Research Article

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Analysis of positive solutions for classes of quasilinear singular problems on exterior domains

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Abstract: We consider the problem

$$\begin{cases} -\Delta_p u = K(x) \frac{f(u)}{u^\delta} & \text{in } \Omega^e, \\ u(x) = 0 & \text{on } \partial\Omega, \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ ($N > 2$) is a simply connected bounded domain containing the origin with C^2 boundary $\partial\Omega$, $\Omega^e := \mathbb{R}^N \setminus \overline{\Omega}$ is the exterior domain, $1 < p < N$ and $0 \leq \delta < 1$. In particular, under an appropriate decay assumption on the weight function K at infinity and a growth restriction on the nonlinearity f , we establish the existence of a positive weak solution $u \in C^1(\overline{\Omega^e})$ with $u = 0$ pointwise on $\partial\Omega$. Further, under an additional assumption on f , we conclude that our solution is unique. Consequently, when Ω is a ball in $\mathbb{R}^N$, for certain classes of $K(x) = K(|x|)$, we observe that our solution must also be radial.

Keywords: Exterior domain, singular problem, positive solution, decay, existence, uniqueness**MSC 2010:** Primary 35J60, 35J92; secondary 35J75, 35J66

1 Introduction

We consider problems of the form

$$\begin{cases} -\Delta_p u = K(x) \frac{f(u)}{u^\delta} & \text{in } \Omega^e, \\ u(x) = 0 & \text{on } \partial\Omega, \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty, \end{cases} \quad (1.1)$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian, $\Omega \subset \mathbb{R}^N$ ($N > 2$) is a simply connected bounded domain containing the origin with C^2 boundary $\partial\Omega$, $\Omega^e := \mathbb{R}^N \setminus \overline{\Omega}$ is the exterior domain, $1 < p < N$ and $0 \leq \delta < 1$. The weight function $K: \Omega^e \rightarrow (0, \infty)$ and the nonlinearity $f: [0, \infty) \rightarrow (0, \infty)$ satisfy the following hypotheses:

(H1) There exists $K^* > 0$ such that $0 < K(x) < \frac{K^*}{|x|^\alpha}$ for $x \in \overline{\Omega^e}$, where $\alpha > N + \delta \frac{N-p}{p-1}$.

(H2) f is continuous and $\lim_{s \rightarrow \infty} \frac{f(s)}{s^{p-1+\delta}} = 0$.

In particular, from (H1) it follows that $K \in L^\infty(\Omega^e) \cap L^1(\Omega^e)$.

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In this paper, we study positive weak solutions

$$u \in X := \left\{ v \in L^{p^*}(\Omega^e) : \int_{\Omega^e} |\nabla v|^p dx < \infty \text{ and } v = 0 \text{ on } \partial\Omega \text{ in the sense of traces} \right\}$$

of (1.1). The space X is a Banach space endowed with the norm

$$\|v\|_X := \left(\int_{\Omega^e} |\nabla v|^p dx \right)^{\frac{1}{p}},$$

where $p^* := \frac{Np}{N-p}$ is the critical Sobolev exponent, and by a *weak solution* of (1.1) we mean a function $u \in X$ which satisfies

$$\int_{\Omega^e} |\nabla u|^{p-2} \nabla u \cdot \nabla \psi dx = \int_{\Omega^e} K(x) \frac{f(u)}{u^\delta} \psi dx \quad \text{for all } \psi \in X. \quad (1.2)$$

Remark 1.1. Each term in (1.2) is well defined for our weak solution $u \in X$. Indeed, the left-hand side is finite by Hölder's inequality. The well-definedness of the right-hand side can be justified by observing that $\frac{f(u)\psi}{u^\delta} \in L^{p^*/p}(\Omega^e)$ and $K \in L^q(\Omega^e)$ for any $q \geq 1$.

We let $R_0 > 0$ be such that $\Omega \subset B_{R_0} := \{x \in \mathbb{R}^N : |x| < R_0\}$ and express $\Omega^e = \Omega_{R_0} \cup \overline{B_{R_0}^e}$, when it is convenient to do so, where $\Omega_{R_0} := \Omega^e \cap B_{R_0}$, $B_{R_0}^e := \Omega^e \setminus \overline{B_{R_0}}$ and $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$. We will also use the splitting $\Omega^e := \Omega_{1-}^e \cup \Omega_{1+}^e$, where $\Omega_{1-}^e := \{x \in \Omega^e : |v(x)| < 1\}$ and $\Omega_{1+}^e := \{x \in \Omega^e : |v(x)| \geq 1\}$ for a real valued function v . Let $\rho(x) := \text{dist}(x, \partial\Omega)$ for $x \in \Omega^e$ and $\rho_0(x) := \text{dist}(x, \partial\Omega_{R_0})$ for $x \in \Omega_{R_0}$. By $C_c^\infty(\Omega^e)$ we mean the space of C^∞ functions from Ω^e to $\mathbb{R}$ with compact support. We note that $X = \overline{C_c^\infty(\Omega^e)}$, where the closure is taken with respect to the norm $\|\cdot\|_X$. We will use the letter C to denote different positive constants at different occurrences.

First, we establish the existence of a positive weak solution of (1.1).

Theorem 1.2. Assume that (H1) and (H2) hold. Then, there exists a positive weak solution $u = u(x)$ of (1.1). Moreover, $u \in C^1(\overline{\Omega^e})$, $u = 0$ on $\partial\Omega$ pointwise, and for $i = 1, 2$, there exist $C_i, \hat{C}_i > 0$ such that

$$\frac{C_1}{|x|^{\frac{N-p}{p-1}}} \leq u(x) \leq \frac{C_2}{|x|^{\frac{N-p}{p-1}}} \quad \text{for } x \in B_{R_0}^e \quad (1.3)$$

and

$$\hat{C}_1 \rho(x) \leq u(x) \leq \hat{C}_2 \rho(x) \quad \text{for } x \in \Omega_{R_0}. \quad (1.4)$$

In [13], Ko, Lee and Shivaji, using sub- and super solutions, studied the existence of positive solutions of $-\Delta_p u = \lambda \frac{f(u)}{u^\delta}$ in Ω , $u = 0$ on $\partial\Omega$, where λ is a positive parameter and Ω is a bounded domain. Again using sub- and supersolutions, when Ω is a ball in $\mathbb{R}^N$ and for classes of $K(x) = K(|x|)$, the same authors in [14] analyzed the positive radial solution of (1.1) by reducing it to a two-point boundary value problem via radial and Kelvin transformations. See also [18], where (1.1) with a nonsingular reaction term of the form $K(x)u^\alpha$, $0 \leq \alpha < p-1$, was studied via critical point theory. Theorem 1.2 generalizes this result for classes of singular reaction terms.

We also establish the following uniqueness result.

Theorem 1.3. Assume that (H1) and (H2) hold. Suppose also that $\frac{f(s)}{s^{p-1+\delta}}$ is strictly decreasing in $(0, \infty)$. Then, (1.1) has a unique positive weak solution $u \in C^1(\overline{\Omega^e})$ with $u = 0$ on $\partial\Omega$ pointwise and which satisfies the growth restrictions (1.3)–(1.4).

For the nonsingular case ($\delta = 0$) in a bounded domain, see [3, Theorem 1] and [2, 5], where such a uniqueness result was established for $p = 2$ and for $1 < p < \infty$, respectively. We establish Theorem 1.3 using the idea developed in [2]. We believe that Theorem 1.3 is new even for the case $p = 2$.

When Ω is a ball in $\mathbb{R}^N$ and the weight function $K(x)$ is radial, Theorem 1.3 implies that (1.1) cannot have a nonradial positive weak solution satisfying (1.3) and (1.4). In particular, we have the following corollary.

Corollary 1.4. *Let Ω be a ball in $\mathbb{R}^N$ centered at the origin with radius $r_0 > 0$, $K(x) = K(|x|)$ and the hypotheses of Theorem 1.3 hold. Then, the unique positive weak solution from Theorem 1.3 must be radial.*

In Section 2, we discuss an eigenvalue problem and a singular problem whose solutions will be used in the sequel for the construction of a subsolution and a supersolution of (1.1), respectively. In Section 3, we prove Theorem 1.2, in Section 4, we prove Theorem 1.3 and in Section 5, we prove Corollary 1.4. Finally, in the Appendix, we prove a weak comparison principle (Proposition 2.2) for the singular problem (2.1), which is crucial in proving existence and uniqueness results.

2 Auxiliary problems

Let $(\lambda_1, \varphi_1) \in \mathbb{R} \times X$ be the first eigenpair of

$$\begin{cases} -\Delta_p u = \lambda K(x)|u|^{p-2}u & \text{in } \Omega^e, \\ u(x) = 0 & \text{on } \partial\Omega, \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty, \end{cases}$$

where K satisfies (H1). Then, λ_1 and φ_1 satisfy the following crucial properties (see [4] for proofs and additional properties):

(EVP1) $\lambda_1 > 0$ and $\varphi_1 > 0$ in Ω^e ,

(EVP2) there exist $D_1, D_2 > 0$ such that

$$\frac{D_1}{|x|^{\frac{N-p}{p-1}}} \leq \varphi_1(x) \leq \frac{D_2}{|x|^{\frac{N-p}{p-1}}} \quad \text{for } x \in B_{R_0}^e,$$

(EVP3) there exist $\hat{D}_1, \hat{D}_2 > 0$ such that

$$\hat{D}_1 \rho(x) \leq \varphi_1(x) \leq \hat{D}_2 \rho(x) \quad \text{for } x \in \Omega_{R_0},$$

(EVP4) for all $u \in X$,

$$\int_{\Omega^e} |\nabla u|^p dx \geq \lambda_1 \int_{\Omega^e} K(x)|u|^p dx,$$

and the equality holds if and only if $u = t\varphi_1$ for some $t \in \mathbb{R}$.

Next, we consider the singular problem

$$\begin{cases} -\Delta_p u = \frac{K(x)}{u^\delta} & \text{in } \Omega^e, \\ u(x) = 0 & \text{on } \partial\Omega, \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty, \end{cases} \quad (2.1)$$

and establish the following theorem.

Theorem 2.1. *Assume that (H1) holds and $0 \leq \delta < 1$. Then, (2.1) has a unique positive weak solution $u_0 = u_0(x)$. Moreover, $u_0 \in C^1(\bar{\Omega}^e)$, $u_0 = 0$ on $\partial\Omega$ pointwise, and there exist $c_1, c_2 > 0$ satisfying*

$$c_1 \varphi_1(x) \leq u_0(x) \leq c_2 \varphi_1(x) \quad \text{for } x \in \Omega^e. \quad (2.2)$$

A similar result to Theorem 2.1 was discussed in [10] for the case of a bounded domain, and was recently extended to a coupled system of equations in [11]. For the case $p = 2$, we refer the reader to the book by Ghergu and Rădulescu [9], where singular elliptic problems are studied in detail, and a rich list of references is provided. Therefore, this auxiliary result contributes independently to the part of the literature dealing with elliptic problems on exterior domains.

To prove Theorem 2.1, we use the following *weak comparison principle* which extends the result proved in [8] for nonsingular problems (see the Appendix for the proof of Proposition 2.2).

Proposition 2.2. Let $f_1, f_2 \in L^1(\Omega_{R_0})$ and $g_1, g_2 \in W^{\frac{1}{p'}, p}(\partial B_{R_0})$ be such that $f_1 \leq f_2$ a.e. in Ω_{R_0} and $0 < g_1 \leq g_2$ a.e. on ∂B_{R_0} , where $\frac{1}{p} + \frac{1}{p'} = 1$. Suppose that $u_1, u_2 \in W^{1,p}(\Omega_{R_0}) \cap L^\infty(\Omega_{R_0})$ are positive weak solutions of

$$\begin{cases} -\Delta_p u_1 - \frac{K(x)}{u_1^\delta} = f_1(x) & \text{in } \Omega_{R_0}, \\ u_1(x) = 0 & \text{on } \partial\Omega, \\ u_1(x) = g_1(x) & \text{on } \partial B_{R_0}, \end{cases} \quad (2.3)$$

and

$$\begin{cases} -\Delta_p u_2 - \frac{K(x)}{u_2^\delta} = f_2(x) & \text{in } \Omega_{R_0}, \\ u_2(x) = 0 & \text{on } \partial\Omega, \\ u_2(x) = g_2(x) & \text{on } \partial B_{R_0}, \end{cases} \quad (2.4)$$

respectively, satisfying $u_i(x) > k_i \rho_0(x)$ for some $k_i > 0$, $i = 1, 2$. Then, $u_1 \leq u_2$ a.e. in Ω_{R_0} .

Proof of Theorem 2.1. Define the functional $E: X \rightarrow \mathbb{R}$ by

$$E(u) := \frac{1}{p} \int_{\Omega^e} |\nabla u|^p \, dx - \frac{1}{1-\delta} \int_{\Omega^e} K(x)(u^+)^{1-\delta} \, dx.$$

Due to our assumption on $K(x)$, using Hölder's inequality combined with (EVP4), one can assert that

$$\begin{aligned} \int_{\Omega^e} K(x)(u^+)^{1-\delta} \, dx &= \int_{\Omega^e} K(x)^{\frac{p-1+\delta}{p}} K(x)^{\frac{1-\delta}{p}} (u^+)^{1-\delta} \, dx \\ &\leq \left(\int_{\Omega^e} K(x) \, dx \right)^{\frac{p-1+\delta}{p}} \left(\int_{\Omega^e} K(x)(u^+)^p \, dx \right)^{\frac{1-\delta}{p}} \\ &\leq \|K\|_{L^1(\Omega^e)}^{\frac{p-1+\delta}{p}} \left(\frac{1}{\lambda_1} \int_{\Omega^e} |\nabla u|^p \, dx \right)^{\frac{1-\delta}{p}} < \infty. \end{aligned}$$

Therefore, E is well defined. Also E is coercive since

$$E(u) \geq \frac{1}{p} \int_{\Omega^e} |\nabla u|^p \, dx - \frac{\|K\|_{L^1(\Omega^e)}^{\frac{p-1+\delta}{p}}}{(1-\delta)\lambda_1^{\frac{1-\delta}{p}}} \left(\int_{\Omega^e} |\nabla u|^p \, dx \right)^{\frac{1-\delta}{p}} \rightarrow \infty \quad \text{as } \|u\|_X \rightarrow \infty.$$

Since $K \in L^\infty(\Omega^e) \cap L^1(\Omega^e)$, the mapping $u \mapsto - \int_{\Omega^e} K(x)(u^+)^{1-\delta} \, dx$ from X to $\mathbb{R}$ is weakly continuous (see [6, p. 164] for a similar proof in $\mathbb{R}^N$). Therefore, E is weakly lower-semicontinuous and so there exists a global minimizer $u_0 \in X$ of E such that $E(u_0) = \min_{u \in X} E(u)$. But $E(0) = 0$ and

$$E(t\varphi_1) = \frac{\lambda_1 t^p}{p} \int_{\Omega^e} K(x)\varphi_1^p \, dx - \frac{t^{1-\delta}}{1-\delta} \int_{\Omega^e} K(x)\varphi_1^{1-\delta} \, dx < 0$$

for $t \approx 0$, and hence $E(u_0) = \min_{u \in X} E(u) < 0 = E(0)$. This implies $u_0 \neq 0$ in Ω^e . The uniqueness of u_0 , the fact that $u_0 \geq c_1 \varphi_1$ in Ω^e for sufficiently small $c_1 > 0$ and that u_0 satisfies

$$\int_{\Omega^e} |\nabla u_0|^{p-2} \nabla u_0 \cdot \nabla \psi \, dx = \int_{\Omega^e} \frac{K(x)}{u_0^\delta} \psi \, dx \quad \text{for all } \psi \in C_c^\infty(\Omega^e) \quad (2.5)$$

follow from arguments similar to those used in the proof of [10, Lemma 3.1] applied to our functional framework in the exterior domain Ω^e .

Following the first part of the proof of [10, Lemma A.6], which does not depend on the boundedness of the domain, we observe that u_0 satisfies

$$\int_{u_0 > 1} |\nabla(u_0 - 1)|^{p-2} \nabla(u_0 - 1) \cdot \nabla \psi \, dx \leq \int_{u_0 > 1} K(x) \psi \, dx \quad \text{for all } \psi \in C_c^\infty(\Omega^e) \, (\psi \geq 0). \quad (2.6)$$

Since $K \in L^\infty(\Omega^e) \cap L^1(\Omega^e)$, (2.6) holds for all $\psi \in X$ ($\psi \geq 0$). In particular, we have

$$\int_{u_0 > 1} |\nabla(u_0 - 1)|^{p-2} \nabla(u_0 - 1) \cdot \nabla \psi \, dx \leq \int_{u_0 > 1} K(x) u_0^{p-1} \psi \, dx \quad \text{for all } \psi \in X \text{ } (\psi \geq 0). \quad (2.7)$$

For $M > 0$, we set $v_M(x) := \min\{M, u_0(x) - 1\}$ and for $\kappa > 0$ let $\psi = v_M^{\kappa p+1} \in X$ be the test function in (2.7). Now, we can start the bootstrap argument (attributed to Moser) with respect to κ , which leads to the existence of a constant $C > 0$ such that $\|u_0\|_{L^\infty(\Omega^e)} \leq C \|u_0\|_{L^{p^*}(\Omega^e)}$. (See [6, Lemma 4.5], where the argument is carried out on the entire $\mathbb{R}^N$.) With this L^∞ estimate at hand, ∇u_0 is locally Hölder continuous by the interior regularity, see [17, Theorem 1]. Thus, u_0 is locally $C^{1,\eta}$ function in Ω^e for some $\eta \in (0, 1)$.

The estimate from below follows since $u_0 \geq c_1 \varphi_1$ in Ω^e . We now establish the upper bound in (2.2) in the following two steps.

Step 1: First we establish the rate of the decay of $u_0(x) \rightarrow 0$ as $|x| \rightarrow \infty$. In particular, we show that there exists $\tilde{C} > 0$ such that $u_0(x) \leq \tilde{C} \varphi_1(x)$ for $x \in B_{R_0}^e$. To this end, we employ the result of [16, Theorem 1]. Indeed, due to the left inequality in (2.2) and (H1), we have

$$0 < \frac{K(x)}{u_0^\delta(x)} \leq \frac{\hat{K}}{|x|^{\alpha - \delta \frac{N-p}{p-1}}}, \quad \text{where } \hat{K} := \frac{K^*}{(c_1 D_1)^\delta}.$$

Thanks to the assumption $\alpha > \delta \frac{N-p}{p-1}$,

$$\left\| \frac{K}{u_0^\delta} \right\|_{L^{\gamma'}(B_{R_0}^e)} < \infty \quad \text{for any } \gamma > \max \left\{ 1, \frac{N}{\alpha - \delta \frac{N-p}{p-1}} \right\}.$$

Now let

$$\gamma_1 > \max \left\{ \frac{N}{p}, \alpha - \delta \frac{N-p}{p-1} \right\}$$

and $B_j(x) := \{y \in \mathbb{R}^N : |y - x| < j\} \subset B_{R_0}^e$ for $j = 1, 2$ and $x \in B_{R_0}^e$. Then, by [16, Theorem 1], we have

$$\|u_0\|_{L^\infty(B_1(x))} \leq \left(\|u_0\|_{L^{p^*}(B_2(x))} + \left\| \frac{K}{u_0^\delta} \right\|_{L^{\gamma_1}(B_2(x))} \right).$$

This in turn implies that $u_0(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Then, since $\alpha > \delta \frac{N-p}{p-1} + N$, from [1, Theorem 4] it follows that there exists $C > 0$ such that $u_0(x) \leq C/|x|^{\frac{N-p}{p-1}}$ for $x \in B_{R_0}^e$. Combining this fact with (EVP3) gives $u_0(x) \leq \tilde{C} \varphi_1(x)$ for $x \in B_{R_0}^e$ and for some $\tilde{C} > 0$.

Step 2: It remains to show that there exists $\hat{C} > 0$ such that $u_0(x) \leq \hat{C} \varphi_1(x)$ for $x \in \Omega_{R_0}$. To achieve this, we follow the approach used in the proof of [10, Lemma A.7], and construct a suitable supersolution $w = w(x)$, $x \in \Omega_{R_0}$, satisfying

$$\begin{cases} -\Delta_p w \geq \frac{K(x)}{w^\delta} & \text{in } \Omega_{R_0}, \\ w(x) = 0 & \text{on } \partial\Omega, \\ w(x) \geq u_0(x) & \text{on } \partial B_{R_0}. \end{cases} \quad (2.8)$$

With this in hand, we will apply Proposition 2.2 to prove that $u_0(x) \leq w(x)$ for $x \in \Omega_{R_0}$. Therefore, to achieve our goal, we aim to construct w in such a way that for appropriate $\hat{C} > 0$, $w(x) \leq \hat{C} \varphi_1(x)$ for $x \in \Omega_{R_0}$.

Let $\theta_\beta : [0, R_\beta] \rightarrow [0, \infty)$ be a solution of the initial value problem

$$\begin{cases} -(|\theta'_\beta(r)|^{p-2} \theta'_\beta(r))' = \theta_\beta^{-\delta}(r), & 0 < r < R_\beta, \\ \theta_\beta(0) = 0, & \theta'_\beta(0) = \beta > 0, \end{cases}$$

where $R_\beta := \sup\{s \in (0, \infty) : \theta'_\beta(r) > 0 \text{ for all } r \in (0, s)\}$. It can be shown, as in [10, p. 29], that for $\sigma > 0$, the function $w(x) := \sigma \theta_\beta(\varphi_1(x))$ satisfies

$$-\Delta_p w \geq \sigma^{p-1+\delta} \{|\nabla \varphi_1|^p + \lambda_1(\theta'_\beta(\|\varphi_1\|_{L^\infty(\Omega_{R_0})}) \varphi_1)^{p-1+\delta}\} w^{-\delta}.$$

Similarly, again as in [10], using the strong maximum principle on φ_1 at $\partial\Omega$ and the strict positivity of φ_1 in Ω_{R_0} , we conclude that

$$\alpha_1 := \min_{\bar{\Omega}_{R_0}} \{ |\nabla \varphi_1|^p + \lambda_1 (\theta'_\beta (\|\varphi_1\|_{L^\infty(\Omega_{R_0})}) \varphi_1)^{p-1+\delta} \} > 0.$$

We choose $\sigma > 0$ large enough so that $\sigma^{p-1+\delta} \alpha_1 \geq \|K\|_{L^\infty(\Omega_{R_0})}$. In particular, we get

$$-\Delta_p w \geq \frac{K(x)}{w^\delta} \quad \text{in } \Omega_{R_0}.$$

Choosing σ larger if necessary, thanks to the fact that $\varphi_1 > 0$ on ∂B_{R_0} , the continuity of u_0 on ∂B_{R_0} and the compactness of ∂B_{R_0} , we also get $w(x) \geq u_0(x)$ for $x \in \Omega_{R_0}$. Therefore, w satisfies (2.8). Applying Proposition 2.2, we get $u_0(x) \leq w(x) = \sigma \theta_\beta(\varphi_1(x))$. It is easy to infer from the properties of θ_β that $\theta_\beta(s) \leq \beta s$ for $s \in (0, R_\beta)$, which in turn implies that $\sigma \theta_\beta(\varphi_1(s)) \leq \sigma \beta \varphi_1(x)$. Hence, $u_0(x) \leq \hat{C} \varphi_1(x)$ for $x \in \Omega_{R_0}$ with $\hat{C} := \sigma \beta$. This completes Step 2.

Now from Steps 1 and 2, it follows that there exists $c_2 > 0$ such that $u_0(x) \leq c_2 \varphi_1(x)$ for $x \in \Omega^e$, and hence (2.2) holds.

It remains to show that $u_0 \in X$ is indeed a weak solution of (2.1). To this end, we show that we can take arbitrary $\psi \in X$ as a test function in (2.5) instead of $\psi \in C_c^\infty(\Omega^e)$ to satisfy the definition of a weak solution in (1.2).

Let $\psi \in X$ be arbitrary. Then, there exists $\psi_n \in C_c^\infty(\Omega^e)$ such that $\lim_{n \rightarrow \infty} \|\psi_n - \psi\|_X = 0$. Then, since $u_0 \in X$, using Hölder's inequality and the relation $\frac{1}{p} + \frac{1}{p'} = 1$, we obtain

$$\left| \int_{\Omega^e} |\nabla u_0|^{p-2} \nabla u_0 \cdot \nabla (\psi_n - \psi) \, dx \right| \leq \left(\int_{\Omega^e} |\nabla u_0|^p \, dx \right)^{\frac{1}{p'}} \left(\int_{\Omega^e} |\nabla (\psi_n - \psi)|^p \, dx \right)^{\frac{1}{p}} \rightarrow 0, \quad (2.9)$$

and hence

$$\int_{\Omega^e} |\nabla u_0|^{p-2} \nabla u_0 \cdot \nabla \psi_n \, dx \rightarrow \int_{\Omega^e} |\nabla u_0|^{p-2} \nabla u_0 \cdot \nabla \psi \, dx \quad \text{as } n \rightarrow \infty.$$

Now we show

$$\left| \int_{\Omega^e} \frac{K(x)}{u_0^\delta} (\psi_n - \psi) \, dx \right| \leq \left| \int_{\Omega_{R_0}} \frac{K(x)}{u_0^\delta} (\psi_n - \psi) \, dx \right| + \left| \int_{B_{R_0}^e} \frac{K(x)}{u_0^\delta} (\psi_n - \psi) \, dx \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (2.10)$$

Indeed, using Hölder's inequality, we get

$$\left| \int_{B_{R_0}^e} \frac{K(x)}{u_0^\delta} (\psi_n - \psi) \, dx \right| \leq \left(\int_{B_{R_0}^e} \frac{K^{(p^*)'}(x)}{u_0^{\delta(p^*)'}} \, dx \right)^{\frac{1}{(p^*)'}} \left(\int_{B_{R_0}^e} |\psi_n - \psi|^{p^*} \, dx \right)^{\frac{1}{p^*}}, \quad (2.11)$$

where $(p^*)' = \frac{Np}{Np-N+1}$ is the conjugate exponent of p^* . Further, since

$$0 < K(x) < \frac{K^*}{|x|^\alpha} \quad \text{and} \quad u_0(x) \geq c_1 \varphi_1(x) \geq \frac{c_1 D_1}{|x|^{\frac{N-p}{p-1}}}$$

for $x \in B_{R_0}^e$ and $\alpha > N + \delta \frac{N-p}{p-1} > \frac{N}{(p^*)'} + \delta \frac{N-p}{p-1}$, there exists a constant $C > 0$ such that

$$\int_{B_{R_0}^e} \frac{K^{(p^*)'}(x)}{u_0^{\delta(p^*)'}} \, dx \leq \frac{1}{C} \int_{B_{R_0}^e} \frac{1}{|x|^{(\alpha - \delta \frac{N-p}{p-1})(p^*)'}} \, dx < \infty. \quad (2.12)$$

The second integral in (2.11) tends to zero as $n \rightarrow \infty$ by the Sobolev inequality. Therefore,

$$\left| \int_{B_{R_0}^e} \frac{K(x)}{u_0^\delta} (\psi_n - \psi) \, dx \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Next we show that the first integral in (2.10) tends to zero as $n \rightarrow \infty$. By [15, Section 8.8], we have the embedding

$$W^{1,p}(\Omega_{R_0}) \hookrightarrow L^p(\Omega_{R_0}; \rho_0^{-1+\omega}(x)) \quad (2.13)$$

for arbitrary $\omega > 0$. Note that $L^p(\Omega_{R_0}; \rho_0^{-1+\omega})$ is equipped with the norm

$$\int_{\Omega_{R_0}} \frac{|u(x)|^p}{\rho_0^{1-\omega}(x)} dx.$$

Since $\lim_{n \rightarrow \infty} \|\psi_n - \psi\|_X = 0$, we have $\lim_{n \rightarrow \infty} \|\psi_n - \psi\|_{W^{1,p}(\Omega_{R_0})} = 0$, and hence by (2.13), for $0 < \omega < 1$,

$$\int_{\Omega_{R_0}} \frac{|\psi_n - \psi|^p}{\rho^{1-\omega}(x)} dx \leq C \int_{\Omega_{R_0}} \frac{|\psi_n - \psi|^p}{\rho_0^{1-\omega}(x)} dx \rightarrow 0, \quad (2.14)$$

where $C > 0$ does not depend on n . Using Hölder's inequality, we have

$$\begin{aligned} \left| \int_{\Omega_{R_0}} \frac{K(x)}{u_0^\delta} (\psi_n - \psi) dx \right| &= \left| \int_{\Omega_{R_0}} \frac{K(x)}{u_0^\delta} \rho^{\frac{1-\omega}{p}}(x) \frac{\psi_n - \psi}{\rho^{\frac{1-\omega}{p}}(x)} dx \right| \\ &\leq \left(\int_{\Omega_{R_0}} \frac{K^{p'}(x)}{u_0^{\delta p'}} \rho^{p' \frac{1-\omega}{p}}(x) dx \right)^{\frac{1}{p'}} \left(\int_{\Omega_{R_0}} \frac{|\psi_n - \psi|^p}{\rho^{1-\omega}(x)} dx \right)^{\frac{1}{p}}. \end{aligned} \quad (2.15)$$

Thus, using the boundedness of K in Ω_{R_0} and the estimate $u_0(x) \geq c_1 \varphi_1(x) \geq c_1 \hat{D}_1 \rho(x)$, there exists $C > 0$ such that

$$\int_{\Omega_{R_0}} \frac{K^{p'}(x)}{u_0^{\delta p'}} \rho^{p' \frac{1-\omega}{p}}(x) dx \leq C \int_{\Omega_{R_0}} \rho^{p'(\frac{1-\omega}{p}-\delta)}(x) dx < \infty, \quad (2.16)$$

by choosing $0 < \omega < p(1 - \delta)$. Now from (2.10), (2.11), (2.12), (2.14), (2.15) and (2.16), it follows that

$$\left| \int_{\Omega^e} \frac{K(x)}{u_0^\delta} (\psi_n - \psi) dx \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (2.17)$$

From (2.9) and (2.17), we conclude that (2.5) holds for arbitrary $\psi \in X$, that is, u_0 is a weak solution of (2.1).

It remains to show that $u_0 \in C^1(\overline{\Omega^e})$. From (2.2), it follows that $0 < u_0(x) \leq C\rho(x)$ for all $x \in \Omega_{R_0}$, where $0 \leq C < \infty$ is a constant. Applying [10, Theorem B.1], we get that $u_0 \in C^{1,\eta}(\overline{\Omega_{R_0}})$ for some $\eta = \eta(R_0) \in (0, 1)$. But since $R_0 > 0$ can be chosen such that $\overline{\Omega} \subset B_{R_0}$, we obtain that $u_0 \in C^1(\overline{\Omega^e})$ as desired. This completes the proof of Theorem 2.1. $\square$

3 Proof of Theorem 1.2

We prove Theorem 1.2 using the method of sub- and supersolutions. By a *subsolution* of (1.1) we mean a function $\underline{u} \in C(\overline{\Omega^e}) \cap C^1(\Omega^e)$ such that $\underline{u} > 0$ in Ω^e , $\underline{u} = 0$ on $\partial\Omega$ and which satisfies

$$\int_{\Omega^e} |\nabla \underline{u}|^{p-2} \nabla \underline{u} \cdot \nabla \psi dx \leq \int_{\Omega^e} K(x) \frac{f(\underline{u})}{\underline{u}^\delta} \psi dx \quad \text{for all } 0 \leq \psi \in C_c^\infty(\Omega^e).$$

By a *supersolution* of (1.1), we mean a function $\overline{u} \in C(\overline{\Omega^e}) \cap C^1(\Omega^e)$ such that $\overline{u} > 0$ in Ω^e , $\overline{u} = 0$ on $\partial\Omega$ and which satisfies

$$\int_{\Omega^e} |\nabla \overline{u}|^{p-2} \nabla \overline{u} \cdot \nabla \psi dx \geq \int_{\Omega^e} K(x) \frac{f(\overline{u})}{\overline{u}^\delta} \psi dx \quad \text{for all } 0 \leq \psi \in C_c^\infty(\Omega^e).$$

We establish the existence of a positive weak solution by employing the following result.

Proposition 3.1 ([7]). Let $\underline{u}$ and $\bar{u}$ be a subsolution and a supersolution of (1.1), respectively, such that $\underline{u} \leq \bar{u}$ in Ω^e . Then, there exists $u \in C(\bar{\Omega}^e) \cap C^1(\Omega^e)$, with $\underline{u} \leq u \leq \bar{u}$ in Ω^e , satisfying

$$\int_{\Omega^e} |\nabla u|^{p-2} \nabla u \cdot \nabla \psi \, dx = \int_{\Omega^e} K(x) \frac{f(u)}{u^\delta} \psi \, dx \quad \text{for all } \psi \in C_c^\infty(\Omega^e). \quad (3.1)$$

Following the approach used in [13, 14], we construct a subsolution $\underline{u}$ and a supersolution $\bar{u}$.

First, we construct a supersolution $\bar{u}$ as a sufficiently large multiple of u_0 from Theorem 2.1. Define $f^*(s) := \max_{t \in [0, s]} f(t)$. Then, $f^*(s)$ is nondecreasing, and from (H2) it follows that

$$\lim_{s \rightarrow \infty} \frac{f^*(s)}{s^{p-1+\delta}} = 0. \quad (3.2)$$

Let $u_0 > 0$ be a solution of (2.1). Then, from (3.2) it follows that there exists $M \gg 1$ such that

$$\frac{f^*(M\|u_0\|_\infty)}{(M\|u_0\|_\infty)^{p-1+\delta}} \leq \frac{1}{\|u_0\|_\infty^{p-1+\delta}},$$

that is,

$$f^*(M\|u_0\|_\infty) \leq M^{p-1+\delta}. \quad (3.3)$$

Then, by (3.3), for $M \gg 1$, $\bar{u} := M^{1+\frac{\delta}{p-1}} u_0 > 0$ is a supersolution. Indeed, $\bar{u} = 0$ on $\partial\Omega$ and formally in Ω^e , we have

$$-\Delta_p(\bar{u}) = M^{p-1+\delta} \frac{K(x)}{u_0^\delta} \geq \frac{K(x)f^*(M\|u_0\|_\infty)}{u_0^\delta} \geq K(x) \frac{f^*(Mu_0)}{(Mu_0)^\delta} \geq K(x) \frac{f(Mu_0)}{(Mu_0)^\delta} = K(x) \frac{f(\bar{u})}{\bar{u}^\delta}.$$

Second, we construct a subsolution $\underline{u}$ as a sufficiently small multiple of the eigenfunction φ_1 discussed in Section 2. Since $\lim_{s \rightarrow 0^+} \frac{f(s)}{s^\delta} = \infty$, there exists a positive constant $m \approx 0$ such that $\frac{f(m\varphi_1)}{(m\varphi_1)^\delta} \geq \lambda_1(m\varphi_1)^{p-1}$. Then, it is easy to see that $\underline{u} := m\varphi_1$ is a subsolution for $m \approx 0$. Indeed, $\underline{u} > 0$ in Ω^e , $\underline{u} = 0$ on $\partial\Omega$ and formally, we have

$$-\Delta_p \underline{u} = \lambda_1(m\varphi_1)^{p-1} \leq \frac{f(m\varphi_1)}{(m\varphi_1)^\delta} \quad \text{in } \Omega^e.$$

By Theorem 2.1, $u_0 \geq c_1\varphi_1$, so we can choose m sufficiently small so that $\bar{u} \geq \underline{u}$ in Ω^e . Thus, by Proposition 3.1, there exists a solution $u \in C(\bar{\Omega}^e) \cap C^1(\Omega^e)$ of (1.1) satisfying (3.1).

To complete the proof of Theorem 1.2, it remains to show that u is a weak solution of (1.1). We will show that:

(A) $u \in X$,

(B) we can take arbitrary $\psi \in X$ as test functions in (3.1), (hence u satisfies (1.2)).

Proof of (A). We carefully inspect the proof of [7, Theorem 1.1]. Let $j \in \mathbb{N}$ be large enough and

$$\Omega_j := \left\{ x \in \Omega^e : |x| < j \text{ and } \rho(x) \geq \frac{1}{j} \right\}.$$

Then, $\Omega^e = \bigcup \Omega_j$ and $\bar{\Omega}_j \subset \Omega_{j+1}$. Let u_j be the minimal weak solution of

$$\begin{cases} -\Delta_p u = K(x) \frac{f(u)}{u^\delta} & \text{in } \Omega_j, \\ u = \underline{u} & \text{on } \partial\Omega_j, \end{cases}$$

that is, $u_j \in W^{1,p}(\Omega_j)$, $u_j = \underline{u}$ on $\partial\Omega_j$ in the sense of traces and

$$\int_{\Omega_j} |\nabla u_j|^{p-2} \nabla u_j \cdot \nabla \psi \, dx = \int_{\Omega_j} K(x) \frac{f(u_j)}{u_j^\delta} \psi \, dx \quad \text{for all } \psi \in W_0^{1,p}(\Omega_j). \quad (3.4)$$

See [7, p. 150] for the construction of the minimal solution u_j on Ω_j . Our goal is to show that $\int_{\Omega_j} |\nabla u_j|^p \, dx$ are uniformly bounded, independent of $j \in \mathbb{N}$. Choosing $\psi = u_j - \underline{u} \in W_0^{1,p}(\Omega_j)$ in (3.4), we get

$$\int_{\Omega_j} |\nabla u_j|^{p-2} \nabla u_j \cdot \nabla (u_j - \underline{u}) \, dx = \int_{\Omega_j} K(x) \frac{f(u_j)}{u_j^\delta} (u_j - \underline{u}) \, dx. \quad (3.5)$$

Now Young's inequality, namely: $ab \leq \frac{\tau^p}{p} a^p + \frac{\tau^{p'}}{p'} b^{p'}$ with $\frac{1}{p} + \frac{1}{p'} = 1$ yields (choosing $\tau = 1$)

$$\left(\int_{\Omega_j} |\nabla u_j|^{(p-1)\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left(\int_{\Omega_j} |\nabla \underline{u}|^p dx \right)^{\frac{1}{p}} \leq \frac{1}{p'} \int_{\Omega_j} |\nabla u_j|^p dx + \frac{1}{p} \int_{\Omega_j} |\nabla \underline{u}|^p dx.$$

Then, using Hölder's and Young's inequalities, we estimate the left-hand side of (3.5) from below as follows:

$$\begin{aligned} \int_{\Omega_j} |\nabla u_j|^{p-2} \nabla u_j \cdot \nabla (u_j - \underline{u}) dx &= \int_{\Omega_j} |\nabla u_j|^p dx - \int_{\Omega_j} |\nabla u_j|^{p-2} \nabla u_j \cdot \nabla \underline{u} dx \\ &\geq \int_{\Omega_j} |\nabla u_j|^p dx - \left(\int_{\Omega_j} |\nabla u_j|^{(p-1)\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left(\int_{\Omega_j} |\nabla \underline{u}|^p dx \right)^{\frac{1}{p}} \\ &\geq \int_{\Omega_j} |\nabla u_j|^p dx - \frac{1}{p'} \int_{\Omega_j} |\nabla u_j|^p dx - \frac{1}{p} \int_{\Omega_j} |\nabla \underline{u}|^p dx \\ &= \frac{1}{p} \left[\int_{\Omega_j} |\nabla u_j|^p dx - \int_{\Omega_j} |\nabla \underline{u}|^p dx \right] \\ &\geq \frac{1}{p} \int_{\Omega_j} |\nabla u_j|^p dx - C, \end{aligned} \quad (3.6)$$

where $\int_{\Omega_j} |\nabla \underline{u}|^p dx \leq \int_{\Omega^e} |\nabla \underline{u}|^p dx \leq C$ for some $C > 0$ independent of $j \in \mathbb{N}$.

We extend u_j to entire $\overline{\Omega^e}$ by setting $u_j(x) = \underline{u}(x)$ for all $x \in \overline{\Omega^e} \setminus \Omega_j$. Notice that $u_j \in C(\overline{\Omega^e})$ as shown in [7, p. 150].

In order to estimate the right-hand side of (3.5) from above, we write

$$\left| \int_{\Omega_j} K(x) \frac{f(u_j)}{u_j^\delta} (u_j - \underline{u}) dx \right| \leq \int_{\Omega^e} K(x) \frac{f(u_j)}{u_j^\delta} u_j dx + \int_{\Omega^e} K(x) \frac{f(u_j)}{u_j^\delta} \underline{u} dx. \quad (3.7)$$

Since $0 \leq \delta < 1$ and $f(u_j)$ is bounded in Ω_{1-}^e , there exists $a_1 > 0$ such that $\frac{f(u_j)}{u_j^\delta} u_j \leq a_1$ in Ω_{1-}^e . Also, there exist $b_1, b_2 > 0$ such that

$$\frac{f(u_j)}{u_j^\delta} \leq b_1 |u_j|^{p-1} + b_2 \quad \text{in } \Omega_{1+}^e. \quad (3.8)$$

Therefore, $u_j \in L^\infty(\Omega^e)$ and $K \in L^1(\Omega^e)$ yield

$$\int_{\Omega^e} K(x) \frac{f(u_j)}{u_j^\delta} u_j dx \leq a_1 \int_{\Omega_{1-}^e} K(x) dx + b_1 \int_{\Omega_{1+}^e} K(x) u_j^p dx + b_2 \int_{\Omega_{1+}^e} K(x) u_j dx < \infty. \quad (3.9)$$

Next, we estimate the second integral on the right-hand side of (3.7) as follows:

$$\int_{\Omega^e} K(x) \frac{f(u_j)}{u_j^\delta} \underline{u} dx \leq \int_{B_{R_0}^e} K(x) \frac{f(u_j)}{u_j^\delta} \underline{u} dx + \int_{\Omega_{R_0}} K(x) \frac{f(u_j)}{u_j^\delta} \underline{u} dx. \quad (3.10)$$

First we observe that $f(u_j)$ is bounded and $\hat{C}_1 \rho(x) \leq \underline{u}(x) \leq u_j(x) \leq \bar{u}(x) \leq \hat{C}_2 \rho(x)$ in Ω_{R_0} . Then, $K \in L^\infty(\Omega^e)$ yields

$$\int_{\Omega_{R_0}} K(x) \frac{f(u_j)}{u_j^\delta} \underline{u} dx \leq a_2 \int_{\Omega_{R_0}} \frac{\underline{u}}{u_j^\delta} dx \leq a_3 \int_{\Omega_{R_0}} \rho^{1-\delta}(x) dx < \infty \quad (3.11)$$

for some positive constants a_2 and a_3 . In the unbounded part $B_{R_0}^e$, taking advantage of the decay rates of $K(x)$, u_j and $\underline{u}$ at infinity, we find that there exists $a_4 > 0$ such that

$$\int_{B_{R_0}^e} K(x) \frac{f(u_j)}{u_j^\delta} \underline{u} dx \leq a_4 \int_{B_{R_0}^e} \frac{1}{|x|^\alpha} |x|^{\delta \frac{N-p}{p-1}} \frac{1}{|x|^{\frac{N-p}{p-1}}} dx = a_4 \int_{B_{R_0}^e} \frac{1}{|x|^{\alpha+(1-\delta)\frac{N-p}{p-1}}} dx < \infty, \quad (3.12)$$

since $\alpha > N + \delta \frac{N-p}{p-1}$. Combining (3.6)–(3.12), there exists $a_5 > 0$, independent of j , such that $\int_{\Omega_j} |\nabla u_j|^p dx \leq a_5$.

From [7, p. 151], it follows that for any fixed $k \in \mathbb{N}$ and $j \geq k + 1$, there exists a subsequence of $\{u_j\}$, defined again as $\{u_j\}$, such that $u_j|_{\overline{\Omega}_k}$ converges to u^k in C^1 -topology. Then, the function u defined as $u(x) = u^k(x)$ if $x \in \overline{\Omega}_k$ for all $k \in \mathbb{N}$ is a solution of (1.1) and satisfies

$$\int_{\Omega_k} |\nabla u|^p dx = \lim_{j \rightarrow \infty} \int_{\Omega_k} |\nabla u_j|^p dx \leq \limsup_{j \rightarrow \infty} \int_{\Omega_j} |\nabla u_j|^p dx \leq a_5.$$

Hence, $\int_{\Omega^e} |\nabla u|^p dx \leq a_5$. We can then infer, using Sobolev's inequality, that $\int_{\Omega^e} |u|^{p^*} dx < \infty$ and, since $u = 0$ on $\partial\Omega$, we conclude that $u \in X$. $\square$

Proof of (B). Following the same argument as in the proof of Theorem 2.1, it can be shown that (3.1) holds for arbitrary $\psi \in X$. $\square$

4 Proof of Theorem 1.3

As discussed in the introduction, the proof follows the arguments used in [2] for nonsingular problems in bounded domains. Let $H_p: X \rightarrow \mathbb{R}$ be defined by

$$H_p(u) := \frac{1}{p} \int_{\Omega^e} |\nabla u|^p dx - \int_{\Omega^e} K(x) \tilde{F}(u) dx,$$

where $\tilde{F}(s) := \int_0^{|s|} \tilde{f}(z) dz$ and $\tilde{f}(s) = \frac{f(s)}{s^\delta}$. First, we wish to show that the functional H_p is well defined in X . The first integral is clearly finite. As for the second case, since each term involved is nonnegative, $K \in L^q(\Omega^e)$ for any $q > 1$ (by interpolation) and $\int_{\Omega_{1+}^e} |u|^{p^*} dx < \infty$, using (3.8) and the characterization of $\Omega_{1-}^e, \Omega_{1+}^e$, we obtain

$$\begin{aligned} \int_{\Omega^e} K(x) \tilde{F}(u) dx &= \int_{\Omega_{1-}^e \cup \Omega_{1+}^e} K(x) \left[\int_0^1 \tilde{f}(z) dz + \int_1^{|u|} \tilde{f}(z) dz \right] dx \\ &\leq \int_{\Omega_{1-}^e} K(x) \left[\int_0^1 \tilde{f}(z) dz + \int_1^{|u|} \tilde{f}(z) dz \right] dx + \int_{\Omega_{1+}^e} K(x) \left[\int_0^1 \tilde{f}(z) dz + \int_1^{|u|} \tilde{f}(z) dz \right] dx \\ &\leq \int_{\Omega^e} K(x) \int_0^1 \tilde{f}(z) dz dx + \int_{\Omega_{1+}^e} K(x) \int_1^{|u|} \tilde{f}(z) dz dx \\ &\leq \max_{0 \leq z \leq 1} |f(z)| \int_{\Omega^e} K(x) \int_0^1 \frac{1}{z^\delta} dz dx + \int_{\Omega_{1+}^e} K(x) \left[\int_1^{|u|} (b_1 |z|^{p-1} + b_2) dz \right] dx \\ &\leq C^* + C \int_{\Omega_{1+}^e} K(x) |u(x)|^p dx + b_2 \int_{\Omega_{1+}^e} K(x) |u(x)| dx \\ &\leq C^* + C \left(\int_{\Omega_{1+}^e} [K(x)]^p dx \right)^{\frac{1}{p}} \left(\int_{\Omega_{1+}^e} |u|^{p^*} dx \right)^{\frac{p}{p^*}} + b_2 \left(\int_{\Omega_{1+}^e} [K(x)]^{p^*} dx \right)^{\frac{1}{p^*}} \left(\int_{\Omega_{1+}^e} |u|^{p^*} dx \right)^{\frac{1}{p^*}} \\ &< \infty \end{aligned}$$

for some positive constants C and C^* , where $(p^*)'$ is the conjugate exponent of p^* . Hence, H_p is a well-defined functional on X .

Let u_i ($i = 1, 2$) be two positive weak solutions of (1.1) such that $u_i \in C^1(\overline{\Omega^e})$ with $u_i = 0$ on $\partial\Omega$ pointwise and u_i satisfy (1.3)–(1.4). Then, from (1.3) and (1.4) it follows that

$$\psi_1 := \frac{u_2^p - u_1^p}{u_1^{p-1}} \in X \quad \text{and} \quad \psi_2 := \frac{u_2^p - u_1^p}{u_2^{p-1}} \in X.$$

We set

$$\tilde{X} := \{v: \Omega^e \rightarrow (0, \infty), v^{\frac{1}{p}} \in X\},$$

and we introduce the functional $J: \tilde{X} \rightarrow \mathbb{R}$ defined by

$$J(v) := H_p(v^{\frac{1}{p}}).$$

The real valued function $g: [0, 1] \rightarrow \mathbb{R}$, defined by $g(t) := J((1-t)v_1 + tv_2)$, where $v_i = u_i^p$ ($i = 1, 2$), is strictly convex in $[0, 1]$ (see the arguments in [2] and [5]) by our assumption that $\tilde{f}(s)/s^{p-1}$ is strictly decreasing in $(0, \infty)$. Since u_i are weak solutions of (1.1) and $\psi_i \in X$ ($i = 1, 2$), we can calculate the one sided derivatives $g'_+(0)$ and $g'_-(1)$. In particular, we obtain

$$\begin{aligned} g'_+(0) &= \frac{d}{dt} J((1-t)v_1 + tv_2)|_{t=0} \\ &= \frac{1}{p} \int_{\Omega^e} |\nabla v_1^{\frac{1}{p}}|^{p-2} \nabla v_1^{\frac{1}{p}} \cdot \nabla \left(\frac{v_2 - v_1}{v_1^{\frac{1}{p}}} \right) dx - \frac{1}{p} \int_{\Omega^e} \tilde{f}(v_1^{\frac{1}{p}}) \frac{v_2 - v_1}{v_1^{\frac{1}{p}}} dx \\ &= \frac{1}{p} \int_{\Omega^e} |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla \psi_1 dx - \frac{1}{p} \int_{\Omega^e} \tilde{f}(u_1) \psi_1 dx = 0 \end{aligned}$$

and

$$\begin{aligned} g'_-(1) &= \frac{d}{dt} J((1-t)v_1 + tv_2)|_{t=1} \\ &= \frac{1}{p} \int_{\Omega^e} |\nabla v_2^{\frac{1}{p}}|^{p-2} \nabla v_2^{\frac{1}{p}} \cdot \nabla \left(\frac{v_2 - v_1}{v_2^{\frac{1}{p}}} \right) dx - \frac{1}{p} \int_{\Omega^e} \tilde{f}(v_2^{\frac{1}{p}}) \frac{v_2 - v_1}{v_2^{\frac{1}{p}}} dx \\ &= \frac{1}{p} \int_{\Omega^e} |\nabla u_2|^{p-2} \nabla u_2 \cdot \nabla \psi_2 dx - \frac{1}{p} \int_{\Omega^e} \tilde{f}(u_2) \psi_2 dx = 0. \end{aligned}$$

However, the strictly convex function g cannot have zero one sided derivative at the end points. This in turn implies that $u_1 = u_2$. Therefore, (1.1) has unique positive weak solution $u \in C^1(\overline{\Omega^e})$ with $u = 0$ on $\partial\Omega$ pointwise, and which satisfies (1.3)–(1.4).

5 Proof of Corollary 1.4

It follows from hypothesis (H1) that $\alpha > N + \delta \frac{N-p}{p-1} > N > p$. Hence, taking any $\mu \in (p-1, \alpha-1)$, our assumption $0 < K(x) < \frac{K^*}{|x|^\alpha}$ yields $\int_{r_0}^\infty r^\mu K(r) dr < \infty$. In other words, the assumptions in [14, Theorem 1] are satisfied and hence there exists a positive radial solution $u \in C^1(\overline{\Omega^e})$ with $u = 0$ on $\partial\Omega$ pointwise. Further, there exist sub- and supersolutions $\underline{u}, \bar{u} \in C^1(\overline{\Omega^e})$ (exactly as in Proposition 3.1 but now radial) such that $\underline{u}(r) \leq u(r) \leq \bar{u}(r)$ with both $\underline{u}$ and $\bar{u}$ decaying faster than $1/r^{\frac{N-p}{p-1}}$ as $r \rightarrow \infty$. Now taking $R_0 > r_0$ and repeating the arguments used in the proof of Theorem 1.2 (see the proof of (A)), we can conclude that $u \in X$. Hence, Corollary 1.4 follows from Theorem 1.3.

A Proof of Proposition 2.2

Set $w = u_1 - u_2$ and write $w = w^+ - w^-$. Then, $w^+ \in W^{1,p}(\Omega_{R_0}) \cap L^\infty(\Omega_{R_0})$ and it has zero trace on $\partial\Omega_{R_0}$. Hence, w^+ can be taken as a test function in the weak formulation of both (2.3) and (2.4), yielding

$$\int_{\Omega_{R_0}} |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla w^+ dx - \int_{\Omega_{R_0}} \frac{K(x)}{u_1^\delta} w^+ dx = \int_{\Omega_{R_0}} f_1 w^+ dx, \quad (\text{A.1})$$

$$\int_{\Omega_{R_0}} |\nabla u_2|^{p-2} \nabla u_2 \cdot \nabla w^+ dx - \int_{\Omega_{R_0}} \frac{K(x)}{u_2^\delta} w^+ dx = \int_{\Omega_{R_0}} f_2 w^+ dx. \quad (\text{A.2})$$

Notice that all integrals in (A.1) and (A.2) are finite. More specifically, the first integrals on the left-hand side are finite by Hölder's inequality, the second integrals on the left-hand side and the integrals on the right-hand side are finite due to the decay estimates $u_i(x) > k_i \rho_0(x)$ for $i = 1, 2$, and the fact that $K, w^+ \in L^\infty(\Omega_{R_0})$ and $f_i \in L^1(\Omega_{R_0})$ for $i = 1, 2$. Subtracting (A.2) from (A.1), we have

$$\int_{\Omega_{R_0}} (|\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2) \cdot \nabla w^+ \, dx - \int_{\Omega_{R_0}} \left(\frac{K(x)}{u_1^\delta} - \frac{K(x)}{u_2^\delta} \right) w^+ \, dx = - \int_{\Omega_{R_0}} (f_1 - f_2) w^+ \, dx \leq 0.$$

Let

$$\Omega_{R_0}^+ := \{x \in \Omega_{R_0} : u_1(x) > u_2(x)\} \quad \text{and} \quad \Omega_{R_0}^- := \{x \in \Omega_{R_0} : u_1(x) \leq u_2(x)\}.$$

Notice that $w^+ = u_1 - u_2 > 0$ and $\nabla w^+ = \nabla u_1 - \nabla u_2$ in $\Omega_{R_0}^+$ whereas $w^+ = 0$ and $\nabla w^+ = 0$ in $\Omega_{R_0}^-$, by [12, Lemma 7.6]. Hence,

$$\int_{\Omega_{R_0}^+} (|\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2) \cdot \nabla (u_1 - u_2) \, dx \leq \int_{\Omega_{R_0}^+} \left(\frac{K(x)}{u_1^\delta} - \frac{K(x)}{u_2^\delta} \right) (u_1 - u_2) \, dx. \quad (\text{A.3})$$

Now assume $\text{meas}(\Omega_{R_0}^+) > 0$. Then, on $\Omega_{R_0}^+$ we have $u_1(x) > u_2(x)$, and hence

$$\int_{\Omega_{R_0}^+} \left(\frac{K(x)}{u_1^\delta} - \frac{K(x)}{u_2^\delta} \right) (u_1 - u_2) \, dx < 0. \quad (\text{A.4})$$

At the same time, using reasoning similar to that used in the proof of [8, Proposition 4.1] leads to

$$\int_{\Omega_{R_0}^+} (|\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2) \cdot \nabla (u_1 - u_2) \, dx \geq 0. \quad (\text{A.5})$$

But then (A.4) and (A.5) contradict (A.3), implying $\text{meas}(\Omega_{R_0}^+) = 0$. Therefore, it must be that $u_1 \leq u_2$ on Ω_{R_0} , as desired.

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