Research Article

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A refined bound on the dimension of $\mathbb{R}^N$ for an elliptic system involving critical terms with infinitely many solutions

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Abstract: In this paper, we extend the result of Yan and Yang [16] on equations to an elliptic system involving critical Sobolev and Hardy–Sobolev exponents in bounded domains satisfying some geometric condition. In addition, we weaken the conditions on the dimension N and on the potential $a(x)$ set in [16]. Our main result asserts, by a variational global-compactness argument, that the condition on the dimension N can be refined from $N \geq 7$ to $N > \max(4, \lfloor 2s \rfloor + 2)$, where $0 < s < 2$ and still end up with infinitely many solutions.

Keywords: Elliptic system, infinitely many solutions, critical exponent, variational method

MSC 2010: 35J60, 35B33

1 Introduction and main result

In this paper, we are concerned with the study of existence of infinitely many solutions for the following system:

$$\mathcal{S}_{\alpha, \beta, 2^*(\cdot)}(s, a(\cdot), b(\cdot)) := \begin{cases} -\Delta u = \frac{2\alpha}{2^*} |u|^{\alpha-2} u |v|^\beta + \frac{|u|^{2^*(s)-2} u}{|x|^s} + a(x)u & \text{in } \Omega, \\ -\Delta v = \frac{2\beta}{2^*} |u|^\alpha |v|^{\beta-2} v + \frac{|v|^{2^*(s)-2} v}{|x|^s} + b(x)v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $N \geq 3$, $0 < s < 2$, Ω is an open bounded domain in $\mathbb{R}^N$ whose boundary is in $\mathcal{C}^3$, $0 \in \overline{\Omega}$, and the real valued function $2^*(\cdot)$ is defined on $[0, 2[$ by setting $2^*(t) := \frac{2(N-t)}{(N-2)}$ for $0 \leq t < 2$. Let $\alpha, \beta > 1$ such that $\alpha + \beta = 2^* := 2^*(0)$. We assume that $a(\cdot), b(\cdot)$ are in $C^1(\overline{\Omega})$, and Ω satisfies the geometric condition:

(C) If $0 \in \partial\Omega$, then all the principle curvatures of $\partial\Omega$ at 0 are negative.

When $a(\cdot) = b(\cdot)$ system (1.1) reduces to the scalar semi-linear elliptic problem

$$\mathcal{E}(s, \mu, a(\cdot)) := \begin{cases} -\Delta u = |u|^{2^*-2} u + \mu \frac{|u|^{2^*(s)-2} u}{|x|^s} + a(x)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $\mu = 1$. Devillanova and Solimini in [7] proved that the problem $\mathcal{E}(0, 0, \lambda)$, $\lambda > 0$ admits infinitely many solutions if $N \geq 7$. Afterwards, Liu and Han in [12] generalized the result of [7] to a system without Hardy terms, i.e., for $s = 0$. They showed that $\mathcal{S}_{\alpha, \beta, 2^*(\cdot)}(s, a(\cdot), b(\cdot))$ has infinitely many solutions if $N \geq 7$ for all $\lambda \geq 0$ and $\mu \geq 0$ with $\lambda + \mu > 0$. In [6] Cao and Yan extended the result obtained in [7] for the equation $\mathcal{E}(2, \mu, a)$

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(i.e., when $s = 2$ and $a(\cdot)$ is constant) involving critical Sobolev and Hardy–Sobolev exponents, under the conditions $N \geq 7$, $\mu \in [0, \frac{(N-2)^2}{4} - 4)$, $a > 0$ and $0 \in \Omega$. Later on, Yan and Yang proved in [16] that the problem $\mathcal{E}(s, 1, a(\cdot))$ admits infinitely many solutions if $N \geq 7$, $0 < s < 2$, $\mu > 0$, $0 \in \overline{\Omega}$, $a(x) \in \mathcal{C}^1(\overline{\Omega})$, $a(0) > 0$ and $\Omega \in \mathcal{C}^3$ satisfies condition (C). This kind of problem, that involves a critical inverse power singular potential, may arise in many fields of science such as cosmology [2], molecular physics [10] and the linearization of combustion models [14]. Currently, many research projects are dealing with these problems, especially with the system $\mathcal{S}_{\alpha, \beta, 2^*(\cdot)}(s, a(\cdot), b(\cdot))$. The landscape we have just drawn above is of course far from being complete (see [4, 9, 11, 13, 16]).

One of the motivations of this work is the fact that the result obtained in [16] was given under the condition on N obtained in [7], i.e., $N \geq 7$, and this could be further refined since the equation in [16] (on which system (1.1) was built) introduced an extra Hardy term. We have also found that it is possible to relax the condition on the potential $a(x)$ given in [16] ($a(0) > 0$). The aim of this paper is to prove that $\mathcal{S}_{\alpha, \beta, 2^*(\cdot)}(s, a(\cdot), b(\cdot))$ has infinitely many solutions under the following weaker conditions: $N > \max(4, \lfloor 2s \rfloor + 2)$, where $\lfloor 2s \rfloor$ is the integer part of $2s$, $a(x), b(x) \in \mathcal{C}^1(\overline{\Omega})$ and Ω satisfies the geometric condition (C). Note that if $0 < s < \frac{3}{2}$, we obtain $N > 4$. A similar case of the problem of systems (without Hardy terms and with constant potentials) has been worked out in [12], using a straightforward application of an argument in [7], to get compactness in large dimension ($N \geq 7$) and an application of Bartsch fountain theorem [15, Theorem 3.6] (see also [5, 6, 8]). Perturbing the problem $\mathcal{S}_{\alpha, \beta, 2^*(\cdot)}(s, a(\cdot), b(\cdot))$ into a subcritical one whose corresponding functional satisfies Palais–Smale (PS for short) at all energy levels (denoted by $\mathcal{S}_{\alpha-\varepsilon, \beta-\varepsilon, 2^*(\cdot)-\varepsilon}(s, a(\cdot), b(\cdot))$). Estimates on controlled and balanced sequences (see Definition 2.2 below), global compactness argument and a (local) Pohozaev identity are used to establish the strong convergence. Finally, the use of min-max theorems on a genus homotopic class produces infinitely many critical values.

The paper is organized as follows. In Section 2, notation, some preliminary results and estimates are established for suitable norms of the controlled concentrating sequences. In Section 3, we establish a local Pohozaev identity, which allows the proof of compactness for bounded balanced sequences. In Section 4, we show how this technique allows the application of classical min-max arguments to $\mathcal{S}_{\alpha, \beta, 2^*(\cdot)}(s, a(\cdot), b(\cdot))$ proving, in this way, the existence of infinitely many solutions. Finally, in Appendix A, we prove some auxiliary results. Our main result is the following.

Theorem 1.1. *Let $a(x), b(x) \in \mathcal{C}^1(\overline{\Omega})$ and $0 \in \overline{\Omega}$. Suppose that Ω , whose boundary is in $\mathcal{C}^3$, satisfies condition (C). If $N > \max(4, \lfloor 2s \rfloor + 2)$, then $\mathcal{S}_{\alpha, \beta, 2^*(\cdot)}(s, a(\cdot), b(\cdot))$ has infinitely many solutions.*

Remark 1.2. If $0 < s < \frac{3}{2}$, then $\mathcal{S}_{\alpha, \beta, 2^*(\cdot)}(s, a(\cdot), b(\cdot))$ has infinitely many solutions if $N > 4$.

2 Notation and preliminaries

We denote with X the Sobolev space $H_0^1(\Omega) \times H_0^1(\Omega)$ equipped with its usual norm

$$\|(u, v)\| := \left(\int_{\Omega} (|\nabla u|^2 + |\nabla v|^2) dx \right)^{1/2}.$$

We always denote positive constants (possibly different) by C .

Solutions to problem (1.1) will be obtained as critical points of the corresponding energy functional

$$I(u, v) := \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + |\nabla v|^2 - a(x)u^2 - b(x)v^2) dx - \frac{2}{2^*} \int_{\Omega} |u|^{\alpha} |v|^{\beta} dx - \frac{1}{2^*(s)} \int_{\Omega} \frac{|u|^{2^*(s)} + |v|^{2^*(s)}}{|x|^s} dx.$$

By the Sobolev, Young and Caffarelli–Kohn–Nirenberg inequality [3], we see that the functional I is well defined and is $\mathcal{C}^1$ in X .

Let us point out that in the case $0 \in \partial\Omega$, the principle curvature of $\partial\Omega$ at 0 plays an important role in the existence of the mountain pass solution if $N > 4$, see [3, 8, 9, 11]. One of the major difficulties to prove of the existence of infinitely many solutions to (1.1), by using the variational methods, is that I does not

satisfy the PS condition at every energy level, since problem (1.1) involves both critical Sobolev and Hardy–Sobolev exponents. Following the original idea in [7], employed in a closer setting in [16], we deal first with the following perturbed problem:

$$\mathcal{S}_{\alpha-\varepsilon, \beta-\varepsilon, 2^*(\cdot)-\varepsilon}(s, a(\cdot), b(\cdot)) := \begin{cases} -\Delta u = \frac{2(\alpha-\varepsilon)}{2^*-2\varepsilon} |u|^{\alpha-2-\varepsilon} |v|^{\beta-\varepsilon} + \frac{|u|^{2^*(s)-2-\varepsilon} u}{|x|^s} + a(x)u & \text{in } \Omega, \\ -\Delta v = \frac{2(\beta-\varepsilon)}{2^*-2\varepsilon} |u|^{\alpha-\varepsilon} |v|^{\beta-2-\varepsilon} v + \frac{|v|^{2^*(s)-2-\varepsilon} v}{|x|^s} + b(x)v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.1)$$

where $\varepsilon > 0$ is suitably small, namely $0 < \varepsilon < \min\{\alpha - 1, \beta - 1, 2^*(s) - 1\}$. The functional corresponding to (2.1) is

$$I_\varepsilon(u, v) := \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + |\nabla v|^2 - a(x)u^2 - b(x)v^2) dx - \frac{2}{2^*} \int_{\Omega} |u|^{\alpha-\varepsilon} |v|^{\beta-\varepsilon} dx - \frac{1}{2^*(s)} \int_{\Omega} \frac{|u|^{2^*(s)-\varepsilon} + |v|^{2^*(s)-\varepsilon}}{|x|^s} dx.$$

Now I_ε (is even and) satisfies the Palais–Smale condition at all energy levels. From the symmetric mountain pass lemma [1], it follows that (2.1) has infinitely many solutions, see [9, Theorem 6.1]. More precisely, for any fixed $\varepsilon > 0$ small enough, there are positive numbers $c_{\varepsilon, l}$, $l \in \mathbb{N}$, which are critical values of I_ε , and satisfy $c_{\varepsilon, l} \rightarrow +\infty$ as $l \rightarrow +\infty$. As first step, we show the following crucial lemma.

Lemma 2.1. *Suppose that $a(x), b(x) \in C^1(\bar{\Omega})$ and Ω satisfies condition (C). Let (u, v) be a solution to problem (2.1) for some $\varepsilon > 0$. Then $\tilde{u} + \tilde{v}$ verifies the following inequality:*

$$-\Delta w \leq 2w^{2^*-1} + 2 \frac{w^{2^*(s)-1}}{|x|^s} + \frac{A}{|x|^s} \quad \text{in } \mathbb{R}^N, \quad (2.2)$$

where $\tilde{u} = |u|$, $\tilde{v} = |v|$ in Ω , $\tilde{u} = \tilde{v} = 0$ in $\mathbb{R}^N \setminus \Omega$ and A is a positive constant independent on n .

Proof. Let $\varphi \in H^1(\mathbb{R}^N)$ with $\varphi \geq 0$. By integrating by parts we get

$$\int_{\mathbb{R}^N} \nabla \tilde{u} \nabla \varphi dx = \int_{\Omega} \nabla |u| \nabla \varphi dx = \int_{\partial\Omega} \varphi \frac{\partial |u|}{\partial n} d\sigma - \int_{\Omega} |u|^{-1} u \Delta u \varphi dx.$$

By using φ as test function in the first equation of (2.1), we have

$$\begin{aligned} \int_{\mathbb{R}^N} \nabla \tilde{u} \nabla \varphi dx &\leq \int_{\Omega} \left[\frac{2(\alpha-\varepsilon)}{2^*-2\varepsilon} |u|^{\alpha-2-\varepsilon} |v|^{\beta-\varepsilon} \varphi + \frac{|u|^{2^*(s)-2-\varepsilon} u}{|x|^s} + a(x)u \right] |u|^{-1} u \varphi dx \\ &= \int_{\Omega} \left[\frac{2(\alpha-\varepsilon)}{2^*-2\varepsilon} |u|^{\alpha-1-\varepsilon} |v|^{\beta-\varepsilon} + \frac{|u|^{2^*(s)-1-\varepsilon}}{|x|^s} + a(x)|u| \right] \varphi dx \\ &\leq \int_{\mathbb{R}^N} \left[\frac{2(\alpha-\varepsilon)}{2^*-2\varepsilon} (\tilde{u} + \tilde{v})^{2^*-1-2\varepsilon} + \frac{|\tilde{u}|^{2^*(s)-1-\varepsilon}}{|x|^s} + a(x)\tilde{u} \right] \varphi dx. \end{aligned}$$

This implies that

$$-\Delta \tilde{u} \leq \frac{2(\alpha-\varepsilon)}{2^*-2\varepsilon} (\tilde{u} + \tilde{v})^{2^*-1-2\varepsilon} + \frac{(\tilde{u})^{2^*(s)-1-\varepsilon}}{|x|^s} + a(x)\tilde{u}.$$

Analogously, we obtain

$$-\Delta \tilde{v} \leq \frac{2(\beta-\varepsilon)}{2^*-2\varepsilon} (\tilde{u} + \tilde{v})^{2^*-1-2\varepsilon} + \frac{(\tilde{v})^{2^*(s)-1-\varepsilon}}{|x|^s} + b(x)\tilde{v}.$$

Since $\alpha + \beta = 2^*$, we conclude that

$$-\Delta (\tilde{u} + \tilde{v}) \leq 2c(\tilde{u} + \tilde{v})^{2^*-1} + 2c \frac{(\tilde{u} + \tilde{v})^{2^*(s)-1}}{|x|^s} + \frac{A}{|x|^s}$$

for sufficiently large constant $A > 0$, independent of ε , and $c > 1$. We can normalize c and always take $c = 1$. This completes the proof. $\square$

Definition 2.2. Let $(u_n, v_n)_{n \geq 1}$ be a given sequence and $(\varepsilon_n)_{n \geq 1}$ a sequence of positive numbers. We say, related to $(\varepsilon_n)_{n \geq 1}$, that:

- $(u_n, v_n)_{n \geq 1}$ is a controlled sequence if for any $n \geq 1$, $\tilde{u}_n + \tilde{v}_n$ is a solution to inequality (2.2),
- $(u_n, v_n)_{n \geq 1}$ is a balanced sequence if for any $n \geq 1$, (u_n, v_n) is a solution of problem (2.1).

In the remainder of the paper, we shall briefly say that $(u_n, v_n)_{n \geq 1}$ is a balanced or controlled sequence with $\varepsilon_n \rightarrow 0$ if $(u_n, v_n)_{n \geq 1}$ is a balanced or controlled sequence related to $(\varepsilon_n)_{n \geq 1}$ and $\varepsilon_n \rightarrow 0$.

Remark 2.3. According to Definition 2.2, Lemma 2.1 states that every balanced sequence is a controlled sequence, and thus $(\tilde{u}_n, \tilde{v}_n)_{n \geq 1}$ is also a controlled sequence.

We need the following notation. For any $\lambda > 0$ and $x \in \mathbb{R}^N$, we write

$$\rho_{x,\lambda}(u, v) = \lambda^{(N-2)/2} (u(\lambda(\cdot - x)), v(\lambda(\cdot - x))), \quad (u, v) \in X.$$

Let $(u_n, v_n)_{n \geq 1}$ be a noncompact bounded balanced sequence. By Proposition A.2 in the Appendix, there exist $m, k \in \mathbb{N}$ ($1 \leq m \leq k$) independent on n , $\lambda_{n,j} \subset (0, +\infty)$, $j = 1, 2, \dots, k$, with $\lambda_{n,j} \rightarrow +\infty$ as $n \rightarrow +\infty$, $x_{n,j} \in \Omega$ with $\lambda_{n,j} d(x_{n,j}, \partial\Omega) \rightarrow +\infty$, $j = m+1, \dots, k$, and nontrivial solutions $(U_1, V_1), \dots, (U_m, V_m)$ of problem (A.2) and nontrivial solutions $(U_{m+1}, V_{m+1}), \dots, (U_k, V_k)$ of problem (A.1) such that a modulo subsequence exists. Each term (u_n, v_n) of this subsequence can be decomposed as follows:

$$(u_n, v_n) = (u_0, v_0) + \sum_{j=1}^m \rho_{0,\lambda_{n,j}}(U_j, V_j) + \sum_{j=m+1}^k \rho_{x_{n,j},\lambda_{n,j}}(U_j, V_j) + (\sigma_n^1, \sigma_n^2), \quad (2.3)$$

where (u_0, v_0) (the weak limit of $(u_n, v_n)_{n \geq 1}$) is a solution of (1.1), $\|(\sigma_n^1, \sigma_n^2)\| \rightarrow 0$,

$$\frac{\lambda_{n,j}}{\lambda_{n,i}} + \frac{\lambda_{n,i}}{\lambda_{n,j}} + \lambda_{n,i} \lambda_{n,j} |x_{n,i} - x_{n,j}|^2 \rightarrow +\infty \quad \text{for any } i \neq j \quad (2.4)$$

and $\rho_{0,\lambda_{n,j}}(U_j, V_j), \rho_{x_{n,j},\lambda_{n,j}}(U_j, V_j)$ concentrate (i.e., the scale parameter $\lambda_{n,j} \rightarrow +\infty$) as $n \rightarrow +\infty$ (here $x_{n,j} = 0$ if $j = 1, \dots, m$).

We call bubbles the terms $\rho_{0,\lambda_{n,j}}(U_j, V_j)$ and $\rho_{x_{n,j},\lambda_{n,j}}(U_j, V_j)$ which appear in (2.3), that, is in the decomposition of the modulo subsequence $(u_n, v_n)_{n \geq 1}$. As in [5–7], we shall state some facts which are essentials to prove the strong convergence of $(u_n, v_n)_{n \geq 1}$ in X . Among all the bubbles in (2.3), we can choose one which has a “slowest” concentration rate, denoted by λ_n , which concentrates in its corresponding concentration point x_n at a slowest rate. That is, the corresponding λ is the lowest order infinity among the ones appearing in the bubbles. Since the number of the bubbles in each (u_n, v_n) is finite, and independent of n , we can always choose a constant $\bar{C} > 0$, independent of n , such that the region

$$\mathcal{A}_n^1 := (B_{(\bar{C}+5)\lambda_n^{-1/2}}(x_n) \setminus B_{\bar{C}\lambda_n^{-1/2}}(x_n)) \cap \Omega$$

does not contain any concentration point of (u_n, v_n) for every n . (Though the set $\mathcal{A}_n^1$ is not used in the following, it is useful to get a pointwise estimate as in [7, Proposition 3.2]). We set

$$\mathcal{A}_n^2 := (B_{(\bar{C}+4)\lambda_n^{-1/2}}(x_n) \setminus B_{(\bar{C}+1)\lambda_n^{-1/2}}(x_n)) \cap \Omega,$$

$$\mathcal{A}_n^3 := (B_{(\bar{C}+3)\lambda_n^{-1/2}}(x_n) \setminus B_{(\bar{C}+2)\lambda_n^{-1/2}}(x_n)) \cap \Omega.$$

We may call the regions $\mathcal{A}_n^1, \mathcal{A}_n^2$ and $\mathcal{A}_n^3$ safe regions.

Definition 2.4. We say that $(u_n, v_n)_{n \geq 1}$ is a concentrating sequence if the limit in (2.3) holds.

For any $p_2 < 2^* < p_1$, $\alpha > 0$ and $\lambda > 0$, we consider the system of inequalities

$$\begin{cases} \|u_1\|_{*,p_1} \leq \alpha \\ \|u_2\|_{*,p_2} \leq \alpha \lambda^{N/2^* - N/p_2}, \end{cases} \quad (2.5)$$

where for a fixed $s \in (0, 2)$

$$\|u\|_{*,p} := \left(\int_{\Omega} |u|^p dx \right)^{1/p} + \left(\int_{\Omega} \frac{|u|^{p2^*(s)/2^*}}{|x|^s} dx \right)^{2^*/(p2^*(s))}.$$

This will let us define the following norm depending on p_1, p_2 and λ , as in [6, Section 2]:

$$\|u\|_{*,p_1,p_2,\lambda} := \inf\{\alpha > 0 : \text{there exist } u_1, u_2 \text{ such that (2.5) holds and } |u| \leq u_1 + u_2\}.$$

We have the following lemma.

Lemma 2.5. *Under the hypotheses of Lemma 2.1, let $(u_n, v_n)_{n \geq 1}$ be a controlled concentrating sequence, and λ_n being (one of) the slowest concentration rate of $(u_n, v_n)_{n \geq 1}$ in X (see the paragraph after formula (2.4)). Then, for any p_1, p_2 satisfying the inequality $\frac{2^*}{2} < p_2 < 2^* < p_1$, there exists a constant $C > 0$, depending only on p_1 and p_2 , such that*

$$\| |u_n| + |v_n| \|_{*,p_1,p_2,\lambda_n} \leq C. \quad (2.6)$$

Moreover, there exists a constant $C > 0$, independent of n , such that for all $p \geq 2$,

$$\int_{\mathcal{A}_n^2} (|u_n(x)|^p + |v_n(x)|^p) dx \leq C \lambda_n^{-N/2}, \quad (2.7)$$

$$\int_{\mathcal{A}_n^3} \frac{|u_n(x)|^{2^*(s)} + |v_n(x)|^{2^*(s)}}{|x|^s} dx \leq C \lambda_n^{-(N/2)2^*(s)/2^*} \quad (2.8)$$

and

$$\int_{\mathcal{A}_n^3} (|\nabla u_n|^2 + |\nabla v_n|^2) dx \leq C \lambda_n^{-N/2^*}. \quad (2.9)$$

Proof. We apply the estimates given by [16, Propositions 2.1, 3.1 and 3.3] to get, respectively, the estimates (2.6), (2.7) and (2.9), while (2.8) follows by applying the same argument employed to get the proof of [16, formula (3.10)]. $\square$

3 Compactness results for balanced sequences

Proposition 3.1. *Let $a(x), b(x) \in C^1(\overline{\Omega})$ and $0 \in \overline{\Omega}$. Suppose that Ω , whose boundary is $\mathcal{O}^3$, satisfies the condition (C). Let $(u_n, v_n)_{n \geq 1}$ be a balanced sequence bounded in X with $\varepsilon_n \rightarrow 0$. Then $(u_n, v_n)_{n \geq 1}$ is relatively compact for $N > \max(4, \lfloor 2s \rfloor + 2)$.*

Proof. We argue by contradiction and we assume that $(u_n, v_n)_{n \geq 1}$ is not compact. Then, by Proposition A.2 in the Appendix, we have the decomposition of $(u_n, v_n)_{n \geq 1}$ modulo subsequence given in formula (2.3), and by Lemma 2.5, inequalities (2.6), (2.7), (2.8) and (2.9) hold. As in proof of [16, the main result], we take $t_n \in [\overline{C} + 2, \overline{C} + 3]$ (here $\overline{C}$ is that which appears in the definition of the safe regions $\mathcal{A}_n^i$, where $i = 1, 2, 3$). Recall also that in Lemma 2.1, the involved sequences were extended to zero out of Ω such that

$$\begin{aligned} & \int_{\partial B_{t_n \lambda_n^{-1/2}}(x_n)} \left(|u_n|^{\alpha-\varepsilon_n} |v_n|^{\beta-\varepsilon_n} + |u_n|^2 + |v_n|^2 + \lambda_n^{-1} (|\nabla u_n|^2 + |\nabla v_n|^2) + \lambda_n^{-s/2} \frac{|u_n|^{2^*(s)-\varepsilon_n} + |v_n|^{2^*(s)-\varepsilon_n}}{|x|^s} \right) d\sigma \\ & \leq C \lambda_n^{1/2} \int_{\mathcal{A}_n^3} \left(|u_n|^{\alpha-\varepsilon_n} |v_n|^{\beta-\varepsilon_n} + |u_n|^2 + |v_n|^2 + \lambda_n^{-1} (|\nabla u_n|^2 + |\nabla v_n|^2) \right. \\ & \quad \left. + \lambda_n^{-s/2} \frac{|u_n|^{2^*(s)-\varepsilon_n} + |v_n|^{2^*(s)-\varepsilon_n}}{|x|^s} \right) dx, \end{aligned} \quad (3.1)$$

where C is independent of n and not necessarily the same as the one in the above statement. Using Young's inequality, (2.7), (2.8), (2.9) and (3.1), we obtain

$$\begin{aligned} & \int_{\partial B_{t_n \lambda_n^{-1/2}}(x_n)} \left(|u_n|^{\alpha-\varepsilon_n} |v_n|^{\beta-\varepsilon_n} + |u_n|^2 + |v_n|^2 + \lambda_n^{-1} (|\nabla u_n|^2 + |\nabla v_n|^2) \right. \\ & \quad \left. + \lambda_n^{-s/2} \frac{|u_n|^{2^*(s)-\varepsilon_n} + |v_n|^{2^*(s)-\varepsilon_n}}{|x|^s} \right) d\sigma \leq C \lambda_n^{(1-N)/2}. \end{aligned} \quad (3.2)$$

Let

$$F(u_n, v_n, x, x_0, v) := \left[\frac{2}{2^* - 2\varepsilon_n} |u_n|^{\alpha - \varepsilon_n} |v_n|^{\beta - \varepsilon_n} + \frac{1}{2} (a(x)u_n^2 + b(x)v_n^2) + \frac{1}{2^*(s) - \varepsilon_n} \frac{|u_n|^{2^*(s) - \varepsilon_n} + |v_n|^{2^*(s) - \varepsilon_n}}{|x|^s} \right] \\ \times (x - x_0) \cdot v + (\nabla u_n \cdot v)(\nabla u_n \cdot (x - x_0)) + (\nabla v_n \cdot v)(\nabla v_n \cdot (x - x_0)) \\ + \frac{N-2}{2} ((\nabla u_n \cdot v)u_n + (\nabla v_n \cdot v)v_n) - \frac{1}{2} (|\nabla u_n|^2 + |\nabla v_n|^2)((x - x_0) \cdot v). \quad (3.3)$$

We establish the Pohozaev identity for (u_n, v_n) on $B_n := B_{t_n \lambda_n^{-1/2}}(x_n) \cap \Omega$ by employing the classical strategy of testing the equation against the test functions $(\nabla u_n \cdot (x - x_0))$ and $(\nabla v_n \cdot (x - x_0))$, and then using divergence theorems. We finally get the following:

$$\frac{4\varepsilon_n N}{2^* - 2\varepsilon_n} \int_{B_n} |u_n|^{\alpha - \varepsilon_n} |v_n|^{\beta - \varepsilon_n} dx + \int_{B_n} \left[\left(a(x) - \frac{1}{2} \nabla a \cdot (x - x_0) \right) u_n^2 + \left(b(x) - \frac{1}{2} \nabla b \cdot (x - x_0) \right) v_n^2 \right] dx \\ + \frac{N}{2^*} \left(\frac{2^*(s)}{2^*(s) - \varepsilon_n} - 1 \right) \int_{B_n} \frac{|u_n|^{2^*(s) - \varepsilon_n} + |v_n|^{2^*(s) - \varepsilon_n}}{|x|^s} dx + \frac{s}{2^*(s) - \varepsilon_n} \int_{B_n} \frac{|u_n|^{2^*(s) - \varepsilon_n} + |v_n|^{2^*(s) - \varepsilon_n}}{|x|^{s+2}} (x_0 \cdot x) dx \\ = \int_{\partial B_n} F(u_n, v_n, x, x_0, v) d\sigma,$$

where v is the outward unit normal to ∂B_n and x_0 is a point in $\mathbb{R}^N$. Since for n large enough $\frac{4\varepsilon_n N}{2^* - 2\varepsilon_n} > 0$, we obtain

$$\int_{B_n} \left[\left(a(x) - \frac{1}{2} \nabla a(x) \cdot (x - x_0) \right) u_n^2 + \left(b(x) - \frac{1}{2} \nabla b(x) \cdot (x - x_0) \right) v_n^2 \right] dx \\ + \frac{N}{2^*} \left(\frac{2^*(s)}{2^*(s) - \varepsilon_n} - 1 \right) \int_{B_n} \frac{|u_n|^{2^*(s) - \varepsilon_n} + |v_n|^{2^*(s) - \varepsilon_n}}{|x|^s} dx + \frac{s}{2^*(s) - \varepsilon_n} \int_{B_n} \frac{|u_n|^{2^*(s) - \varepsilon_n} + |v_n|^{2^*(s) - \varepsilon_n}}{|x|^{s+2}} (x_0 \cdot x) dx \\ \leq \int_{\partial B_n} F(u_n, v_n, x, x_0, v) d\sigma. \quad (3.4)$$

We have to consider the following two cases:

- (i) $x_n \rightarrow 0$,
- (ii) $x_n \rightarrow x^* \neq 0$.

Case (i): By setting $x_0 = 0$ in (3.4), we obtain

$$\int_{B_n} \left[\left(a(x) - \frac{1}{2} \nabla a(x) \cdot x \right) u_n^2 + \left(b(x) - \frac{1}{2} \nabla b(x) \cdot x \right) v_n^2 \right] dx + \frac{N}{2^*} \left(\frac{2^*(s)}{2^*(s) - \varepsilon_n} - 1 \right) \int_{B_n} \frac{|u_n|^{2^*(s) - \varepsilon_n} + |v_n|^{2^*(s) - \varepsilon_n}}{|x|^s} dx \\ \leq \int_{\partial B_n} F(u_n, v_n, x, 0, v) d\sigma. \quad (3.5)$$

We decompose

$$\partial B_n = \partial_i B_n \cup \partial_e B_n, \quad \text{where } \partial_i B_n := \partial B_n \cap \Omega \text{ and } \partial_e B_n := \partial B_n \cap \partial \Omega.$$

Let us consider the case $0 \in \partial \Omega$ first. We have $u_n = 0$ and $v_n = 0$, and so it follows that on $\partial \Omega \supset \partial_e B_n$, ∇u_n , ∇v_n and v have the same direction. Then expression (3.3) becomes

$$\int_{\partial B_n} F(u_n, v_n, x, 0, v) d\sigma = \int_{\partial_i B_n} F(u_n, v_n, x, 0, v) d\sigma + \frac{1}{2} \int_{\partial_e B_n} (|\nabla u_n|^2 + |\nabla v_n|^2)(x \cdot v) d\sigma, \quad (3.6)$$

and since $a(x), b(x) \in C^1(\overline{\Omega})$, there exists $C > 0$ such that

$$\left| \int_{B_n} \left[\left(a(x) - \frac{1}{2} \nabla a(x) \cdot x \right) u_n^2 + \left(b(x) - \frac{1}{2} \nabla b(x) \cdot x \right) v_n^2 \right] dx \right| \leq C \int_{B_n} (u_n^2 + v_n^2) dx. \quad (3.7)$$

By combining (3.5), (3.6) and (3.7), we obtain

$$\begin{aligned} & \frac{N}{2^*} \left(\frac{2^*(s)}{2^*(s) - \varepsilon_n} - 1 \right) \int_{B_n} \frac{|u_n|^{2^*(s) - \varepsilon_n} + |v_n|^{2^*(s) - \varepsilon_n}}{|x|^s} dx - \frac{1}{2} \int_{\partial_e B_n} (|\nabla u_n|^2 + |\nabla v_n|^2)(x \cdot \nu) d\sigma \\ & \leq C \int_{B_n} (|u_n|^2 + |v_n|^2) dx + \int_{\partial_i B_n} F(u_n, v_n, x, 0, \nu) d\sigma. \end{aligned} \quad (3.8)$$

Since we suppose that $\Omega \in \mathcal{C}^3$ and $0 \in \partial\Omega$, then up to a rotation, we can find a small constant $\delta > 0$ and a function $\phi \in \mathcal{C}^3$ such that

$$\Omega \cap B_\delta(0) = \{x : x_N > \phi(x')\}, \quad \partial\Omega \cap B_\delta(0) = \{x : x_N = \phi(x')\}.$$

Here we take $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$, where $\phi(x')$ has the following expansion:

$$\phi(x') = \sum_{j=1}^{N-1} a_j x_j^2 + O(|x'|^3) \quad \text{for } |x'| \text{ small enough}$$

with $a_j < 0$, $j = 1, \dots, N-1$. This follows from condition (C). Then the outward unit normal ν of $\partial\Omega$ is given by

$$\nu(x) = \frac{(D\phi(x'), -1)}{\sqrt{1 + |D\phi(x')|^2}} \quad \text{for } x \in \partial\Omega \cap B_\delta(0).$$

Thus,

$$x \cdot \nu = \sum_{j=1}^{N-1} a_j x_j^2 + O(|x'|^3) \quad \text{for } x \in \partial\Omega \cap B_\delta(0),$$

and since each $a_j < 0$, we have

$$(x \cdot \nu) \leq 0 \quad \text{for } x \in \partial\Omega \cap B_\delta(0).$$

Then, since $x_n \rightarrow 0$ and $t_n \lambda_n^{-1/2} \rightarrow 0$,

$$\int_{\partial_e B_n} (|\nabla u_n|^2 + |\nabla v_n|^2)(x \cdot \nu) d\sigma \leq 0. \quad (3.9)$$

Since we have assumed $0 \in \partial\Omega$, there exists $C > 0$ such that for all $x \in \Omega$, $|x|^s \leq C^{-1}$. By (3.8), (3.9), and since $\frac{N}{2^*} \left(\frac{2^*(s)}{2^*(s) - \varepsilon_n} - 1 \right) > 0$, we have

$$\frac{N}{2^*} \left(\frac{2^*(s)}{2^*(s) - \varepsilon_n} - 1 \right) C \int_{B_n} (|u_n|^{2^*(s) - \varepsilon_n} + |v_n|^{2^*(s) - \varepsilon_n}) dx \leq C \int_{B_n} (|u_n|^2 + |v_n|^2) dx + \int_{\partial_i B_n} F(u_n, v_n, x, 0, \nu) d\sigma. \quad (3.10)$$

For all $x \in \partial B_n$, we have $|x| \leq C \lambda_n^{-1/2}$ and, by using the expression of $F(u_n, v_n, x, 0, \nu)$ given in (3.3) and the inequality (3.2), we obtain

$$\begin{aligned} \int_{\partial_i B_n} F(u_n, v_n, x, 0, \nu) d\sigma & \leq C \lambda_n^{-1/2} \int_{\partial_i B_n} \left[|u_n|^{\alpha - \varepsilon_n} |v_n|^{\beta - \varepsilon_n} + |u_n|^2 + |v_n|^2 + |\nabla u_n|^2 + |\nabla v_n|^2 \right. \\ & \quad \left. + \frac{|u_n|^{2^*(s) - \varepsilon_n} + |v_n|^{2^*(s) - \varepsilon_n}}{|x|^s} \right] d\sigma + C \int_{\partial_i B_n} |\nabla u_n| |u_n| + |\nabla v_n| |v_n| d\sigma \\ & \leq C \lambda_n^{(2-N)/2}. \end{aligned}$$

Set $B'_n = B_{\lambda_n^{-1}}(x_n) \cap \Omega$. Recall that by (2.3), we have the decomposition

$$(u_n, v_n) = (u_0, v_0) + (u_{n,1}, v_{n,1}) + (u_{n,2}, v_{n,2}),$$

where

$$(u_{n,1}, v_{n,1}) = \sum_{j=1}^m \rho_{0, \lambda_{n,j}}(U_j, V_j) + \sum_{j=m+1}^k \rho_{x_{n,j}, \lambda_{n,j}}(U_j, V_j),$$

and

$$(u_{n,2}, v_{n,2}) = (u_n - u_0 - u_{n,1}, v_n - v_0 - v_{n,1})$$

with $\|(u_{n,2}, v_{n,2})\| \rightarrow 0$ as $n \rightarrow \infty$. Then, since for n large enough $B'_n \subset B_n$,

$$\begin{aligned} \int_{B_n} (|u_n|^{2^*(s)-\varepsilon_n} + |v_n|^{2^*(s)-\varepsilon_n}) &\geq \int_{B'_n} (|u_n|^{2^*(s)-\varepsilon_n} + |v_n|^{2^*(s)-\varepsilon_n}) \\ &\geq \int_{B'_n} \left[\frac{1}{2} (|u_{n,1}|^{2^*(s)-\varepsilon_n} + |v_{n,1}|^{2^*(s)-\varepsilon_n}) \right. \\ &\quad \left. - 2(|u_0|^{2^*(s)-\varepsilon_n} + |v_0|^{2^*(s)-\varepsilon_n} + |u_{n,2}|^{2^*(s)-\varepsilon_n} + |v_{n,2}|^{2^*(s)-\varepsilon_n}) \right]. \end{aligned} \quad (3.11)$$

Furthermore, by Hölder's inequality,

$$\begin{aligned} \int_{B'_n} (|u_0|^{2^*(s)-\varepsilon_n} + |v_0|^{2^*(s)-\varepsilon_n}) &\leq C \left[\left(\int_{B'_n} |u_0|^{2^*} \right)^{(2^*(s)-\varepsilon_n)/2^*} + \left(\int_{B'_n} |v_0|^{2^*} \right)^{(2^*(s)-\varepsilon_n)/2^*} \right] \lambda_n^{-N(1-[2^*(s)-\varepsilon_n]/2^*)} \\ &\leq C \|(u_0, v_0)\| \lambda_n^{-N(1-[2^*(s)-\varepsilon_n]/2^*)}, \end{aligned} \quad (3.12)$$

$$\begin{aligned} \int_{B'_n} (|u_{n,2}|^{2^*(s)-\varepsilon_n} + |v_{n,2}|^{2^*(s)-\varepsilon_n}) &\leq C \left[\left(\int_{B'_n} |u_{n,2}|^{2^*} \right)^{(2^*(s)-\varepsilon_n)/2^*} + \left(\int_{B'_n} |v_{n,2}|^{2^*} \right)^{(2^*(s)-\varepsilon_n)/2^*} \right] \lambda_n^{-N(1-[2^*(s)-\varepsilon_n]/2^*)} \\ &\leq o(1) \lambda_n^{-N(1-[2^*(s)-\varepsilon_n]/2^*)}. \end{aligned} \quad (3.13)$$

By (2.4), the definition of $(u_{n,1}, v_{n,1})$ and the fact that the λ_n is the slowest concentration rate of blow-up, there exists a constant $C > 0$ such that

$$\int_{B'_n} (|u_{n,1}|^{2^*(s)-\varepsilon_n} + |v_{n,1}|^{2^*(s)-\varepsilon_n}) \geq C \lambda_n^{-N(1-[2^*(s)-\varepsilon_n]/2^*)}. \quad (3.14)$$

Then (3.11)–(3.14) imply that

$$\int_{B_n} (|u_n|^{2^*(s)-\varepsilon_n} + |v_n|^{2^*(s)-\varepsilon_n}) \geq C \lambda_n^{-N(1-[2^*(s)-\varepsilon_n]/2^*)}. \quad (3.15)$$

By applying (2.6) with $p_1 > \frac{2N}{N-4} > 2^*$ and $p_2 = 2 > \frac{2^*}{2}$ (since $N > 4$), there exist $v_1 \geq 0$ and $v_2 \geq 0$ such that $|u_n| + |v_n| \leq v_1 + v_2$ and

$$\|v_1\|_{*,p_1} \leq C, \quad \|v_2\|_{*,2} \leq C \lambda_n^{(N-2)/2-N/2} = C \lambda_n^{-1}.$$

By applying Hölder's inequality,

$$\int_{B_n} (|u_n|^2 + |v_n|^2) dx \leq 2 \int_{B_n} |v_1|^2 + 2 \int_{B_n} |v_2|^2 \leq C \lambda_n^{-N/2(1-2/p_1)} + C \lambda_n^{-2} \leq C \lambda_n^{-2}.$$

Then

$$\int_{B_n} (|u_n|^2 + |v_n|^2) dx \leq C \lambda_n^{-2}. \quad (3.16)$$

From (3.10), (3.15) and (3.16), we deduce

$$\lambda_n^{-N(1-[2^*(s)-\varepsilon_n]/2^*)} \leq C(\lambda_n^{(2-N)/2} + \lambda_n^{-2}).$$

Since $\lambda_n \rightarrow +\infty$ and $\varepsilon_n \rightarrow 0$ as $n \rightarrow +\infty$, the above inequality can not be verified under the assumption that

$$N > \max(4, \lfloor 2s \rfloor + 2).$$

We now consider the case $0 \in \Omega$. In such a case, for large enough n , $\partial_e B_n = \emptyset$, and so (3.10) holds, as well as $\partial_i B_n = \partial B_n$. Arguing as above, we get a contradiction when $N > \max(4, \lfloor 2s \rfloor + 2)$. Thus, Proposition 3.1 is proved for case (i).

Case (ii): We have two possibilities: either $B_{t_n\lambda_n^{-1/2}}(x_n) \subset\subset \Omega$ or $B_{t_n\lambda_n^{-1/2}}(x_n) \cap (\mathbb{R}^N \setminus \Omega) \neq \emptyset$.

Assume $B_{t_n\lambda_n^{-1/2}}(x_n) \subset\subset \Omega$. Then $B_n = B_{t_n\lambda_n^{-1/2}}(x_n)$. We take $x_0 = x_n$ in (3.4) and since $\frac{N}{2^*}(\frac{2^*(s)}{2^*(s)-\varepsilon_n} - 1) > 0$, we obtain

$$\frac{s}{2^*(s) - \varepsilon_n} \int_{B_n} \frac{|u_n|^{2^*(s)-\varepsilon_n} + |v_n|^{2^*(s)-\varepsilon_n}}{|x|^{s+2}} (x_n \cdot x) dx \leq C \int_{B_n} (|u_n|^2 + |v_n|^2) dx + \int_{\partial B_n} F(u_n, v_n, x, x_0, v) d\sigma. \quad (3.17)$$

Note that $(x_n \cdot x) \geq \frac{1}{2}|x_n| \geq \frac{1}{4}|x^*| > 0$ in B_n , since $x^* = \lim_{n \rightarrow \infty} x_n \neq 0$. In this case by (3.17),

$$C \int_{B_n} (|u_n|^{2^*(s)-\varepsilon_n} + |v_n|^{2^*(s)-\varepsilon_n}) dx \leq C \int_{B_n} (|u_n|^2 + |v_n|^2) dx + \int_{\partial B_n} F(u_n, v_n, x, x_0, v) d\sigma.$$

By arguing as in case (i), we get

$$C\lambda_n^{-N(1-[2^*(s)-\varepsilon_n]/2^*)} \leq C\lambda_n^{(2-N)/2} + C\lambda_n^{-2}, \quad (3.18)$$

which is impossible since $N > \max(4, [2s] + 2)$.

It remains to consider the case $B_{t_n\lambda_n^{-1/2}}(x_n) \cap (\mathbb{R}^N \setminus \Omega) \neq \emptyset$. In this case, we choose $x_0 \in \mathbb{R}^N \setminus \Omega$ such that $|x_n - x_0| \leq 2t_n\lambda_n^{-1/2}$ and $v \cdot (x - x_0) \leq 0$ for all $x \in \partial\Omega \cap B_n$, and then from (3.4) we get

$$\frac{s}{2^*(s) - \varepsilon_n} \int_{B_n} \frac{|u_n|^{2^*(s)-\varepsilon_n} + |v_n|^{2^*(s)-\varepsilon_n}}{|x|^{s+2}} (x_0 \cdot x) dx \leq C \int_{B_n} (|u_n|^2 + |v_n|^2) dx + \int_{\partial B_n} F(u_n, v_n, x, x_n, v) d\sigma.$$

Note that $(x_0 \cdot x) \geq \frac{1}{4}|x^*| > 0$ in B_n , so (3.18) still holds, and we get a contradiction. The proof of Proposition 3.1 is thus completed. $\square$

4 Proof of Theorem 1.1

For any positive integer k , define the Z_2 -homotopy class $\mathcal{F}_k$ as follows:

$$\mathcal{F}_k := \{A : A \subset X \text{ is compact, } Z_2\text{-invariant and } \gamma(A) \geq k\},$$

where the genus $\gamma(A)$ is the smallest integer m such that there exists an odd map $\phi \in \mathcal{C}(A, \mathbb{R}^m \setminus \{0\})$. For $k = 1, 2, \dots$, we define the min-max value (see [8, p. 134])

$$c_{k,\varepsilon} = \min_{A \in \mathcal{F}_k} \max_{(u,v) \in A} I_\varepsilon(u, v).$$

From [8, Corollary 7.12] it follows that for each small $\varepsilon > 0$, $c_{k,\varepsilon}$ is a critical value of I_ε , since I_ε satisfies the Palais–Smale condition. Thus, problem $\mathcal{S}_{\alpha-\varepsilon, \beta-\varepsilon, 2^*(\cdot)-\varepsilon}(s, a(\cdot), b(\cdot))$ has a solution $(u_{k,\varepsilon}, v_{k,\varepsilon})$ such that $I_\varepsilon(u_{k,\varepsilon}, v_{k,\varepsilon}) = c_{k,\varepsilon}$.

In order to show that $c_{k,\varepsilon}$ is uniformly bounded for suitably small ε , we fix $\alpha_0 > 1$, $\beta_0 > 1$ and $\sigma_0 > 0$ such that $\varepsilon_0 = \min\{\alpha - \alpha_0, \beta - \beta_0, 2^*(s) - \sigma_0\} > 0$. Let $0 < \varepsilon < \varepsilon_0$. Then it is easy to verify that for all $x, y \in \mathbb{R}$,

$$\begin{cases} |x|^{\alpha_0}|y|^{\beta_0} - 1 \leq |x|^{\alpha-\varepsilon}|y|^{\beta-\varepsilon} + |x|^{\alpha_0} + |y|^{\beta_0}, \\ |x|^{\sigma_0} - 1 \leq |x|^{2^*(s)-\varepsilon}. \end{cases}$$

Let us consider the following auxiliary functional:

$$\begin{aligned} I_0(u, v) := & \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + |\nabla v|^2 - a(x)|u|^2 - b(x)|v|^2) dx + \frac{2}{2^*} \int_{\Omega} |u|^{\alpha_0} + |v|^{\beta_0} dx \\ & - \frac{2}{2^*} \int_{\Omega} |u|^{\alpha_0}|v|^{\beta_0} dx - \frac{1}{2^*(s)} \int_{\Omega} \frac{|u|^{\sigma_0} + |v|^{\sigma_0}}{|x|^s} dx. \end{aligned}$$

In the sequel, for all $0 < \varepsilon < \varepsilon_0$, we have

$$I_\varepsilon(u, v) \leq I_0(u, v) + c_0,$$

where $c_0 = \frac{4}{2^*}|\Omega|$. For $k = 1, 2, \dots$, we set the minimax value

$$a_k = \min_{A \in \mathcal{F}_k} \max_{(u,v) \in A} I_0(u, v).$$

By the definitions of $c_{k,\varepsilon}$ and a_k , we have

$$c_{k,\varepsilon} \leq a_k + c_0.$$

Thus, for k fixed, $c_{k,\varepsilon}$ is uniformly bounded for $0 < \varepsilon < \varepsilon_0$. By a direct calculation, we find $\|(u_{k,\varepsilon}, v_{k,\varepsilon})\| \leq C$ uniformly with respect to ε for each fixed k . Fixing an infinitesimal sequence of real positive numbers $(\varepsilon_n)_{n \geq 1}$, we get that the corresponding sequence $(u_{k,\varepsilon_n}, v_{k,\varepsilon_n})_{n \geq 1}$ is such that each $(u_{k,\varepsilon_n}, v_{k,\varepsilon_n})$ is a solution to $\mathcal{S}_{\alpha-\varepsilon_n, \beta-\varepsilon_n, 2^*(\cdot)-\varepsilon}(s, a(\cdot), b(\cdot))$ at level c_{k,ε_n} . Hence, we can now apply Proposition 3.1 and obtain that $(u_{k,\varepsilon_n}, v_{k,\varepsilon_n}) \rightarrow (u_k, v_k)$ strongly in X as $n \rightarrow +\infty$ for some (u_k, v_k) , and $c_{k,\varepsilon_n} \rightarrow c_k = I(u_k, v_k)$. Then (u_k, v_k) is solution of $\mathcal{S}_{\alpha, \beta, 2^*(\cdot)}(s, a(\cdot), b(\cdot))$. We are now ready to show that I has infinitely many critical point solutions. Note that c_k is non-decreasing in k . We distinguish two cases.

Case 1: Suppose that there exist $1 < k_1 < \dots < k_i < \dots$ such that

$$c_{k_1} < \dots < c_{k_i} < \dots.$$

In this case, we have infinitely many distinct critical points and therefore infinitely many solutions.

Case 2: We assume that for some positive integer m , $c_k = c$ for all $k \geq m$. Suppose that for any $\delta > 0$, I has a critical point (u, v) with $I(u, v) \in (c - \delta, c + \delta)$ and $I(u, v) \neq c$. In this case, we are done. So we assume that there exists a $\delta > 0$ such that I has no critical point (u, v) with $I(u, v) \in (c - \delta, c) \cup (c, c + \delta)$. In this case, using the deformation argument as described in the proof of [5, Theorem 1.1], we can prove that

$$\gamma(K_c) \geq 2, \quad \text{where } K_c = \{(u, v) \in X : I(u, v) = c, I'(u, v) = 0\}.$$

As a consequence, I has infinitely many critical points. Thus, we can obtain infinitely many solutions for problem $\mathcal{S}_{\alpha, \beta, 2^*(\cdot)}(s, a(\cdot), b(\cdot))$.

A Appendix

We use $\mathbb{R}_*^N$ to denote either $\mathbb{R}^N$ or $\mathbb{R}_+^N$.

Lemma A.1. *Let (U, V) be a solution of the system*

$$\begin{cases} -\Delta u = \frac{2(\alpha - \varepsilon)}{2^* - 2\varepsilon} |u|^{\alpha-2-\varepsilon} |v|^{\beta-\varepsilon} + \frac{|u|^{2^*(s)-2-\varepsilon} u}{|x|^s} + a(x)u & \text{in } \mathbb{R}_*^N, \\ -\Delta v = \frac{2(\beta - \varepsilon)}{2^* - 2\varepsilon} |u|^{\alpha-\varepsilon} |v|^{\beta-2-\varepsilon} v + \frac{|v|^{2^*(s)-2-\varepsilon} v}{|x|^s} + b(x)v & \text{in } \mathbb{R}_*^N, \\ u, v \in W_0^{1,2}(\mathbb{R}_*^N). \end{cases}$$

Then we have

$$|U(x)| + |V(x)| \leq \frac{C}{1 + |x|^{N-2}} \quad \text{for all } |x| \geq 2.$$

Proof. Using the Kelvin transformation $u(x) = |x|^{2-N} U(\frac{x}{|x|^2})$, $v(x) = |x|^{2-N} V(\frac{x}{|x|^2})$, we know that (u, v) satisfies

$$\begin{cases} -\Delta u = \frac{2(\alpha - \varepsilon)}{2^* - 2\varepsilon} |u|^{\alpha-2-\varepsilon} |v|^{\beta-\varepsilon} + \frac{|u|^{2^*(s)-2-\varepsilon} u}{|x|^s} + a(x)u & \text{in } B_1(0) \cap \mathbb{R}_*^N, \\ -\Delta v = \frac{2(\beta - \varepsilon)}{2^* - 2\varepsilon} |u|^{\alpha-\varepsilon} |v|^{\beta-2-\varepsilon} v + \frac{|v|^{2^*(s)-2-\varepsilon} v}{|x|^s} + b(x)v & \text{in } B_1(0) \cap \mathbb{R}_*^N, \\ u, v \in W_0^{1,2}(B_1(0) \cap \mathbb{R}_*^N). \end{cases}$$

So,

$$|u(x)| + |v(x)| \leq C, \quad x \in B_{1/2}(0) \cap \mathbb{R}_*^N,$$

and therefore

$$|U(x)| + |V(x)| \leq \frac{C}{|x|^{N-2}}, \quad |x| \geq 2. \quad \square$$

Let $\rho_{x,\lambda}(u, v) = \lambda^{(N-2)/2}(u(\lambda(\cdot - x)), v(\lambda(\cdot - x)))$ with $x \in \mathbb{R}^N$ and $\lambda > 0$. We give a global compactness result in the following proposition.

Proposition A.2. *Let $(u_n, v_n)_{n \geq 1}$ be a noncompact balanced sequence bounded in X with $\varepsilon_n \rightarrow 0$. Then there exists a subsequence of $(u_n, v_n)_{n \geq 1}$ (still denoted by $(u_n, v_n)_{n \geq 1}$) that can be decomposed as follows:*

$$(u_n, v_n) = (u_0, v_0) + \sum_{j=1}^m \rho_{0, \lambda_{n,j}}(U_j, V_j) + \sum_{j=m+1}^k \rho_{x_{n,j}, \lambda_{n,j}}(U_j, V_j) + (\sigma_n^1, \sigma_n^2),$$

where $m \leq k$ are independent on n , (u_0, v_0) is a weak solution of (1.1) and $(\sigma_n^1, \sigma_n^2) \rightarrow 0$ strongly in X as $n \rightarrow +\infty$. Moreover, for $j = m+1, \dots, k$, $x_{n,j} \in \Omega$ with $\lambda_{n,j}d(x_{n,j}, \partial\Omega) \rightarrow +\infty$ and $\lambda_{n,j}|x_{n,j}| \rightarrow +\infty$, and (U_j, V_j) is solution of

$$\begin{cases} -\Delta u = \frac{2\alpha b_j}{2^*} |u|^{\alpha-2} u |v|^\beta, \\ -\Delta v = \frac{2\beta b_j}{2^*} |u|^\alpha |v|^{\beta-2} v, \\ (u, v) \in (D^{1,2}(\mathbb{R}^N))^2, \quad b_j \in (0, 1]. \end{cases} \quad (\text{A.1})$$

On the other hand, for $j = 1, \dots, m$, $\lambda_{n,j} \rightarrow +\infty$ as $n \rightarrow +\infty$ and $(U_j, V_j) \neq (0, 0)$ is a solution to

$$\begin{cases} -\Delta u = \frac{2\alpha b_j}{2^*} |u|^{\alpha-2} u |v|^\beta + b_j \frac{|u|^{2^*(s)-2} u}{|x|^s}, \\ -\Delta v = \frac{2\beta b_j}{2^*} |u|^\alpha |v|^{\beta-2} v + b_j \frac{|v|^{2^*(s)-2} v}{|x|^s}, \\ (u, v) \in (D_0^{1,2}(\mathbb{R}_*^N))^2, \quad b_j \in (0, 1], \end{cases} \quad (\text{A.2})$$

where $\mathbb{R}_*^N = \mathbb{R}^N$ if $0 \in \Omega$, while $\mathbb{R}_*^N = \mathbb{R}_+^N$ if $0 \in \partial\Omega$. (The constants b_j in the two systems (A.1) and (A.2) are not necessarily the same.)

Finally, set $x_{n,i} = 0$ for $i = 1, \dots, m$. For $i, j = 1, \dots, k$, if $i \neq j$, then as $n \rightarrow +\infty$,

$$\frac{\lambda_{n,j}}{\lambda_{n,i}} + \frac{\lambda_{n,i}}{\lambda_{n,j}} + \lambda_{n,i} \lambda_{n,j} |x_{n,i} - x_{n,j}|^2 \rightarrow +\infty.$$

Proof. The proof of this proposition is standard and thus we omit it. The reader can refer to [5, 6, 12, 16] for the details. $\square$

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