\xi_i(u - k_{n+1})_+|^t d\sigma \\ &\leq C_3(t, \delta) \left[\left(\int_{\Omega} |\xi_i(u - k_{n+1})_+|^{\delta} dx \right)^{\frac{1}{\delta}} + \left(\int_{\Omega} |\nabla[\xi_i(u - k_{n+1})_+]|^{\delta} dx \right)^{\frac{1}{\delta}} \right]^t. \end{aligned}$$

On the other hand, it follows from the Hölder inequality that

$$\begin{aligned} &\left(\int_{\Omega} |\xi_i(u - k_{n+1})_+|^{\delta} dx \right)^{\frac{1}{\delta}} + \left(\int_{\Omega} |\nabla[\xi_i(u - k_{n+1})_+]|^{\delta} dx \right)^{\frac{1}{\delta}} \\ &\leq \left[\left(\int_{\Omega} |\xi_i(u - k_{n+1})_+|^{p_{s,i}^-} dx \right)^{\frac{1}{p_{s,i}^-}} + \left(\int_{\Omega} |\nabla[\xi_i(u - k_{n+1})_+]|^{p_{s,i}^-} dx \right)^{\frac{1}{p_{s,i}^-}} \right] |A_{k_{n+1}}|^{\frac{1}{\delta}(1-\frac{\delta}{p_{s,i}^-})}. \end{aligned}$$

Invoking (3.14) with an equivalent norm, we have

$$\begin{aligned} &\left(\int_{\Omega} |\xi_i(u - k_{n+1})_+|^{p_{s,i}^-} dx \right)^{\frac{1}{p_{s,i}^-}} + \left(\int_{\Omega} |\nabla[\xi_i(u - k_{n+1})_+]|^{p_{s,i}^-} dx \right)^{\frac{1}{p_{s,i}^-}} \\ &\leq \tilde{c}_i \left(\int_{\Omega} |\xi_i(u - k_{n+1})_+|^{p_i^-} dx + \int_{\Omega} w(x) |\nabla[\xi_i(u - k_{n+1})_+]|^{p_i^-} dx \right)^{\frac{1}{p_i^-}}, \end{aligned}$$

where $\tilde{c}_i$ is the imbedding constant in (3.14).

Combining the last three estimates above with (3.16)–(3.17), and (3.6), we obtain

$$\begin{aligned} \int_{\Gamma_{k_{n+1}}} (u - k_{n+1})^t \xi_i^t d\sigma &\leq \frac{C_4(t, \delta, \bar{c}_i, p_i^-, p_{s,i}^-)}{(k_*^q)^{\frac{t}{\delta}(1-\frac{\delta}{p_{s,i}})}} \left(a^{\frac{t}{p_i^-} + \frac{t}{\delta}(1-\frac{\delta}{p_{s,i}})} \right)^n Y_n^{\frac{t}{p_i^-}} Y_n^{\frac{t}{\delta}(1-\frac{\delta}{p_{s,i}})} \\ &\leq \frac{C_4(t, \delta, \bar{c}_i, p_i^-, p_{s,i}^-)}{k_*^{q-\tilde{\eta}_*}} (a^{r^+})^n \left(Y_n + Y_n^{\frac{r^+}{p^-}} \right) \left(Y_n^{\tilde{\eta}_*} + Y_n^{\tilde{\eta}_*} \right), \end{aligned}$$

where

$$0 < \tilde{\eta}_* := \min_{i \in I} \left\{ \frac{r^-}{\delta(r_i^+)} - \frac{r^-}{p_{s,i}^-} \right\} \leq \tilde{\eta}^* := \max_{i \in I} \left\{ \frac{r^+}{\delta(r_i^-)} - \frac{r^+}{p_{s,i}^-} \right\}$$

(noting that $\delta(r_i^+) \geq \delta(r_i^-)$ for all $i \in I$). Finally, by the last estimate and (3.19), we infer

$$\tilde{Z}_{n+1} \leq \frac{d_6}{k_*^{q-\tilde{\eta}_*}} (a^{r^+})^n \left(Y_n^{1+\tilde{\eta}_*} + Y_n^{1+\tilde{\eta}^*} + Y_n^{\frac{r^+}{p^-}+\tilde{\eta}_*} + Y_n^{\frac{r^+}{p^-}+\tilde{\eta}^*} \right), \quad (3.21)$$

where

$$d_6 = m^{r^+-1} \sum_{i \in I} [C_4(r_i^-, \delta(r_i^-), \bar{c}_i, p_i^-, p_{s,i}^-) + C_4(r_i^+, \delta(r_i^+), \bar{c}_i, p_i^-, p_{s,i}^-)].$$

(vi) Estimating Y_{n+1} by Y_n . It follows from (3.18) and (3.21) that

$$Y_{n+1} \leq \frac{d_7 b^n}{8k_*^\theta} \left(Y_n^2 + Y_n^{2-\eta} + Y_n^{1+\frac{q^+}{p^-}} + Y_n^{1+\frac{q^+}{p^-}-\eta} + Y_n^{1+\tilde{\eta}_*} + Y_n^{1+\tilde{\eta}^*} + Y_n^{\frac{r^+}{p^-}+\tilde{\eta}_*} + Y_n^{\frac{r^+}{p^-}+\tilde{\eta}^*} \right),$$

where $\theta := \min(q^-(1-\eta), q^-\tilde{\eta}_*) > 0$, $b := \max(a^{r^+}, a^{1+\frac{q^+}{p^-}}) > 1$, and $d_7 = 8 \max(d_5, d_6) > 0$ independent of u and n . Arguing the cases $Y_n \leq 1$ and $Y_n > 1$ and noting $\min(2-\eta, 1+\tilde{\eta}_*)$ and $\max(1+\frac{q^+}{p^-}, \tilde{\eta}^* + \frac{r^+}{p^-})$ as the smallest and largest exponents, respectively, we obtain

$$Y_{n+1} \leq \frac{d_7}{k_*^\theta} b^n (Y_n^{1+\delta_1} + Y_n^{1+\delta_2}), \quad (3.22)$$

where $0 < \delta_1 := \min(1-\eta, \tilde{\eta}_*) < 1 \leq \delta_2 := \max(\frac{q^+}{p^-}, \tilde{\eta}^* + \frac{r^+}{p^-} - 1)$.

Applying Lemma 3.6 to (3.22), we obtain

$$Y_n = \int_{\Omega} (u - k_n)_+^{q(x)} dx + \int_{\Gamma} (u - k_n)_+^{r(x)} d\sigma \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (3.23)$$

provided that

$$Y_0 \leq \min \left(\left(\frac{2d_7}{k_*^\theta} \right)^{\frac{-1}{\delta_1}} b^{\frac{-1}{\delta_1}}, \left(\frac{2d_7}{k_*^\theta} \right)^{\frac{-1}{\delta_2}} b^{-\frac{1}{\delta_1\delta_2} - \frac{\delta_2 - \delta_1}{\delta_2^2}} \right).$$

Thus, choosing

$$k_* = \left(1 + (2d_7)^{\frac{1}{\theta}} b^{\frac{1}{\theta\delta_1} + \frac{\delta_2 - \delta_1}{\theta\delta_2^2}} \right) \left[1 + \left(\int_{\Omega} u_+^{q(x)} dx + \int_{\Gamma} u_+^{r(x)} d\sigma \right)^{\frac{\delta_2}{\theta}} \right],$$

we have (3.23) since $Y_0 \leq \int_{\Omega} u_+^{q(x)} dx + \int_{\Gamma} u_+^{r(x)} d\sigma$, hence we deduce

$$\int_{\Omega} (u - 2k_*)_+^{q(x)} dx = \int_{\Gamma} (u - 2k_*)_+^{r(x)} d\sigma = 0.$$

Consequently, $(u - 2k_*)_+ = 0$ a.e. in Ω , and $(u - 2k_*)_+ = 0$ a.e. on Γ ; this means that

$$\operatorname{ess\,sup}_{\Omega} u \leq 2k_*, \quad \operatorname{ess\,sup}_{\Gamma} u \leq 2k_*.$$

The boundedness from below of a weak supersolution u for (1.2) can be determined analogously, by replacing u with $-u$, A_k with $\bar{A}_k$, and Γ_k with $\bar{\Gamma}_k$ and using (3.4) instead of (3.3). This completes the proof. $\square$

To obtain an L^∞ -bound for solutions of the Dirichlet problem (1.1), we apply Lemma 3.6 with the crucial properties of the weighted variable exponent Sobolev spaces $W_0^{1,p(x)}(w, \Omega)$, which were obtained in [10].

Proof of Theorem 3.1. Define $A_{k_n}, \tilde{A}_{k_n}, Z_n, \tilde{Z}_n$ ($n \in \mathbb{N}$) as in the proof of Theorem 3.3 and note that, for $u \in W_0^{1,p(x)}(w, \Omega)$, we have $(u - k_n)_+ \in W_0^{1,p(x)}(w, \Omega)$ (see [10, Lemma 4.1]). Using the same argument to show step (i) in the proof of Theorem 3.3, there exists a positive constant d_1 independent of u and n such that, if u is a weak subsolution of (1.1), then

$$\int_{A_{k_n}} w(x) |\nabla u|^{p(x)} dx \leq d_1 \int_{A_{k_n}} u^{q(x)} dx,$$

and if u is a weak supersolution of (1.1), then

$$\int_{\tilde{A}_{k_n}} w(x) |\nabla(-u)|^{p(x)} dx \leq d_1 \int_{\tilde{A}_{k_n}} (-u)^{q(x)} dx.$$

Repeating the argument used in steps (ii)–(iv) in the proof of Theorem 3.3 with Y_n and a being replaced by Z_n and $\tilde{a} := 2^{q^+}$, respectively, we obtain for a weak subsolution u ,

$$Z_{n+1} \leq \frac{1}{4} \frac{d}{k_*^{q^-(1-\eta)}} \tilde{b}^n \left(Z_n^2 + Z_n^{2-\eta} + Z_n^{1+\frac{q^+}{p^-}} + Z_n^{1+\frac{q^+}{p^-}-\eta} \right),$$

where $\tilde{b} := \tilde{a}^{1+\frac{q^+}{p^-}} > 1$, $\eta := \max_{1 \leq i \leq m} \frac{q_i^+}{q_i} < 1$, and d is a positive constant independent of u and n . From this, we get

$$Z_{n+1} \leq \frac{d}{k_*^{q^-(1-\eta)}} \tilde{b}^n \left(Z_n^{1+\tilde{\delta}_1} + Z_n^{1+\tilde{\delta}_2} \right),$$

where $0 < \tilde{\delta}_1 := 1 - \eta < 1 \leq \tilde{\delta}_2 := \frac{q^+}{p^-}$. In a similar fashion to the proof of Theorem 3.3, we obtain the conclusion. $\square$

4 Existence of solutions

In this section, we show the existence of solutions of both the Dirichlet problem (1.4) and the Neumann problem (1.5). The following are the main results of this section.

Theorem 4.1. Assume that (w1), (A1), ($\tilde{A}2$), (A3), and ($\tilde{B}1$) hold, in which $p(x) \leq q(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$. Assume also that $\text{ess inf}_{\Omega} b \geq 0$. Then (1.4) has at least one weak solution.

Theorem 4.2. Assume that (w1), (A1), ($\tilde{A}2$), (A3), ($\tilde{B}1$) and ($\tilde{C}1$) hold, in which $p(x) \leq q(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$. Assume also that $\text{ess inf}_{\Omega} b > 0$. Then (1.5) has at least one weak solution.

When $h(x), m(x) \equiv \text{constant}$ and $p(x) \leq q(x)$ for all $x \in \overline{\Omega}$, conditions ($\tilde{A}2$), ($\tilde{B}1$) and ($\tilde{C}1$) imply (A2), (B1) and (C1), respectively. Consequently, we have two corollaries from Corollary 3.2 and Theorem 4.1 for the Dirichlet problem and from Corollary 3.4 and Theorem 4.2 for the Neumann problem.

Corollary 4.3. Assume that (w2), (A1), ($\tilde{A}2$), (A3), and ($\tilde{B}1$) hold, in which $h \equiv 1$, and $p(x) \leq q(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$. Assume also that $\text{ess inf}_{\Omega} b \geq 0$. Then (1.4) has a weak solution $u \in L^\infty(\Omega)$.

Corollary 4.4. Assume that (w2), (A1), ($\tilde{A}2$), (A3), ($\tilde{B}1$) and ($\tilde{C}1$) hold, in which $h \equiv 1$, $m \equiv 1$, $s(x) > \frac{N}{p(x)-1}$ for all $x \in \Gamma$, and $p(x) \leq q(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$. Assume also that $\text{ess inf}_{\Omega} b > 0$. Then (1.5) has a weak solution $u \in L^\infty(\Omega) \cap L^\infty(\Gamma)$.

To prove Theorems 4.1 and 4.2 we use Brezis' theorem for pseudomonotone operators in the separable reflexive space (see [21, Theorem 27.A]). As in Section 3, we focus on the proof for the existence of solutions to the Neumann problem (1.5).

Proof of Theorem 4.2. Denote $X := W^{1,p(x)}(w, \Omega)$, then X is the separable reflexive Banach space. By Proposition 2.6, Proposition 2.8 and the assumptions on exponents, it is easy to see that X is compactly imbedded in spaces $L^{p(x)}(\Omega)$, $L^{q(x)}(\Omega)$, $L^{\eta(x)+1}(\Omega)$, $L^{a(x)}(\Omega)$, $L^{\eta(x)q'(x)}(\Omega)$ and $L^{r(x)}(\Gamma)$ with the imbedding constants C_p , C_q , $C_{\eta+1}$, C_a , $C_{\eta q'}$ and C_r , respectively. The assumptions on p , q , δ imply that $\delta(x) < p(x)$ for all $x \in \overline{\Omega}$. From this and the assumption $w \in L^1(\Omega)$, we have

$$L^{p(x)}(w, \Omega) \hookrightarrow L^{\delta(x)}(w, \Omega)$$

with the imbedding constant $C_{p,\delta}$.

Let us define the operator $T : X \rightarrow X^*$ by

$$\langle Tu, v \rangle = \int_{\Omega} A(x, u, \nabla u) \cdot \nabla v \, dx + \int_{\Omega} b(x)|u|^{p(x)-2}uv \, dx - \int_{\Omega} B(x, u, \nabla u)v \, dx - \int_{\Gamma} C(x, u)v \, d\sigma, \quad \forall u, v \in X.$$

We verify that T satisfies all conditions in Brezis' theorem.

(i) T is bounded. We first show that

Claim 1. *If $\mathcal{U}$ is a bounded subset in X , then*

$$\{|w|^{-\frac{1}{p}}A(\cdot, u, \nabla v) : u, v \in \mathcal{U}\}, \quad \{b|u|^{p-2}u : u \in \mathcal{U}\}, \quad \{B(\cdot, u, \nabla u) : u \in \mathcal{U}\}, \quad \{C(\cdot, u) : u \in \mathcal{U}\}$$

are bounded in $L^{p'(x)}(\Omega)$, $L^{p'(x)}(\Omega)$, $L^{q'(x)}(\Omega)$, and $L^{r(x)}(\Gamma)$, respectively.

Indeed, by (A1) and Jensen's inequality, for any measurable subset $H \subset \Omega$ and $u, v \in X$, we have

$$\begin{aligned} \int_H |w|^{-\frac{1}{p(x)}} A(x, u, \nabla v) |^{p'(x)} \, dx &\leq \int_H \alpha_1^{p'(x)} \left[|w|^{-\frac{1}{p(x)}} k(x) + |u|^{\frac{q(x)}{p'(x)}} + w^{\frac{1}{p'(x)}} |\nabla v|^{p(x)-1} \right]^{p'(x)} \, dx \\ &\leq C_1 \left[\int_H |w|^{-\frac{1}{p(x)}} k |^{p'(x)} \, dx + \int_H |u|^{q(x)} \, dx + \int_H w |\nabla v|^{p(x)} \, dx \right], \end{aligned} \quad (4.1)$$

where $C_1 = (1 + \alpha_1^{(p')^+}) 3^{(p')^+-1}$. By this and Proposition 2.2, we get

$$\begin{aligned} \int_{\Omega} |w|^{-\frac{1}{p(x)}} A(x, u, \nabla v) |^{p'(x)} \, dx &\leq C_1 \left[3 + |w|^{-\frac{1}{p}} k |_{p'}^{(p')^+} + |u|_q^{q^+} + \|v\|^{p^+} \right] \\ &\leq C_1 \left[3 + |w|^{-\frac{1}{p}} k |_{p'}^{(p')^+} + C_q^{q^+} \|u\|^{q^+} + \|v\|^{p^+} \right]. \end{aligned}$$

Thus, invoking Proposition 2.2, we obtain

$$\left| w|^{-\frac{1}{p}} A(\cdot, u, \nabla v) \right|_{p'} \leq \left\{ 1 + C_1 \left[3 + |w|^{-\frac{1}{p}} k |_{p'}^{(p')^+} + C_q^{q^+} \|u\|^{q^+} + \|v\|^{p^+} \right] \right\}^{\frac{1}{(p')^-}}. \quad (4.2)$$

Similarly, using (B1), we have

$$\begin{aligned} \int_{\Omega} |B(x, u, \nabla u)|^{q'(x)} \, dx &\leq \int_{\Omega} \beta^{q'(x)} \left[h(x) + |u|^{\eta(x)} + w(x)^{\frac{1}{q'(x)}} |\nabla u|^{\frac{\delta(x)}{q'(x)}} \right]^{q'(x)} \, dx \\ &\leq C_2 \left[\int_{\Omega} |h(x)|^{q'(x)} \, dx + \int_{\Omega} |u|^{\eta(x)q'(x)} \, dx + \int_{\Omega} w(x) |\nabla u|^{\delta(x)} \, dx \right] \\ &\leq C_2 \left[3 + |h|_{q'}^{(q')^+} + |u|_{\eta q'}^{(\eta q')^+} + \|\nabla u\|_{\delta, w}^{\delta^+} \right] \\ &\leq C_2 \left[3 + |h|_{q'}^{(q')^+} + C_{\eta q'}^{(\eta q')^+} \|u\|^{(\eta q')^+} + C_{p,\delta}^{\delta^+} \|u\|^{\delta^+} \right], \end{aligned}$$

where $C_2 = (1 + \beta^{(q')^+}) 3^{(q')^+-1}$. Hence we infer

$$|B(\cdot, u, \nabla u)|_{q'} \leq \left\{ 1 + C_2 \left[3 + |h|_{q'}^{(q')^+} + C_{\eta q'}^{(\eta q')^+} \|u\|^{(\eta q')^+} + C_{p,\delta}^{\delta^+} \|u\|^{\delta^+} \right] \right\}^{\frac{1}{(q')^-}}. \quad (4.3)$$

In a similar way, we estimate from (C1)

$$\begin{aligned} \int_{\Gamma} |C(x, u)|^{r'(x)} d\sigma &\leq \int_{\Gamma} \gamma^{r'(x)} [m(x) + |u|^{r(x)-1}]^{r'(x)} d\sigma \\ &\leq (1 + \gamma^{(r')^+}) 2^{(r')^+-1} \left(\int_{\Gamma} |m(x)|^{r'(x)} d\sigma + \int_{\Gamma} |u|^{r(x)} d\sigma \right) \\ &\leq (1 + \gamma^{(r')^+}) 2^{(r')^+-1} (2 + |m|_{L^{r'(x)}(\Gamma)}^{(r')^+} + |u|_{L^{r(x)}(\Gamma)}^{r^+}) \\ &\leq (1 + \gamma^{(r')^+}) 2^{(r')^+-1} (2 + |m|_{L^{r'(x)}(\Gamma)}^{(r')^+} + C_r^{r^+} \|u\|^{r^+}). \end{aligned}$$

Thus,

$$|C(\cdot, u)|_{L^{r'(x)}(\Gamma)} \leq \left\{ 1 + (1 + \gamma^{(r')^+}) 2^{(r')^+-1} (2 + |m|_{L^{r'(x)}(\Gamma)}^{(r')^+} + C_r^{r^+} \|u\|^{r^+}) \right\}^{\frac{1}{(r')^+}}. \quad (4.4)$$

Finally, by Proposition 2.4, we have

$$|b|u|^{p-2}u|_{p'} \leq |b|_{\infty} (1 + |u|_p^{p^+-1}) \leq |b|_{\infty} (1 + C_p^{p^+-1} \|u\|^{p^+-1}). \quad (4.5)$$

Estimates (4.2)–(4.5) imply the conclusion of Claim 1.

Utilizing the Hölder-type inequality and taking into account the imbeddings, for any $v \in X$ with $\|v\| \leq 1$,

$$\left| \int_{\Omega} A(x, u, \nabla u) \cdot \nabla v dx \right| \leq 2|w^{-\frac{1}{p}} A(\cdot, u, \nabla u)|_{p'} |w^{\frac{1}{p}} \nabla v|_p \leq 2|w^{-\frac{1}{p}} A(\cdot, u, \nabla u)|_{p'},$$

and

$$\begin{aligned} \left| \int_{\Omega} b|u|^{p(x)-2}uv dx \right| &\leq 2C_p |b|u|^{p-2}u|_{p'}, \\ \left| \int_{\Omega} B(x, u, \nabla u)v dx \right| &\leq 2C_q |B(\cdot, u, \nabla u)|_{q'}, \\ \left| \int_{\Gamma} C(x, u)v d\sigma \right| &\leq 2C_r |C(\cdot, u)|_{L^{r'(x)}(\Gamma)}. \end{aligned}$$

From these estimates and Claim 1, we easily determine the boundedness of T .

(ii) T is demicontinuous. Suppose that $u_n \rightarrow u$ in X . We need to show that $Tu_n \rightarrow Tu$ in X^* , i.e.,

$$\begin{aligned} &\int_{\Omega} A(x, u_n, \nabla u_n) \cdot \nabla v dx + \int_{\Omega} b(x)|u_n|^{p(x)-2}u_n v dx - \int_{\Omega} B(x, u_n, \nabla u_n)v dx - \int_{\Gamma} C(x, u_n)v d\sigma \\ &\rightarrow \int_{\Omega} A(x, u, \nabla u) \cdot \nabla v dx + \int_{\Omega} b(x)|u|^{p(x)-2}uv dx - \int_{\Omega} B(x, u, \nabla u)v dx - \int_{\Gamma} C(x, u)v d\sigma, \quad v \in X. \end{aligned} \quad (4.6)$$

Since $u_n \rightarrow u$ in X , up to a subsequence, we have

$$\begin{cases} u_n \rightarrow u & \text{and} & \nabla u_n \rightarrow \nabla u & \text{a.e. in } \Omega, \\ u_n \rightarrow u & \text{and} & |u_n|^{r(x)} \leq l_1(x) & \text{a.e. on } \Gamma, \end{cases} \quad (4.7)$$

for some $l_1 \in L^1(\Gamma)$. Thus,

$$\begin{cases} w^{-\frac{1}{p(x)}} A(x, u_n, \nabla u_n) \rightarrow w^{-\frac{1}{p(x)}} A(x, u, \nabla u), \\ b(x)|u_n|^{p(x)-2}u_n \rightarrow b(x)|u|^{p(x)-2}u, \\ B(x, u_n, \nabla u_n) \rightarrow B(x, u, \nabla u), \end{cases} \quad \text{a.e. in } \Omega. \quad (4.8)$$

Moreover, the boundedness of $\{u_n\}$ in X and Claim 1 imply that $\{b|u_n|^{p-2}u_n\}$, $\{w^{-\frac{1}{p}} A(\cdot, u_n, \nabla u_n)\}$, and $\{B(\cdot, u_n, \nabla u_n)\}$ are bounded in $L^{p'(x)}(\Omega)$, $(L^{p'(x)}(\Omega))^N$, and $L^{q'(x)}(\Omega)$, respectively. Invoking [1, Lemma 3.3]

with these facts and (4.8), we obtain

$$\begin{aligned} \int_{\Omega} A(x, u_n, \nabla u_n) \cdot \nabla v \, dx &\rightarrow \int_{\Omega} A(x, u, \nabla u) \cdot \nabla v \, dx, \\ \int_{\Omega} b(x) |u_n|^{p(x)-2} u_n v \, dx &\rightarrow \int_{\Omega} b(x) |u|^{p(x)-2} u v \, dx, \\ \int_{\Omega} B(x, u_n, \nabla u_n) v \, dx &\rightarrow \int_{\Omega} B(x, u, \nabla u) v \, dx. \end{aligned}$$

For the remaining part, noting

$$|C(x, u_n) v| \leq \gamma(|m| + |u_n|^{r(x)-1})|v| \leq \gamma|m||v| + \frac{\gamma}{(r')^-} |u_n|^{r(x)} + \frac{\gamma}{r^-} |v|^{r(x)}$$

and using the Dominated Convergence Theorem with (4.7), we have

$$\int_{\Gamma} C(x, u_n) v \, d\sigma \rightarrow \int_{\Gamma} C(x, u) v \, d\sigma.$$

(iii) **T is coercive.** First, note that

$$\langle Tu, u \rangle = \int_{\Omega} A(x, u, \nabla u) \cdot \nabla u \, dx + \int_{\Omega} b(x) |u|^{p(x)} \, dx - \int_{\Omega} B(x, u, \nabla u) u \, dx - \int_{\Gamma} C(x, u) u \, d\sigma. \quad (4.9)$$

To estimate the first and second integral terms, using $(\tilde{A}2)$ and Proposition 2.2 with (2.1), we deduce that

$$\begin{aligned} \int_{\Omega} A(x, u, \nabla u) \cdot \nabla u \, dx + \int_{\Omega} b(x) |u|^{p(x)} \, dx &\geq \min(\alpha_2, b_0) \left[\int_{\Omega} w(x) |\nabla u|^{p(x)} \, dx + \int_{\Omega} |u|^{p(x)} \, dx \right] - \alpha_3 \int_{\Omega} (|u|^{\alpha(x)} + 1) \, dx \\ &\geq \min(\alpha_2, b_0) \left(\frac{1}{2p^-} \|u\|^{p^-} - 1 \right) - \alpha_3 (C_{\alpha}^{\alpha^+} \|u\|^{\alpha^+} + 1) - \alpha_3 |\Omega|, \end{aligned}$$

where $b_0 := \text{ess inf}_{\Omega} b > 0$. To estimate the third integral term, we deduce from $(\tilde{B}1)$, the Hölder-type inequality, and Proposition 2.2 that

$$\begin{aligned} \int_{\Omega} B(x, u, \nabla u) u \, dx &\leq \int_{\Omega} \beta \left[|h| |u| + |u|^{\eta(x)+1} + w(x)^{\frac{1}{q'(x)}} |\nabla u|^{\frac{\delta(x)}{q'(x)}} |u| \right] \, dx \\ &\leq 2\beta |h|_{q'} |u|_q + \beta (|u|_{\eta+1}^{\eta+1} + 1) + 2\beta \left| w^{\frac{1}{q'}} |\nabla u|^{\frac{\delta}{q'}} \right|_{q'} |u|_q \\ &\leq \beta + 2\beta C_q |h|_{q'} \|u\| + \beta C_{\eta+1}^{\eta+1} \|u\|^{\eta+1} + 2\beta C_q \|u\| \left(1 + C_{p,\delta}^{\left(\frac{\delta}{q'}\right)^+} \|u\|^{\left(\frac{\delta}{q'}\right)^+} \right). \end{aligned}$$

Here, we have used Proposition 2.4 to estimate that

$$\left| w^{\frac{1}{q'}} |\nabla u|^{\frac{\delta}{q'}} \right|_{q'} \leq 1 + \left| |\nabla u|^{\left(\frac{\delta}{q'}\right)^+} \right|_{\delta,w} \leq 1 + C_{p,\delta}^{\left(\frac{\delta}{q'}\right)^+} \left| |\nabla u|^{\left(\frac{\delta}{q'}\right)^+} \right|_{p,w} \leq 1 + C_{p,\delta}^{\left(\frac{\delta}{q'}\right)^+} \|u\|^{\left(\frac{\delta}{q'}\right)^+}.$$

To estimate the fourth integral term, $(\tilde{C}1)$ and Proposition 2.2 yield

$$\begin{aligned} \int_{\Gamma} C(x, u) u \, d\sigma &\leq \gamma \int_{\Gamma} (|m| |u| + |u|^{r(x)}) \, d\sigma \\ &\leq 2\gamma |m|_{L^{r'(x)}(\Gamma)} |u|_{L^{r(x)}(\Gamma)} + \gamma (1 + |u|_{L^{r(x)}(\Gamma)}^{r^+}) \\ &\leq \gamma + 2\gamma C_r |m|_{L^{r'(x)}(\Gamma)} \|u\| + \gamma C_r^{r^+} \|u\|^{r^+}. \end{aligned}$$

Noting that $r^+ < p^-$, $\left(\frac{\delta}{q'}\right)^+ < p^-$ and $\eta^+ + 1 < p^-$, we deduce the coerciveness of T from (4.9) and the estimates for integrals above.

(iv) T is an (S_+) -type operator. Let $\{u_n\} \subset X$ be such that $u_n \rightharpoonup u$ in X as $n \rightarrow \infty$ and

$$\limsup_{n \rightarrow \infty} \langle Tu_n, u_n - u \rangle \leq 0.$$

First, note that

$$\begin{aligned} \langle Tu_n, u_n - u \rangle &= \int_{\Omega} A(x, u_n, \nabla u_n) \cdot \nabla(u_n - u) \, dx + \int_{\Omega} b(x) |u_n|^{p(x)-2} u_n (u_n - u) \, dx \\ &\quad - \int_{\Omega} B(x, u_n, \nabla u_n) (u_n - u) \, dx - \int_{\Gamma} C(x, u_n) (u_n - u) \, d\sigma. \end{aligned} \quad (4.10)$$

Since $u_n \rightharpoonup u$ in X , up to a subsequence, we have

$$\left\{ \begin{array}{ll} u_n \rightarrow u & \text{in } L^{p(x)}(\Omega), \\ u_n \rightarrow u & \text{in } L^{q(x)}(\Omega), \\ u_n \rightarrow u & \text{in } L^{r(x)}(\Gamma), \\ u_n(x) \rightarrow u(x) & \text{a.e. in } \Omega, \\ |u_n(x)|^{a(x)} \leq l_2(x) & \text{a.e. in } \Omega, \\ |u_n(x)|^{q(x)} \leq l_3(x) & \text{a.e. in } \Omega, \end{array} \right. \quad (4.11)$$

for some $l_2, l_3 \in L^1(\Omega)$. Using the Hölder-type inequality, we have

$$\left| \int_{\Omega} B(x, u_n, \nabla u_n) (u_n - u) \, dx \right| \leq 2 |B(\cdot, u_n, \nabla u_n)|_{q'} |u_n - u|_q.$$

By the boundedness of $\{u_n\}$ in X and (4.11), we infer from the inequality above and Claim 1 that

$$\int_{\Omega} B(x, u_n, \nabla u_n) (u_n - u) \, dx \rightarrow 0. \quad (4.12)$$

In the same fashion, we have

$$\left| \int_{\Omega} b(x) |u_n|^{p(x)-2} u_n (u_n - u) \, dx \right| \leq 2 |b| |u_n|^{p-2} u_n \Big|_p |u_n - u|_p \rightarrow 0, \quad (4.13)$$

and

$$\left| \int_{\Gamma} C(x, u_n) (u_n - u) \, d\sigma \right| \leq 2 |C(\cdot, u_n)|_{L^{r'(x)}(\Gamma)} |u_n - u|_{L^{r(x)}(\Gamma)} \rightarrow 0. \quad (4.14)$$

Furthermore, we also have

$$\lim_{n \rightarrow \infty} \int_{\Omega} A(x, u_n, \nabla u) \cdot \nabla(u_n - u) \, dx = 0. \quad (4.15)$$

Indeed, using (4.1) with (4.11), for any measurable subset H of Ω and all $n \in \mathbb{N}$,

$$\int_H \left| w^{-\frac{1}{p(x)}} A(x, u_n, \nabla u) \right|^{p'(x)} \, dx \leq C_1 \left[\int_H |w^{-\frac{1}{p(x)}} k|^{p'(x)} \, dx + \int_H l_2(x) \, dx + \int_H w |\nabla u|^{p(x)} \, dx \right].$$

Thus, we find that $\{|w^{-\frac{1}{p}} A(\cdot, u_n, \nabla u)|^{p'}\}$ is uniformly integrable; hence $\{|w^{-\frac{1}{p}} (A(\cdot, u_n, \nabla u) - A(\cdot, u, \nabla u))|^{p'}\}$ is also uniformly integrable. From this and (4.11), we obtain by Vitali's theorem that

$$\lim_{n \rightarrow \infty} \int_{\Omega} \left| w(x)^{-\frac{1}{p(x)}} (A(x, u_n, \nabla u) - A(x, u, \nabla u)) \right|^{p'(x)} \, dx = 0.$$

Equivalently,

$$\left| w^{-\frac{1}{p}} (A(\cdot, u_n, \nabla u) - A(\cdot, u, \nabla u)) \right|_{p'} \rightarrow 0. \quad (4.16)$$

Estimating

$$\begin{aligned} \left| \int_{\Omega} A(x, u_n, \nabla u) \cdot \nabla(u_n - u) \, dx \right| &\leq \int_{\Omega} \left| w^{-\frac{1}{p(x)}} (A(x, u_n, \nabla u) - A(x, u, \nabla u)) \right| \left| w^{\frac{1}{p(x)}} \nabla(u_n - u) \right| \, dx \\ &\quad + \left| \int_{\Omega} A(x, u, \nabla u) (\nabla u_n - \nabla u) \, dx \right| \\ &\leq 2 \left\| w^{-\frac{1}{p}} (A(\cdot, u_n, \nabla u) - A(\cdot, u, \nabla u)) \right\|_{p'} \|u_n - u\| \\ &\quad + \left| \int_{\Omega} A(x, u, \nabla u) (\nabla u_n - \nabla u) \, dx \right| \end{aligned}$$

and using (4.16) and the fact that $u_n \rightharpoonup u$, we obtain (4.15). So, from (4.10) and (4.12)–(4.15), we deduce

$$\limsup_{n \rightarrow \infty} \int_{\Omega} [A(x, u_n, \nabla u_n) - A(x, u_n, \nabla u)] \cdot (\nabla u_n - \nabla u) \, dx = \limsup_{n \rightarrow \infty} \langle Tu_n, u_n - u \rangle \leq 0.$$

Combining this with (A3), we obtain

$$\lim_{n \rightarrow \infty} \int_{\Omega} [A(x, u_n, \nabla u_n) - A(x, u_n, \nabla u)] \cdot (\nabla u_n - \nabla u) \, dx = 0. \quad (4.17)$$

Hence, by a standard argument (e.g., see [1, Proof of Lemma 3.4]), we deduce that

$$\nabla u_n(x) \rightarrow \nabla u(x) \quad \text{as } n \rightarrow \infty \quad \text{for a.e. } x \in \Omega. \quad (4.18)$$

Let $\epsilon > 0$ be arbitrary. By (4.17) and (A3), there is $n_0 \in \mathbb{N}$ such that, for any measurable subset H of Ω and all $n \geq n_0$,

$$\int_H [A(x, u_n, \nabla u_n) - A(x, u_n, \nabla u)] \cdot (\nabla u_n - \nabla u) \, dx \leq \frac{\alpha_2 \epsilon}{4}.$$

By this and $(\tilde{A}2)$, we deduce

$$\begin{aligned} \alpha_2 \int_H w(x) |\nabla u_n|^{p(x)} \, dx &\leq \frac{\alpha_2 \epsilon}{4} + \alpha_3 \int_H (|u_n|^{\alpha(x)} + 1) \, dx + \int_H |A(x, u_n, \nabla u_n)| |\nabla u| \, dx \\ &\quad + \int_H |A(x, u_n, \nabla u)| |\nabla u_n| \, dx + \int_H |A(x, u_n, \nabla u)| |\nabla u| \, dx. \end{aligned} \quad (4.19)$$

Note that the boundedness of $\{u_n\}$ in X and Claim 1 imply that there is a positive constant C_3 such that

$$\|w^{-\frac{1}{p}} A(\cdot, u_n, \nabla u_n)\|_{p'} \leq C_3, \quad \|w^{-\frac{1}{p}} A(\cdot, u_n, \nabla u)\|_{p'} \leq C_3 \quad \text{for all } n.$$

Moreover, Young's inequality and the continuity of p on $\bar{\Omega}$ imply $ts \leq \frac{\alpha_2}{2} t^{p(x)} + C_4 s^{p'(x)}$ for all $x \in \Omega$, $t, s \geq 0$, for some constant $C_4 > 0$. Using this fact with (4.11) and the Hölder-type inequality, we deduce from (4.19) that

$$\begin{aligned} \alpha_2 \int_H w(x) |\nabla u_n|^{p(x)} \, dx &\leq \frac{\alpha_2 \epsilon}{4} + \alpha_3 \int_H (l_2(x) + 1) \, dx + 4C_3 \|w^{\frac{1}{p}} \nabla u\|_{L^{p(x)}(H)} \\ &\quad + \frac{\alpha_2}{2} \int_H w(x) |\nabla u_n|^{p(x)} \, dx + C_4 \int_H \left| w^{-\frac{1}{p(x)}} A(x, u_n, \nabla u) \right|^{p'(x)} \, dx. \end{aligned} \quad (4.20)$$

By (4.1) and (4.11), we see that $\{|w^{-\frac{1}{p}} A(\cdot, u_n, \nabla u)|^{p'}\}$ is uniformly integrable. Combining this and the fact that $l_2, |w^{\frac{1}{p}} \nabla u| \in L^1(\Omega)$, there exists a constant $\delta > 0$ such that, for any measurable $H \subset \Omega$ with $|H| < \delta$, we have

$$\alpha_3 \int_H (l_2(x) + 1) \, dx + 4C_3 \|w^{\frac{1}{p}} \nabla u\|_{L^{p(x)}(H)} + C_4 \int_H \left| w^{-\frac{1}{p(x)}} A(x, u_n, \nabla u) \right|^{p'(x)} \, dx < \frac{\alpha_2 \epsilon}{4}.$$

Hence, we obtain from the last inequality and (4.20) that

$$\int_H w(x) |\nabla u_n|^{p(x)} dx < \epsilon \quad \text{for all } n \geq n_0$$

and $|H| < \delta$, i.e., $\{w|\nabla u_n|^p\}_{n \geq n_0}$ is uniformly integrable and hence, so is $\{w|\nabla u_n - \nabla u|^p\}_{n \geq n_0}$. From this and (4.18), we obtain by Vitali's theorem that

$$\lim_{n \rightarrow \infty} \int_{\Omega} w(x) |\nabla u_n - \nabla u|^{p(x)} dx = 0.$$

Thus, $u_n \rightarrow u$ in X as $n \rightarrow \infty$ due to (4.11). This shows the (S_+) -property of T . Finally, invoking Brezis' theorem, we get that $Tu = 0$ has at least one solution u in X , i.e., (1.5) has at least one weak solution u . This completes the proof. $\square$

Proof of Theorem 4.1. The proof is similar to the proof of Theorem 4.2. In this case, we consider the space $X = W_0^{1,p(x)}(w, \Omega)$ with an equivalent norm $|\nabla u|_{p,w}$ (see [13, Corollary 2.12]) and define the operator

$$T : X \rightarrow X^*$$

by

$$\langle Tu, v \rangle = \int_{\Omega} A(x, u, \nabla u) \cdot \nabla v dx + \int_{\Omega} b(x) |u|^{p(x)-2} uv dx - \int_{\Omega} B(x, u, \nabla u) v dx$$

for all $u, v \in X$. It is easy to verify that T is bounded, demicontinuous, coercive, and satisfies the (S_+) -property as in the proof of Theorem 4.2 above. $\square$

Funding: The first author was supported by the project LO1506 of the Czech Ministry of Education, Youth and Sports. The second author was supported by the National Research Foundation of Korea Grant funded by the Korea Government (MEST) (NRF-2015R1D1A3A01019789).

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