Research Article

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Quasilinear equations with indefinite nonlinearity

<https://doi.org/10.1515/anona-2018-0010>

Received January 14, 2018; accepted April 7, 2018

Abstract: In this paper, we are concerned with quasilinear equations with indefinite nonlinearity and explore the existence of infinitely many solutions.

Keywords: Quasilinear equations, indefinite nonlinearity, Sobolev inequality, approximate solution

MSC 2010: 35J62, 35A15, 35B20, 35B38

1 Introduction

Consider the quasilinear equation with the indefinite nonlinearity

$$\begin{cases} \Delta u + \frac{1}{2}u\Delta u^2 - a_-|u|^{r-2}u + a_+|u|^{s-2}u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded smooth domain, $r > 4$, $4 < s < \frac{4N}{N-2}$, and $a_{\pm}$ are nonnegative continuous functions in $\overline{\Omega}$. A great number of theoretical issues concerning nonlinear elliptic equations with indefinite nonlinearity have received considerable attention in the past few decades. In particular, the existence of solutions has been studied extensively. For example, the existence of positive solutions and their multiplicity was studied by variational techniques [2], and the existence of nontrivial solutions was investigated by two different approaches (one involving the Morse theory and the other using the min-max method) [1]. It was shown that the existence of positive solutions, negative solutions and sign-changing solutions could be established by means of the Morse theory [7, 8]. For the results on a priori estimates and more comparable relations among various solutions etc., we refer the reader to [3, 5, 6, 9, 10, 12] and the references therein. However, as far as one can see from the literature, not much has been known about the existence of solutions to quasilinear equations with indefinite nonlinearity. From the variational point of view, there are two main difficulties that arise in the study. One lies in the fact that there is no suitable space in which the corresponding functional enjoys both smoothness and compactness. Compared with quasilinear equations with the definite nonlinearity, the other one is to prove the boundedness of the associated Palais–Smale sequences. In this work, we will get over these obstacles by means of the variational techniques and the perturbation method to study the existence of infinitely many solutions of system (1.1).

Set

$$\Omega_{\pm} = \{x \in \overline{\Omega} \mid a_{\pm}(x) > 0\} \quad \text{and} \quad \Omega_0 = \Omega \setminus (\overline{\Omega}_+ \cup \overline{\Omega}_-).$$

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Assume that

(a) $\Omega_+ \neq \emptyset$ and $\overline{\Omega}_+ \cap \overline{\Omega}_- = \emptyset$.

We are looking for $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ satisfying equation (1.1) in the weak form

$$\int_{\Omega} (1 + u^2) \nabla u \nabla \varphi \, dx + \int_{\Omega} u |\nabla u|^2 \varphi \, dx + \int_{\Omega} a_- |u|^{r-2} u \varphi \, dx = \int_{\Omega} a_+ |u|^{s-2} u \varphi \, dx \quad (1.2)$$

for $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, which is formally the variational formulation of the following functional:

$$I(u) = \frac{1}{2} \int_{\Omega} (1 + u^2) |\nabla u|^2 \, dx + \frac{1}{r} \int_{\Omega} a_- |u|^r \, dx - \frac{1}{s} \int_{\Omega} a_+ |u|^s \, dx.$$

In view of the perturbation (regularization) approach [11], due to the lack of a suitable working space, we introduce the corresponding perturbed functionals, which are smooth functionals in the given space and satisfy the necessary compactness property. For $\mu \in (0, 1]$, we define the perturbed functional I_μ on the Sobolev space $W_0^{1,p}(\Omega)$ with $p > N$ by

$$\begin{aligned} I_\mu(u) &= \frac{\mu}{2} \left(\int_{\Omega} (1 + u^2)^{\frac{p}{2}} |\nabla u|^p \, dx \right)^{\frac{2}{p}} + I(u) \\ &= \frac{\mu}{2} \left(\int_{\Omega} (1 + u^2)^{\frac{p}{2}} |\nabla u|^p \, dx \right)^{\frac{2}{p}} + \frac{1}{2} \int_{\Omega} (1 + u^2) |\nabla u|^2 \, dx + \frac{1}{r} \int_{\Omega} a_- |u|^r \, dx - \frac{1}{s} \int_{\Omega} a_+ |u|^s \, dx. \end{aligned}$$

Note that I_μ is a C^1 -functional. We shall show that I_μ satisfies the Palais–Smale condition. The critical points of I_μ will be used as the approximate solution of problem (1.1).

Now let us briefly summarize our main results of this paper.

Theorem 1.1. Assume that condition (a) holds, $r > 4$, $4 < s < \frac{4N}{N-2}$, $\mu_n > 0$, $\mu_n \rightarrow 0$, $u_n \in W_0^{1,p}(\Omega)$, $DI_{\mu_n}(u_n) = 0$ and $I_{\mu_n}(u_n) \leq C$. Then the following assertions hold:

- (i) There exists a function $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ satisfying equation (1.1).
- (ii) Up to a subsequence, there holds $\|u_n\|_{L^\infty} \leq C$, $u_n \rightarrow u$ for a.e. $x \in \Omega$, $u_n \rightarrow u$ in $H_0^1(\Omega)$, and

$$\mu_n \left[\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p \, dx \right]^{\frac{2}{p}} \rightarrow 0 \quad \text{and} \quad I_{\mu_n}(u_n) \rightarrow I(u) \quad \text{as } n \rightarrow \infty.$$

Theorem 1.2. Assume that condition (a) holds, $r > 4$ and $4 < s < \frac{4N}{N-2}$. Then problem (1.1) has infinitely many solutions.

Consider the more general quasilinear equation

$$\begin{cases} \sum_{i,j=1}^N D_i(a_{ij}(x, u) D_j(u)) - \frac{1}{2} \sum_{i,j=1}^N D_s a_{ij}(x, u) D_i u D_j u, \\ -a_-(x) |u|^{r-2} u + a_+(x) |u|^{s-2} u = 0 \\ u = 0 \end{cases} \quad \begin{array}{l} \text{in } \Omega, \\ \text{on } \partial\Omega. \end{array} \quad (1.3)$$

In the weak form, we look for $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that

$$\begin{aligned} \int_{\Omega} \sum_{i,j=1}^N a_{ij}(x, u) D_i u D_j \varphi \, dx + \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^N D_s a_{ij}(x, u) D_i u D_j u \varphi \, dx + \int_{\Omega} a_-(x) |u|^{r-2} u \varphi \, dx \\ = \int_{\Omega} a_+(x) |u|^{s-2} u \varphi \, dx \end{aligned} \quad (1.4)$$

for $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, where $D_i = \frac{\partial}{\partial x_i}$ and $D_s a_{ij}(x, s) = \frac{\partial}{\partial s} a_{ij}(x, s)$. For the coefficients a_{ij} , $i, j = 1, \dots, N$,

we make the following assumptions:

(a0) $a_{ij}, D_s a_{ij} \in C^1(\overline{\Omega} \times \mathbb{R})$, $a_{ij} = a_{ji}$, $i, j = 1, \dots, N$.

(a1) There exist constants $c_1, c_2 > 0$ such that

$$c_1(1+s^2)|\xi|^2 \leq \sum_{i,j=1}^N a_{ij}(x, s)\xi_i\xi_j \leq c_2(1+s^2)|\xi|^2$$

for $x \in \overline{\Omega}$, $s \in \mathbb{R}$ and $\xi = (\xi_i) \in \mathbb{R}^N$.

(a2) There exist $\delta > 0$ and $0 < q < s$ such that

$$\delta \sum_{i,j=1}^N a_{ij}(x, s)\xi_i\xi_j \leq \sum_{i,j=1}^N \left[a_{ij}(x, s) + \frac{1}{2}sD_s a_{ij}(x, s) \right] \xi_i\xi_j \leq q\left(\frac{1}{2} - \delta\right) \sum_{i,j=1}^N a_{ij}(x, s)\xi_i\xi_j$$

for $x \in \overline{\Omega}$, $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$.

(a3) There exists an $M > 0$ such that

$$\sum_{i,j=1}^N \left[a_{ij}(x, s) + \frac{1}{2}sD_s a_{ij}(x, s) \right] \xi_i\xi_j \geq \frac{2s^2}{M+s^2} \sum_{i,j=1}^N a_{ij}(x, s)\xi_i\xi_j$$

for $x \in \overline{\Omega}$, $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$.

(a4) There holds

$$\lim_{|s| \rightarrow \infty} \frac{a_{ij}(x, s)}{s^2} = A_{ij}(x)$$

uniformly in $x \in \overline{\Omega}$.

Theorem 1.3. Assume that $r > 4$, $4 < s < \frac{4N}{N-2}$ and conditions (a0)–(a4) hold. Then equation (1.3) has infinitely many solutions.

The rest of the paper is organized as follows: The proof of Theorem 1.1 is presented in Section 2, and the proof of Theorem 1.2 is shown in Section 3. Section 4 is dedicated to the existence of infinitely many solutions to the more general quasilinear equation (1.3).

2 Convergence theorem

To prove Theorem 1.1, we need the following two technical lemmas.

Lemma 2.1. There holds that

$$\begin{aligned} & \mu \left(\int_{\Omega} (1+u^2)^{\frac{p}{2}} |\nabla u|^p dx \right)^{\frac{2}{p}} + \int_{\Omega} (1+u^2) |\nabla u|^2 dx + \int_{\Omega} a_- |u|^r dx + \int_{\Omega} a_+ |u|^s dx \\ & \leq C \left\{ I_{\mu}(u) + \|DI_{\mu}(u)\| \cdot \|u\| + \mu \left[\int_{\Omega} (1+u^2)^{\frac{p}{2}} |u|^p dx \right]^{\frac{2}{p}} + \int_{\Omega} (1+u^2) u^2 dx \right\}. \end{aligned}$$

Proof. For $\varphi \in W_0^{1,p}(\Omega)$, we know that

$$\begin{aligned} \langle DI_{\mu}(u), \varphi \rangle &= \mu \left[\int_{\Omega} (1+u^2)^{\frac{p}{2}} |\nabla u|^p dx \right]^{\frac{2}{p}-1} \int_{\Omega} [(1+u^2)^{\frac{p}{2}} |\nabla u|^{p-2} \nabla u \nabla \varphi + (1+u^2)^{\frac{p}{2}-1} u |\nabla u|^p \varphi] dx \\ & \quad + \int_{\Omega} [(1+u^2) \nabla u \nabla \varphi + u |\nabla u|^2 \varphi] dx + \int_{\Omega} a_- |u|^{r-2} u \varphi dx - \int_{\Omega} a_+ |u|^{s-2} u \varphi dx. \end{aligned} \quad (2.1)$$

Since $\overline{\Omega}_- \cap \overline{\Omega}_+ = \emptyset$, we can choose $\psi \in C_0^\infty(\mathbb{R}^N)$ such that $\psi \geq 0$, $\psi(x) = 1$ for $x \in \overline{\Omega}_-$, and $\psi(x) = 0$ for $x \in \overline{\Omega}_+$. Then

$$\int_{\Omega} a_- |u|^r \psi^p dx = \int_{\Omega} a_- |u|^r dx \quad \text{and} \quad \int_{\Omega} a_+ |u|^s \psi^p dx = 0.$$

Taking $\psi = u\psi^p$ as the test function in (2.1), we get

$$\begin{aligned} & \mu \left(\int_{\Omega} (1+u^2)^{\frac{p}{2}} |\nabla u|^p \, dx \right)^{\frac{2}{p}-1} \int_{\Omega} (1+u^2)^{\frac{p}{2}-1} (1+2u^2) |\nabla u|^p \psi^p \, dx + \int_{\Omega} (1+2u^2) |\nabla u|^2 \psi^p \, dx + \int_{\Omega} a_- |u|^r \, dx \\ &= \langle DI_{\mu}(u), u\psi^p \rangle - p \int_{\Omega} (1+u^2) \nabla u u \psi^{p-1} \nabla \psi \, dx \\ &\quad - p \mu \left(\int_{\Omega} (1+u^2)^{\frac{p}{2}} |\nabla u|^p \, dx \right)^{\frac{2}{p}-1} \int_{\Omega} (1+u^2)^{\frac{p}{2}} |\nabla u|^{p-2} \nabla u u \psi^{p-1} \nabla \psi \, dx \\ &\leq C \|DI_{\mu}(u)\| \cdot \|u\| + \varepsilon \int_{\Omega} (1+u^2) |\nabla u|^2 \psi^p \, dx + C \int_{\Omega} (1+u^2) u^2 |\nabla \psi|^2 \, dx \\ &\quad + \mu \left(\int_{\Omega} (1+u^2)^{\frac{p}{2}} |\nabla u|^p \, dx \right)^{\frac{2}{p}-1} \left\{ \varepsilon \int_{\Omega} (1+u^2)^{\frac{p}{2}} |\nabla u|^p \psi^p \, dx + C \int_{\Omega} (1+u^2)^{\frac{p}{2}} |u|^p |\nabla \psi|^p \, dx \right\}. \quad (2.2) \end{aligned}$$

In view of the Sobolev inequality

$$\int_{\Omega} (1+u^2)^{\frac{p}{2}} |u|^p \, dx \leq C \int_{\Omega} (1+u^2)^{\frac{p}{2}} |\nabla u|^p \, dx, \quad (2.3)$$

it follows from (2.2) and (2.3) that

$$\int_{\Omega} a_- |u|^r \, dx \leq C \left\{ \|DI_{\mu}(u)\| \|u\| + \mu \left[\int_{\Omega} (1+u^2)^{\frac{p}{2}} |u|^p \, dx \right]^{\frac{2}{p}} + \int_{\Omega} (1+u^2) u^2 \, dx \right\}.$$

By choosing $q \in (4, s)$, we deduce that

$$\begin{aligned} I_{\mu}(u) - \frac{1}{q} \langle DI_{\mu}(u), u \rangle &= \frac{1}{2} \mu \left(\int_{\Omega} (1+u^2)^{\frac{p}{2}} |\nabla u|^p \, dx \right)^{\frac{2}{p}} + \frac{1}{2} \int_{\Omega} (1+u^2) |\nabla u|^2 \, dx \\ &\quad - \frac{1}{q} \mu \left(\int_{\Omega} (1+u^2)^{\frac{p}{2}} |\nabla u|^p \, dx \right)^{\frac{2}{p}-1} \int_{\Omega} (1+u^2)^{\frac{p}{2}-1} (1+2u^2) |\nabla u|^p \, dx \\ &\quad - \frac{1}{q} \int_{\Omega} (1+2u^2) |\nabla u|^2 \, dx - \left(\frac{1}{q} - \frac{1}{r} \right) \int_{\Omega} a_- |u|^r \, dx + \left(\frac{1}{q} - \frac{1}{s} \right) \int_{\Omega} a_+ |u|^s \, dx \\ &\geq \left(\frac{1}{2} - \frac{2}{q} \right) \mu \left(\int_{\Omega} (1+u^2)^{\frac{p}{2}} |\nabla u|^p \, dx \right)^{\frac{2}{p}} + \left(\frac{1}{2} - \frac{2}{q} \right) \left(\int_{\Omega} (1+u^2)^{\frac{p}{2}} |\nabla u|^2 \, dx \right)^{\frac{2}{p}} \\ &\quad - \left(\frac{1}{q} - \frac{1}{r} \right) \int_{\Omega} a_- |u|^r \, dx + \left(\frac{1}{q} - \frac{1}{s} \right) \int_{\Omega} a_+ |u|^s \, dx, \end{aligned}$$

and thus

$$\begin{aligned} & \mu \left(\int_{\Omega} (1+u^2)^{\frac{p}{2}} |\nabla u|^p \, dx \right)^{\frac{2}{p}} + \int_{\Omega} (1+u^2) |\nabla u|^2 \, dx + \int_{\Omega} a_+ |u|^s \, dx \\ &\leq C \left\{ I_{\mu}(u) + \|DI_{\mu}(u)\| \cdot \|u\| + \int_{\Omega} a_- |u|^r \, dx \right\} \\ &\leq C \left\{ I_{\mu}(u) + \|DI_{\mu}(u)\| \cdot \|u\| + \mu \left[\int_{\Omega} (1+u^2)^{\frac{p}{2}} |u|^p \, dx \right]^{\frac{2}{p}} + \int_{\Omega} (1+u^2) |u|^2 \, dx \right\}. \quad \square \end{aligned}$$

Lemma 2.2. Assume that $\mu_n > 0$, $\mu_n \rightarrow 0$, $u_n \in W_0^{1,p}(\Omega)$, $DI_{\mu_n}(u_n) = 0$ and $I_{\mu_n}(u_n) \leq C_0$. Then there exists a constant $C > 0$ independent of n such that

$$\mu_n \left[\int_{\Omega} (1+u_n^2)^{\frac{p}{2}} |u_n|^p \, dx \right]^{\frac{2}{p}} + \int_{\Omega} (1+u_n^2) u_n^2 \, dx \leq C.$$

Moreover, there holds

$$\mu_n \left[\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p \, dx \right]^{\frac{2}{p}} + \int_{\Omega} (1 + u_n^2) |\nabla u_n|^2 \, dx + \int_{\Omega} a_- |u_n|^r \, dx + \int_{\Omega} a_+ |u_n|^s \, dx \leq C.$$

Proof. Assume that

$$\rho_n^4 = \mu_n \left(\int_{\Omega} (1 + u_n^2)^{\frac{2}{p}} |u_n|^p \, dx \right)^{\frac{2}{p}} + \int_{\Omega} (1 + u_n^2) u_n^2 \, dx \rightarrow \infty \quad \text{as } n \rightarrow \infty. \quad (2.4)$$

By Lemma 2.1, we have

$$\mu_n \left[\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p \, dx \right]^{\frac{2}{p}} + \int_{\Omega} (1 + u_n^2) |\nabla u_n|^2 \, dx + \int_{\Omega} a_- |u_n|^r \, dx + \int_{\Omega} a_+ |u_n|^s \, dx \leq C \rho_n^4.$$

Let $v_n = \rho_n^{-1} u_n$. Then

$$\begin{aligned} \int_{\Omega} v_n^2 |\nabla v_n|^2 \, dx &= \rho_n^{-4} \int_{\Omega} u_n^2 |\nabla u_n|^2 \, dx \leq C, \\ \int_{\Omega} a_- |v_n|^r \, dx &= \rho_n^{-r} \int_{\Omega} a_- |u_n|^r \, dx \leq C \rho_n^{4-r} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \\ \int_{\Omega} a_+ |v_n|^s \, dx &= \rho_n^{-s} \int_{\Omega} a_+ |u_n|^s \, dx \leq C \rho_n^{4-s} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

We have $v_n \rightarrow v$ in $L^q(\Omega)$, $1 \leq q < \frac{4N}{N-2}$, $v_n^2 \rightharpoonup v^2$ in $H_0^1(\Omega)$, $\int_{\Omega} a_- |v|^r \, dx = 0$ and $\int_{\Omega} a_+ |v|^s \, dx = 0$. Thus, $v(x) = 0$ for $x \in \overline{\Omega}_+ \cap \overline{\Omega}_-$ and $v^2 \in H_0^1(\Omega_0) \subset H_0^1(\Omega)$.

If $p > N$, then $W_0^{1,p}(\Omega) \hookrightarrow C^\alpha(\Omega)$ for some $\alpha \in (0, 1)$. Given $\psi \geq 0$ and $\psi \in C_0^\infty(\Omega_0)$, we take

$$\varphi_n = \frac{\psi u_n}{1 + u_n^2} \in W_0^{1,p}(\Omega_0) \subset W_0^{1,p}(\Omega),$$

and have

$$\begin{aligned} 0 &= \langle DI_{\mu_n}(u_n), \varphi_n \rangle \\ &= \mu_n \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p \, dx \right)^{\frac{2}{p}-1} \int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^{p-2} \nabla u_n \nabla \psi \frac{u_n}{1 + u_n^2} \, dx \\ &\quad + \mu_n \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p \, dx \right)^{\frac{2}{p}-1} \int_{\Omega} (1 + u_n^2)^{\frac{p}{2}-2} |\nabla u_n|^p \psi \, dx \\ &\quad + \int_{\Omega} (1 + u_n^2) \nabla u_n \nabla \psi \frac{u_n}{1 + u_n^2} \, dx + \int_{\Omega} (1 + u_n^2)^{-1} |\nabla u_n|^2 \psi \, dx \\ &\geq \mu_n \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p \, dx \right)^{\frac{2}{p}} \int_{\Omega} (1 + u_n^2)^{\frac{p-1}{2}} |\nabla u_n|^{p-2} \nabla u_n \nabla \psi \frac{u_n}{(1 + u_n^2)^{\frac{1}{2}}} \, dx + \int_{\Omega} u_n \nabla u_n \nabla \psi \, dx. \end{aligned}$$

We further obtain the estimates as

$$\begin{aligned} &\rho_n^{-2} \mu_n \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p \, dx \right)^{\frac{2}{p}-1} \left| \int_{\Omega} (1 + u_n^2)^{\frac{p-1}{2}} |\nabla u_n|^{p-2} \nabla u_n \nabla \psi \frac{u_n}{(1 + u_n^2)^{\frac{1}{2}}} \, dx \right| \\ &\leq \rho_n^{-2} \mu_n \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p \, dx \right)^{\frac{1}{p}} \left(\int_{\Omega} |\nabla \psi|^p \, dx \right)^{\frac{1}{p}} \\ &\leq C \mu_n^{\frac{1}{2}} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

and

$$\rho_n^{-2} \int_{\Omega} u_n \nabla u_n \nabla \psi \, dx = \int_{\Omega} v_n \nabla v_n \nabla \psi \, dx \rightarrow \int_{\Omega} v \nabla v \nabla \psi \, dx \quad \text{as } n \rightarrow \infty.$$

Hence, it gives

$$\int_{\Omega} \nabla v^2 \nabla \psi \, dx \leq 0 \quad \text{for } \psi \geq 0 \text{ and } \psi \in C_0^\infty(\Omega_0).$$

Since $v^2 \in H_0^1(\Omega_0)$, we have $v^2 \equiv 0$ in Ω . It follows from (2.4) and Lemma 2.1 that

$$\rho_n^{-4} \left[\mu_n \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p \, dx \right)^{\frac{2}{p}} + \int_{\Omega} (1 + u_n^2) |\nabla u_n|^2 \, dx \right] \leq C,$$

and

$$\rho_n^{-4} \left[\mu_n \left(\int_{\Omega} (1 + u^2)^{\frac{p}{2}} |u_n|^p \, dx \right)^{\frac{2}{p}} + \int_{\Omega} (1 + u_n^2) u_n^2 \, dx \right] = 1. \quad (2.5)$$

In view of $1 \leq q < \frac{4N}{N-2}$, by taking $v_n \rightarrow v = 0$ in $L^q(\Omega)$, we get

$$\rho_n^{-4} \int_{\Omega} u_n^4 \, dx = \int_{\Omega} v_n^4 \, dx \rightarrow \int_{\Omega} v^4 \, dx = 0,$$

and

$$\rho_n^{-4} \int_{\Omega} u_n^2 \, dx \leq \frac{1}{2} \rho_n^{-4} \int_{\Omega} (1 + u_n^4) \, dx \rightarrow 0.$$

Let $z_n \in W_0^{1,p}(\Omega)$ and

$$Dz_n = \rho_n^{-2} \mu_n^{\frac{1}{2}} (1 + u_n^2)^{\frac{1}{2}} Du_n.$$

It is easy to see that

$$\left(\int_{\Omega} |Dz_n|^p \, dx \right)^{\frac{2}{p}} \leq \rho_n^{-4} \mu_n \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p \, dx \right)^{\frac{2}{p}} \leq C. \quad (2.6)$$

So we further deduce that

$$\begin{aligned} C_1 \rho_n^{-2} \mu_n^{\frac{1}{2}} (1 + u_n^2)^{\frac{1}{2}} |u_n| &\leq |z_n| \leq C_2 \rho_n^{-2} \mu_n^{\frac{1}{2}} (1 + u_n^2)^{\frac{1}{2}} |u_n|, \\ |z_n| &\leq C_2 \mu_n^{\frac{1}{2}} (\rho_n^{-2} + v_n^2)^{\frac{1}{2}} |v_n| \rightarrow 0 \quad \text{for a.e. } x \in \Omega, \\ \rho_n^{-4} \mu_n \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |u_n|^p \, dx \right)^{\frac{2}{p}} &\leq C \left(\int_{\Omega} |z_n|^p \, dx \right)^{\frac{2}{p}} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Consequently, the left-hand side of (2.5) converges to zero, which yields a contradiction. $\square$

Proof of Theorem 1.1. By Lemmas 2.1 and 2.2, we have

$$\mu_n \left[\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p \, dx \right]^{\frac{2}{p}} + \int_{\Omega} (1 + u_n^2) |\nabla u_n|^2 \, dx \leq C.$$

Assume that $u_n \rightarrow u$ in $L^q(\Omega)$, $1 \leq q < \frac{4N}{N-2}$ and $u_n^2 \rightarrow u^2$ in $H_0^1(\Omega)$. We separate the proof into three steps.

Step 1. Moser's iteration shows that the sequence $\{u_n\}$ is uniformly bounded.

Note that for $p > N$, the convergence $W_0^{1,p}(\Omega) \hookrightarrow C^\alpha(\overline{\Omega})$ holds for some $\alpha \in (0, 1)$. The function u_n belongs to $L^\infty(\Omega)$. Here we prove that $\|u_n\|_{L^\infty(\Omega)}$ is uniformly bounded. The term $a_+ |u|^{s-2} u$ in equation (1.2) is subcritical ($s < \frac{4N}{N-2}$), while the term of $a_- |u|^{r-2} u$ in the equation does not cause any trouble to us at this step. The sequence $\{u_n\}$ is bounded in $L^{\frac{4N}{N-2}}(\Omega)$. Starting from this $L^{\frac{4N}{N-2}}(\Omega)$ -bound, by Moser's iteration we see the $L^\infty(\Omega)$ -bound.

Step 2. Choose a suitable test function and show that the limit function u satisfies equation (1.2).

Let $\psi \geq 0$, $\psi \in C_0^\infty(\Omega)$ and $\varphi = \psi e^{-u_n} \in W_0^{1,p}(\Omega)$. Take φ as the test function. Then we have

$$\begin{aligned} 0 &= \langle DI_{\mu_n}(u_n), \varphi \rangle \\ &= \mu_n \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p dx \right)^{\frac{2}{p}-1} \int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^{p-2} \nabla u_n \nabla \psi e^{-u_n} dx \\ &\quad - \mu_n \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p dx \right)^{\frac{2}{p}-1} \int_{\Omega} (1 + u_n^2)^{\frac{p}{2}-1} (1 + u_n^2 - u_n) |\nabla u_n|^p \psi e^{-u_n} dx \\ &\quad + \int_{\Omega} (1 + u_n^2) \nabla u_n \nabla \psi e^{-u_n} dx - \int_{\Omega} (1 + u_n^2 - u_n) |\nabla u_n|^2 \psi dx \\ &\quad + \int_{\Omega} a_- |u_n|^{r-2} u_n \psi e^{-u_n} dx - \int_{\Omega} a_+ |u_n|^{s-2} u_n \psi e^{-u_n} dx. \end{aligned} \quad (2.7)$$

We estimate each term in (2.7). From (2.6) we get

$$\begin{aligned} &\left| \mu_n \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p dx \right)^{\frac{2}{p}-1} \int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^{p-2} \nabla u_n \nabla \psi e^{-u_n} dx \right| \\ &\leq \mu_n \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p dx \right)^{\frac{1}{p}} \left(\int_{\Omega} |\nabla \psi|^{\frac{1}{p}} dx \right) \\ &\leq C \mu_n^{\frac{1}{2}} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

By the weak convergence, it gives

$$\int_{\Omega} (1 + u_n^2) \nabla u_n \nabla \psi e^{-u_n} dx \rightarrow \int_{\Omega} (1 + u^2) \nabla u \nabla \psi e^{-u} dx.$$

By the lower semi-continuity, we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} (1 + u_n^2 - u_n) |\nabla u_n|^2 \psi e^{-u_n} dx \geq \int_{\Omega} (1 + u^2 - u) |\nabla u|^2 \psi e^{-u} dx.$$

Using Lebesgue's dominated convergence theorem leads to

$$\int_{\Omega} a_- |u_n|^{r-2} u_n \psi e^{-u_n} dx \rightarrow \int_{\Omega} a_- |u|^{r-2} u \psi e^{-u} dx$$

and

$$\int_{\Omega} a_+ |u_n|^{s-2} u_n \psi e^{-u_n} dx \rightarrow \int_{\Omega} a_+ |u|^{s-2} u \psi e^{-u} dx.$$

It follows from (2.7) and the above estimates that

$$\int_{\Omega} (1 + u^2) \nabla u \nabla \psi e^{-u} dx - \int_{\Omega} (1 + u^2 - u) |\nabla u|^2 \psi e^{-u} dx + \int_{\Omega} a_- |u|^{r-2} u \psi e^{-u} dx - \int_{\Omega} a_+ |u|^{s-2} u \psi e^{-u} dx \geq 0$$

for $\psi \geq 0$ and $\psi \in C_0^\infty(\Omega)$. Moreover, we get

$$\int_{\Omega} (1 + u^2) \nabla u \nabla (\psi e^{-u}) dx + \int_{\Omega} u |\nabla u|^2 \psi e^{-u} dx + \int_{\Omega} a_- |u|^{r-2} u \psi e^{-u} dx - \int_{\Omega} a_+ |u|^{s-2} u \psi e^{-u} dx \geq 0. \quad (2.8)$$

Given $\varphi \geq 0$ and $\varphi \in C_0^\infty(\Omega)$, we choose $\psi_n \in C_0^\infty(\Omega)$ such that $\psi_n \rightarrow \varphi e^u$ in $H_0^1(\Omega)$, $\|\psi_n\| \leq C$ and $\psi_n \rightarrow \varphi e^{-u}$ for a.e. $x \in \Omega$. Taking ψ_n as the test function in (2.8) and letting n tend to infinity, we obtain

$$\int_{\Omega} (1 + u^2) \nabla u \nabla \varphi dx + \int_{\Omega} u |\nabla u|^2 \varphi dx + \int_{\Omega} a_- |u|^{r-2} u \varphi dx - \int_{\Omega} a_+ |u|^{s-2} u \varphi dx \geq 0$$

for $\varphi \geq 0$ and $\varphi \in C_0^\infty(\Omega)$.

Processing in a similar manner, one can also obtain an inequality with an opposite direction. Equation (1.2) holds for all $\varphi \geq 0$ and $\varphi \in C_0^\infty(\Omega)$. By the density argument, equation (1.2) also holds for all functions $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Step 3. Since $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ satisfies equation (1.2), we have

$$\int_{\Omega} (1 + 2u^2) |\nabla u|^2 \, dx + \int_{\Omega} a_- |u|^r \, dx = \int_{\Omega} a_+ |u|^s \, dx.$$

By $DI_{\mu_n}(u_n) = 0$, we have $\langle DI_{\mu_n}(u_n), u_n \rangle = 0$. That is,

$$\begin{aligned} \mu_n \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p \, dx \right)^{\frac{2}{p}-1} \int_{\Omega} (1 + u_n^2)^{\frac{p}{2}-1} (1 + u_n^2) |\nabla u_n|^p \, dx + \int_{\Omega} (1 + 2u_n^2) |\nabla u_n|^2 \, dx + \int_{\Omega} a_- |u_n|^r \, dx \\ = \int_{\Omega} a_+ |u_n|^s \, dx. \end{aligned}$$

In view of $s < \frac{4N}{N-2}$ and $\int_{\Omega} a_+ |u_n|^s \, dx \rightarrow \int_{\Omega} a_+ |u|^s \, dx$, as n tends to infinity, by the lower semi-continuity we deduce that

$$\begin{aligned} \mu_n \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p \, dx \right)^{\frac{2}{p}} \rightarrow 0, \quad \int_{\Omega} |\nabla u_n|^2 \, dx \rightarrow \int_{\Omega} |\nabla u|^2 \, dx, \\ \int_{\Omega} u_n^2 |\nabla u_n|^2 \, dx \rightarrow \int_{\Omega} u^2 |\nabla u|^2 \, dx, \quad \int_{\Omega} a_- |u_n|^r \, dx \rightarrow \int_{\Omega} a_- |u|^r \, dx, \end{aligned}$$

and

$$I_{\mu_n}(u_n) \rightarrow I(u).$$

Hence, we obtain $u_n \rightarrow u$ in $H_0^1(\Omega)$ and $u_n \nabla u_n \rightarrow u \nabla u$ in $L^2(\Omega)$ as $n \rightarrow \infty$. $\square$

In the following context, we call $c \in \mathbb{R}$ a critical value of the functional I , provided there exists a function $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ satisfying equation (1.2) and $I(u) = c$. Theorem 1.1 implies that if $\mu_n \rightarrow 0$, c_n is a critical value of I_{μ_n} and $c = \lim_{n \rightarrow \infty} c_n$, then c is a critical value of I .

3 Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We construct a sequence of critical values of the functional I_μ with $\mu > 0$. The corresponding critical points will be used as the approximate solutions of equation (1.2).

Lemma 3.1. Suppose that $\{u_n\}$ is a Palais–Smale sequence of the functional I_μ with $\mu > 0$. Then u_n is bounded in $W_0^{1,p}(\Omega)$.

Proof. The proof is similar to the one of Lemma 2.2. Since $\mu > 0$ is fixed and $\|DI_\mu(u_n)\| \rightarrow 0$, by Lemma 2.1 we have

$$\begin{aligned} \mu \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p \, dx \right)^{\frac{2}{p}} + \int_{\Omega} (1 + u_n^2) |\nabla u_n|^2 \, dx + \int_{\Omega} a_- |u_n|^r \, dx + \int_{\Omega} a_+ |u_n|^s \, dx \\ \leq C \left[1 + I_\mu(u_n) + \mu \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |u_n|^p \, dx \right)^{\frac{2}{p}} + \int_{\Omega} (1 + u_n^2) u_n^2 \, dx \right]. \end{aligned}$$

As in the proof of Lemma 2.2, we prove it by way of contradiction. Assume that

$$\rho_n^4 = \mu \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |u_n|^p \, dx \right)^{\frac{2}{p}} + \int_{\Omega} (1 + u_n^2) u_n^2 \, dx \rightarrow \infty \quad \text{as } n \rightarrow \infty. \quad (3.1)$$

Then it is easy to see that

$$\mu \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p \, dx \right)^{\frac{2}{p}} + \int_{\Omega} (1 + u_n^2) |\nabla u_n|^2 \, dx + \int_{\Omega} a_- |u_n|^r \, dx + \int_{\Omega} a_+ |u_n|^s \, dx \leq C \rho_n^4.$$

Let $v_n = \rho_n^{-1} u_n$. As in the proof of Lemma 2.1, we have $v_n \rightarrow v$ in $L^q(\Omega)$, $1 \leq q < \frac{4N}{N-2}$, $v_n^2 \rightharpoonup v^2$ in $H_0^1(\Omega)$, and $v = 0$ in $\overline{\Omega}_+ \cup \overline{\Omega}_-$. Since $p > N$ and μ is fixed, we find

$$\mu \left(\int_{\Omega} |v_n|^p |\nabla v_n|^p \, dx \right)^{\frac{2}{p}} \leq \rho_n^{-4} \mu \left(\int_{\Omega} |u_n|^p |\nabla u_n|^p \, dx \right)^{\frac{2}{p}} \leq C.$$

So we see $v_n^2 \rightarrow v^2$ in $C^\alpha(\overline{\Omega})$ for some $\alpha \in (0, 1)$. Given $\psi \geq 0$ and $\psi \in C_0^\infty(\Omega_0)$, we take $\varphi_n = \psi u_n / (1 + u_n^2)$ as the test function, and thus have

$$\begin{aligned} \nabla \varphi_n &= \nabla \psi \frac{u_n}{1 + u_n^2} - \psi \frac{1 - u_n^2}{(1 + u_n^2)^2} \nabla u_n, \\ \|\varphi_n\| &= \left(\int_{\Omega} |\nabla \varphi_n|^p \, dx \right)^{\frac{1}{p}} \leq C \left(\int_{\Omega} |\nabla \psi|^p \, dx \right)^{\frac{1}{p}} + C \|\psi\|_{L^\infty(\Omega)} \left(\int_{\Omega} |\nabla u_n|^p \, dx \right)^{\frac{1}{p}} \leq C \rho_n^2 \|\psi\|, \end{aligned}$$

and

$$\begin{aligned} o(1) \|\psi\| &= \rho_n^{-2} \langle DI_\mu(u_n), \varphi_n \rangle \\ &\geq \rho_n^{-2} \left[\mu \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p \, dx \right)^{\frac{2}{p}-1} \int_{\Omega} (1 + u_n^2)^{\frac{p}{2}-1} |\nabla u_n|^{p-2} \nabla u_n u_n \nabla \psi \, dx + \int_{\Omega} u_n \nabla u_n \nabla \psi \, dx \right] \\ &= \mu \left(\int_{\Omega} (\rho_n^{-2} + v_n^2)^{\frac{p}{2}} |\nabla v_n|^p \, dx \right)^{\frac{2}{p}-1} \int_{\Omega} (\rho_n^{-2} + v_n^2)^{\frac{p}{2}-1} |\nabla v_n|^{p-2} \nabla v_n v_n \nabla \psi \, dx + \int_{\Omega} v_n \nabla v_n \nabla \psi \, dx \quad (3.2) \end{aligned}$$

for $\psi \geq 0$ and $\psi \in C_0^\infty(\Omega_0)$.

If

$$\lim_{n \rightarrow \infty} \int_{\Omega} (\rho_n^{-2} + v_n^2)^{\frac{p}{2}} |\nabla v_n|^p \, dx = C_0 = 0,$$

then

$$\int_{\Omega} |v|^p |\nabla v|^p \, dx \leq \lim_{n \rightarrow \infty} \int_{\Omega} (\rho_n^{-2} + v_n^2)^{\frac{p}{2}} |\nabla v_n|^p \, dx = 0.$$

So one can see $v^2 \equiv 0$. Otherwise, we assume that

$$\lim_{n \rightarrow \infty} \int_{\Omega} (\rho_n^{-2} + v_n^2)^{\frac{p}{2}} |\nabla v_n|^p \, dx = C_0 > 0.$$

By a density argument, inequality (3.2) also holds for $\psi \geq 0$ and $\psi \in W_0^{1,p}(\Omega_0)$.

Let

$$C_n = \max\{v_n^2(x) \mid x \in \partial\Omega_0\}.$$

Since $v_n^2 \rightarrow v^2$ in $C^\alpha(\overline{\Omega})$ and $v^2 = 0$ in $\overline{\Omega}_+ \cup \overline{\Omega}_-$, we have $v^2 \equiv 0$ on $\partial\Omega_0$ and $C_n \rightarrow 0$ as $n \rightarrow \infty$.

Define

$$\psi_n \in W_0^{1,p}(\Omega_0) \quad \text{by} \quad \psi_n(x) = (v_n^2(x) - C_n)_+,$$

for $x \in \overline{\Omega}_0$, $|\nabla \psi_n| \leq 2|v_n \nabla v_n|$ and $\|\psi_n\| \leq C$. Taking ψ_n as the test function in inequality (3.2), we have

$$\begin{aligned} o(1) &\geq \mu C_0^{\frac{2}{p}-1} \int_{\Omega} |v_n|^{p-2} v_n |\nabla v_n|^{p-2} \nabla v_n \nabla (v_n^2 - C_n)_+ \, dx + \int_{\Omega} v_n \nabla v_n \nabla (v_n^2 - C_n)_+ \, dx \\ &= \mu C_0^{\frac{2}{p}-1} 2^{1-p} \int_{\Omega} |\nabla (v_n^2 - C_n)_+|^p \, dx + \frac{1}{2} \int_{\Omega} |\nabla (v_n^2 - C_n)_+|^2 \, dx. \end{aligned}$$

Taking $n \rightarrow \infty$, we obtain

$$0 \geq \mu C_0^{\frac{2}{p}-1} 2^{1-p} \int_{\Omega} |\nabla v^2|^p dx + \frac{1}{2} \int_{\Omega} |\nabla v^2|^2 dx.$$

Hence, $v^2 = 0$ in $\overline{\Omega}$, which leads to a contradiction by virtue of Lemma 2.2. In fact, it follows from (3.1) that

$$\mu \left(\int_{\Omega} |v|^p dx \right)^{\frac{2}{p}} + \int_{\Omega} v^4 dx = 1. \quad \square$$

Lemma 3.2. *The functional I_{μ} with $\mu > 0$ satisfies the Palais–Smale condition.*

Proof. Let $\{u_n\}$ be a Palais–Smale sequence of the functional I_{μ} with $\mu > 0$. By Lemma 3.1, $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. When $p > N$, we have $u_n \rightarrow u$ in $C^{\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$.

Let

$$\lim_{n \rightarrow \infty} \int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p dx = C_0.$$

If $C_0 = 0$, then $u_n \rightarrow 0$ in $W_0^{1,p}(\Omega)$. Otherwise, there holds

$$\begin{aligned} o(1) &= \langle DI_{\mu}(u_n) - DI_{\mu}(u_m), u_n - u_m \rangle \\ &= \mu C_0^{\frac{2}{p}-1} \int_{\Omega} (1 + u^2)^{\frac{p}{2}} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_m|^{p-2} \nabla u_m, \nabla u_n - \nabla u_m) dx \\ &\quad + \int_{\Omega} (1 + u^2) |\nabla u_n - \nabla u_m|^2 dx + o(1) \\ &\geq C \int_{\Omega} |\nabla u_n - \nabla u_m|^p dx + o(1). \end{aligned}$$

So, $\{u_n\}$ is a Cauchy sequence of $W_0^{1,p}(\Omega)$. $\square$

Proof of Theorem 1.2. We define a sequence of critical values of the functional I_{μ} with $\mu \in (0, 1]$ by

$$c_k(\mu) = \inf_{\varphi \in \Gamma_k} \sup_{t \in B_k} I_{\mu}(\varphi(t)), \quad k = 1, 2, \dots,$$

where

$$\Gamma_k = \{\varphi \mid \varphi \in C(B_k, W_0^{1,p}(\Omega)), \varphi \text{ is odd and } I_1(\varphi(t)) < 0 \text{ for } t \in \partial B_k\},$$

and B_k is the unit ball of $\mathbb{R}^k$. Then we have

$$\begin{aligned} I_{\mu}(u) &= \frac{1}{2} \mu \left[\int_{\Omega} (1 + u^2)^{\frac{p}{2}} |\nabla u|^p dx \right]^{\frac{2}{p}} + I(u) \\ &\geq \frac{1}{2} \int_{\Omega} (1 + u^2) |\nabla u|^2 dx + \frac{1}{r} \int_{\Omega} a_- |u|^r dx - \frac{1}{s} \int_{\Omega} a_+ |u|^s dx \\ &\geq \frac{1}{2} \int_{\Omega} |\nabla w|^2 dx - C \int_{\Omega} |w|^{\frac{s}{2}} dx \\ &:= J(w), \end{aligned}$$

where w is defined by $Dw = (1 + u^2)^{1/2} Du$ and $w \in H_0^1(\Omega)$, and J is a C^1 -functional defined on $H_0^1(\Omega)$.

Define the critical values of J by

$$\alpha_k = \inf_{\varphi \in G_k} \sup_{t \in B_k} J(\varphi(t)),$$

where

$$G_k = \{\varphi \mid \varphi \in C(B_k, W_0^{1,p}(\Omega)), \varphi \text{ is odd and } J(\varphi(t)) < 0 \text{ for } t \in \partial B_k\}.$$

By Lemma 3.2 and the symmetric mountain pass lemma [4], α_k , $k = 1, 2, \dots$, are critical values of J and $\alpha_k \rightarrow \infty$ as $k \rightarrow \infty$. We have the estimate $c_k(\mu) \geq \alpha_k$. On the other hand, let $\beta_k = c_k(1)$. Then we get

$$\alpha_k \leq c_k(\mu) \leq \beta_k, \quad k = 1, 2, \dots, \text{ for } \mu \in (0, 1].$$

Let

$$c_k = \lim_{\mu \rightarrow 0} c_k(\mu).$$

By virtue of Theorem 1.1, c_k , $k = 1, 2, \dots$, are the critical values of I and $c_k \rightarrow +\infty$ as $k \rightarrow \infty$. □

4 More general cases

In this section, we consider the more general quasilinear equation (1.3) and prove Theorem 1.3.

Equation (1.3) has a variational structure, given by the functional

$$H(u) = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^N a_{ij}(x, u) D_i u D_j u \, dx + \frac{1}{r} \int_{\Omega} a_- |u|^r \, dx - \frac{1}{s} \int_{\Omega} a_+ |u|^s \, dx.$$

Again we apply the perturbation method and introduce the perturbed functional H_{μ} with $\mu \in (0, 1]$:

$$\begin{aligned} H_{\mu}(u) &= \frac{1}{2} \mu \left(\int_{\Omega} (1 + u^2)^{\frac{p}{2}} |\nabla u|^p \, dx \right)^{\frac{2}{p}} + H(u) \\ &= \frac{1}{2} \mu \left(\int_{\Omega} (1 + u^2)^{\frac{p}{2}} |\nabla u|^p \, dx \right)^{\frac{2}{p}} + \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^N a_{ij}(x, u) D_i u D_j u \, dx + \frac{1}{r} \int_{\Omega} a_- |u|^r \, dx - \frac{1}{s} \int_{\Omega} a_+ |u|^s \, dx. \end{aligned}$$

Note that H_{μ} is defined on the Sobolev space $W_0^{1,p}(\Omega)$ with $p > N$. It is a C^1 -functional on $W_0^{1,p}(\Omega)$, and satisfies

$$\begin{aligned} \langle DH_{\mu}(u), \varphi \rangle &= \mu \left[\int_{\Omega} (1 + u^2)^{\frac{p}{2}} |\nabla u|^p \, dx \right]^{\frac{2}{p}-1} \cdot \int_{\Omega} [(1 + u^2)^{\frac{p}{2}} |\nabla u|^{p-2} \nabla u \nabla \varphi + (1 + u^2)^{\frac{p}{2}-1} u |\nabla u|^p \varphi] \, dx \\ &\quad + \int_{\Omega} \sum_{i,j=1}^N a_{ij}(x, u) D_i u D_j \varphi \, dx + \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^N D_s a_{ij}(x, u) D_i u D_j \varphi \, dx \\ &\quad + \int_{\Omega} a_- |u|^{r-2} u \varphi \, dx - \int_{\Omega} a_+ |u|^{s-2} u \varphi \, dx \end{aligned}$$

for $\varphi \in W_0^{1,p}(\Omega)$.

As we have seen in the preceding section, for the quasilinear equations with indefinite nonlinearity, compared with ones with definite nonlinearity, the difficulty is to prove the boundedness of some associated sequences, either the sequence of approximate solutions (Lemmas 2.1 and 2.2) or the Palais–Smale sequence of the perturbed functional (Lemma 3.1). When we have proved the boundedness of these sequences, we can deal with the quasilinear equations as before to obtain the convergence and the existence results.

In the following, we will prove the boundedness of sequences of the approximate solutions, and the boundedness of the Palais–Smale sequences of the functional H_{μ} .

Lemma 4.1. *For $H_{\mu}(u)$, there holds*

$$\begin{aligned} &\mu \left(\int_{\Omega} (1 + u^2)^{\frac{p}{2}} |\nabla u|^p \, dx \right)^{\frac{2}{p}} + \int_{\Omega} (1 + u^2) |\nabla u|^2 \, dx + \int_{\Omega} a_- |u|^r \, dx + \int_{\Omega} a_+ |u|^s \, dx \\ &\leq C \left[H_{\mu}(u) + \|DH_{\mu}(u)\| \|u\| + \mu \left(\int_{\Omega} (1 + u^2)^{\frac{p}{2}} |u|^p \, dx \right)^{\frac{2}{p}} + \int_{\Omega} (1 + u^2) u^2 \, dx \right]. \end{aligned} \quad (4.1)$$

Proof. The proof is closely analogous to the one of Lemma 2.1. Choose $\psi \in C_0^\infty(\mathbb{R}^N)$ such that $\psi \geq 0$, $\psi(x) = 1$ for $x \in \Omega_-$, and $\psi(x) = 0$ for $x \in \Omega_+$. Taking $\varphi = u\psi^p$ as the test function, we have

$$\begin{aligned} & \mu \left(\int_{\Omega} (1+u^2)^{\frac{p}{2}} |\nabla u|^p dx \right)^{\frac{2}{p}-1} \int_{\Omega} (1+u^2)^{\frac{p}{2}-1} (1+2u^2) |\nabla u|^p \psi^p dx \\ & + \int_{\Omega} \sum_{i,j=1}^N \left(a_{ij}(x, u) + \frac{1}{2} u D_s a_{ij}(x, u) \right) D_i u D_j u \psi^p dx + \int_{\Omega} a_- |u|^r dx \\ & = \langle DH_{\mu}(u), u\psi^p \rangle - p\mu \left(\int_{\Omega} (1+u^2)^{\frac{p}{2}} |\nabla u|^p dx \right)^{\frac{2}{p}-1} \int_{\Omega} (1+u^2)^{\frac{p}{2}} |\nabla u|^{p-2} \nabla u u \psi^{p-1} \nabla \psi dx \\ & - p \int_{\Omega} \sum_{i,j=1}^N a_{ij}(x, u) D_i u u \psi^{p-1} D_j \psi dx \\ & \leq \mu \left(\int_{\Omega} (1+u^2)^{\frac{p}{2}} |\nabla u|^p dx \right)^{\frac{2}{p}-1} \left[\varepsilon \int_{\Omega} (1+u^2)^{\frac{p}{2}} |\nabla u|^p \psi^p dx + C \int_{\Omega} (1+u^2)^{\frac{p}{2}} |u|^p |\nabla \psi|^p dx \right] + C \|CH_{\mu}(u)\| \|u\| \\ & + \varepsilon \int_{\Omega} \sum_{i,j=1}^N a_{ij}(x, u) D_i u D_j u \psi^p dx + C \int_{\Omega} \sum_{i,j=1}^N a_{ij}(x, u) D_i \psi D_j \psi u^2 \psi^{p-2} dx. \end{aligned}$$

In view of assumptions (a2) and (a3), it follows from the Sobolev inequality (2.3) that

$$\int_{\Omega} a_- |u|^r dx \leq C \left[\|DH_{\mu}(u)\| \|u\| + \mu \left(\int_{\Omega} (1+u^2)^{\frac{p}{2}} |u|^p dx \right)^{\frac{2}{p}} + \int_{\Omega} (1+u^2) u^2 dx \right]. \quad (4.2)$$

Hence, we further have

$$\begin{aligned} H_{\mu}(u) - \frac{1}{q} \langle DH_{\mu}(u), u \rangle &= -\frac{1}{q} \mu \left(\int_{\Omega} (1+u^2)^{\frac{p}{2}} |\nabla u|^p dx \right)^{\frac{2}{p}-1} \int_{\Omega} (1+2u^2)^{\frac{2}{p}-1} (1+2u^2) |\nabla u|^p dx \\ &+ \int_{\Omega} \sum_{i,j=1}^N \left(\frac{1}{2} a_{ij}(x, u) - \frac{1}{q} \left(a_{ij}(x, u) + \frac{1}{2} u D_s a_{ij}(x, u) \right) \right) D_i u D_j u dx \\ &+ \frac{1}{2} \mu \left(\int_{\Omega} (1+u^2)^{\frac{p}{2}} |\nabla u|^p dx \right)^{\frac{2}{p}} - \left(\frac{1}{q} - \frac{1}{r} \right) \int_{\Omega} a_- |u|^r dx + \left(\frac{1}{q} - \frac{1}{s} \right) \int_{\Omega} a_+ |u|^s dx \\ &\geq \left(\frac{1}{2} - \frac{2}{q} \right) \left(\int_{\Omega} (1+u^2)^{\frac{p}{2}} |\nabla u|^p dx \right)^{\frac{2}{p}} + \delta \int_{\Omega} \sum_{i,j=1}^N a_{ij}(x, u) D_i u D_j u dx \\ &- \left(\frac{1}{q} - \frac{1}{r} \right) \int_{\Omega} a_- |u|^r dx + \left(\frac{1}{q} - \frac{1}{s} \right) \int_{\Omega} a_+ |u|^s dx. \end{aligned} \quad (4.3)$$

By virtue of (4.2) and (4.3), we arrive at (4.1). $\square$

Lemma 4.2. Assume that $\mu_n > 0$, $\mu_n \rightarrow 0$, $u_n \in W_0^{1,p}(\Omega)$, $DH_{\mu_n}(u_n) = 0$ and $H_{\mu_n}(u_n) \leq C$. Then there exists a constant C independent of n such that

$$\mu_n \left(\int_{\Omega} (1+u_n^2)^{\frac{p}{2}} |u_n|^p dx \right)^{\frac{2}{p}} + \int_{\Omega} (1+u_n^2) u_n^2 dx \leq C.$$

Moreover, we have

$$\mu_n \left(\int_{\Omega} (1+u_n^2)^{\frac{p}{2}} |\nabla u_n|^p dx \right)^{\frac{2}{p}} + \int_{\Omega} (1+u_n^2) |\nabla u_n|^2 dx + \int_{\Omega} a_- |u_n|^r dx + \int_{\Omega} a_+ |u_n|^s dx \leq C.$$

Proof. As in the proof of Lemma 2.2, we apply the indirect argument by assuming that

$$\rho_n^4 = \mu_n \left(\int_{\Omega} (1+u_n^2)^{\frac{p}{2}} |u_n|^p dx \right)^{\frac{2}{p}} + \int_{\Omega} (1+u_n^2) |u_n|^2 dx \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

By Lemma 4.1, we get

$$\mu_n \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p dx \right)^{\frac{2}{p}} + \int_{\Omega} (1 + u_n^2) |\nabla u_n|^2 dx + \int_{\Omega} a_- |u_n|^r dx + \int_{\Omega} a_+ |u_n|^s dx \leq C \rho_n^4. \quad (4.4)$$

Let $v_n = \rho_n^{-1} u_n$. Then $v_n \rightarrow u$ in $L^q(\Omega)$, $1 \leq q < \frac{4N}{N-2}$ and $v_n Dv_n \rightharpoonup v \nabla v$ in $L^2(\Omega)$, where $v^2 \in H_0^1(\Omega_0) \subset H_0^1(\Omega)$. Given $\psi \geq 0$ and $\psi \in C_0^\infty(\Omega_0)$, we set $\varphi_n = \psi u_n / (M + u_n^2)$ with M being the constant given in condition (a₃). From condition (a₃) we have

$$\begin{aligned} 0 &= \langle DH_{\mu_n}(u_n), \varphi_n \rangle \\ &= \mu_n \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p dx \right)^{\frac{2}{p}-1} \int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^{p-2} \nabla u_n \nabla \psi \frac{u_n}{M + u_n^2} dx \\ &\quad + \mu_n \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p dx \right)^{\frac{2}{p}-1} \int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p \psi \\ &\quad \cdot \left(\frac{M}{(M + u_n^2)^2} + \frac{(M-1)u_n^2}{(1 + u_n^2)(M + u_n^2)^2} \right) dx + \int_{\Omega} \sum_{i,j=1}^N \frac{a_{ij}(x, u_n)}{M + u_n^2} u_n D_i u_n D_j \psi dx \\ &\quad + \int_{\Omega} \sum_{i,j=1}^N \left(a_{ij}(x, u_n) \frac{M - u_n^2}{(M + u_n^2)^2} + \frac{1}{2} u_n D_s a_{ij}(x, u_n) \frac{1}{M + u_n^2} \right) D_i u_n D_j u_n \psi dx \\ &\geq \mu_n \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p dx \right)^{\frac{2}{p}-1} \int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^{p-2} \nabla u_n \nabla \psi \frac{u_n}{M + u_n^2} dx \\ &\quad + \int_{\Omega} \sum_{i,j=1}^N \frac{a_{ij}(x, u_n)}{M + u_n^2} u_n D_i u_n D_j u_n \psi dx. \end{aligned}$$

By (4.4), we find

$$\begin{aligned} &\rho_n^{-2} \mu_n \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p dx \right)^{\frac{2}{p}-1} \left| \int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^{p-2} \nabla u_n \nabla \psi \frac{u_n}{M + u_n^2} dx \right| \\ &\leq \rho_n^{-2} \mu_n \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p dx \right)^{\frac{1}{p}} \left(\int_{\Omega} |\nabla \psi|^p dx \right)^{\frac{1}{p}} \\ &\leq C \mu_n^{\frac{1}{2}} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

It follows from Lemma 4.3 that

$$\rho_n^{-2} \int_{\Omega} \sum_{i,j=1}^N \frac{a_{ij}(x, u_n)}{M + u_n^2} u_n \partial_i u_n \partial_j \psi dx = \int_{\Omega} \sum_{i,j=1}^N \frac{a_{ij}(x, u_n)}{M + u_n^2} v_n \partial_i v_n \partial_j \psi dx \rightarrow \int_{\Omega} \sum_{i,j=1}^N A_{ij}(x) v \partial_i v \partial_j \psi dx \quad \text{as } n \rightarrow \infty.$$

That is,

$$\int_{\Omega} \sum_{i,j=1}^N A_{ij}(x) v \partial_i v \partial_j \psi dx \leq 0$$

for $\psi \geq 0$ and $\psi \in C_0^\infty(\Omega_0)$.

Since $v^2 \in H_0^1(\Omega_0) \subset H_0^1(\Omega)$, by the density argument we have

$$\int_{\Omega} \sum_{i,j=1}^N A_{ij}(x) v \partial_i v \partial_j v^2 dx = 0$$

and $v^2 \equiv 0$ in Ω . The remainder is the same as that shown in Lemma 2.2, so we omit it here. $\square$

Lemma 4.3. For $\psi \in H_0^1(\Omega)$, there holds

$$\lim_{n \rightarrow \infty} \rho_n^{-2} \int_{\Omega} \sum_{i,j=1}^N \frac{a_{ij}(x, u_n)}{M + u_n^2} u_n \partial_i u_n \partial_j \psi dx = \int_{\Omega} \sum_{i,j=1}^N A_{ij}(x) v \partial_i v \partial_j \psi dx.$$

Proof. For $T > 0$, let u_n^T be the truncated function of u_n , that is, $u_n^T = u_n$ if $|u_n| \leq T$, and $u_n^T = \pm T$ if $\pm u_n \geq T$. Taking u_n^T as the test function, we have

$$\begin{aligned} 0 &= \langle DH_{\mu_n}(u_n), u_n^T \rangle \\ &= \mu_n \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p dx \right)^{\frac{2}{p}-1} \int_{\Omega} [(1 + u_n^2)^{\frac{p}{2}} |\nabla u_n^T|^p + (1 + u_n^2)^{\frac{p}{2}-1} |\nabla u_n|^2 u_n u_n^T] dx \\ &\quad + \int_{\Omega} \sum_{i,j=1}^N \left(a_{ij}(x, u_n) + \frac{1}{2} u_n D_s a_{ij}(x, u_n) \right) \partial_i u_n^T \partial_j u_n^T dx + \frac{1}{2} \int_{|u_n| \geq T} \sum_{i,j=1}^N u_n^T D_s a_{ij}(x, u_n) D_i u_n D_j u_n dx \\ &\quad + \int_{\Omega} a_- |u_n|^{r-2} u_n u_n^T dx - \int_{\Omega} a_+ |u_n|^{s-2} u_n u_n^T dx. \end{aligned} \quad (4.5)$$

In view of assumption (a_3) , there exists a $T_0 > 0$ such that

$$\sum_{i,j=1}^N s D_s a_{ij}(x, s) \xi_i \xi_j \geq 0$$

for $|s| \geq T > T_0$ and $x \in \overline{\Omega}$, where $\xi \in \mathbb{R}^N$. By condition (a2) and (4.5), we get

$$\begin{aligned} \int_{\Omega} |\nabla u_n^T|^2 dx &\leq C \int_{\Omega} \sum_{i,j=1}^N \left(a_{ij}(x, u_n) + \frac{1}{2} u_n D_s a_{ij}(x, u_n) \right) \partial_i u_n^T \partial_j u_n^T dx \\ &\leq C \int_{\Omega} a_+ |u_n|^{s-2} u_n u_n^T dx \leq CT \int_{\Omega} a_+ |u_n|^{s-1} dx \\ &\leq CT \left(\int_{\Omega} a_+ |u_n|^s dx \right)^{\frac{s-1}{s}} \\ &\leq CT \rho_n^{4-\frac{4}{s}}. \end{aligned}$$

Hence, we obtain the estimate

$$\begin{aligned} &\rho_n^{-2} \int_{\Omega} \sum_{i,j=1}^N \frac{a_{ij}(x, u_n)}{M + u_n^2} u_n \partial_i u_n \partial_j \psi dx \\ &= \rho_n^{-2} \int_{|u_n| \geq T} \sum_{i,j=1}^N \frac{a_{ij}(x, u_n)}{M + u_n^2} u_n \partial_i u_n \partial_j \psi dx + \rho_n^{-2} \int_{|u_n| < T} \sum_{i,j=1}^N \frac{a_{ij}(x, u_n)}{M + u_n^2} u_n \partial_i u_n \partial_j \psi dx \\ &= \rho_n^{-2} \left(\int_{|u_n| \geq T} A_{ij}(x) u_n \partial_i u_n \partial_j \psi dx + o_T(1) \|u_n \nabla u_n\|_{L^2(\Omega)} \right) + \rho_n^{-2} CT \|\nabla u_n^T\|_{L^2(\Omega)} \\ &= \int_{\Omega} A_{ij}(x) v_n \partial_i v_n \partial_j \psi dx + o_T(1) + \rho_n^{-2} CT^{\frac{3}{2}} \rho_n^{2-\frac{2}{s}} \\ &= \int_{\Omega} A_{ij}(x) v_n \partial_i v_n \partial_j \psi dx + o_n(1) \rightarrow \int_{\Omega} A_{ij}(x) v \partial_i v \partial_j \psi dx \quad \text{for } \psi \in H_0^1(\Omega). \end{aligned} \quad \square$$

The following proposition can be regarded as a counterpart of Theorem 1.1.

Proposition 4.4. Assume that $\mu_n > 0$, $\mu_n \rightarrow 0$, $u_n \in C_0^1(\Omega)$, $DH_{\mu_n}(u_n) = 0$ and $H_{\mu_n}(u_n) \leq C$. Then the following assertions hold:

- (i) There exists a function $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ satisfying equation (1.4).
- (ii) Up to a subsequence, there holds

$$\|u_n\|_{L^\infty} \leq C, \quad u_n \rightarrow u \text{ in } H_0^1(\Omega), \quad \mu_n \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p dx \right)^{\frac{2}{p}} \rightarrow 0,$$

and

$$H_{\mu_n}(u_n) \rightarrow H(u_n).$$

By Lemma 4.2, there holds

$$\mu_n \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p \, dx \right)^{\frac{2}{p}} + \int_{\Omega} (1 + u_n^2) |\nabla u_n|^2 \, dx \leq C. \quad (4.6)$$

Proposition 4.4 can be proved similarly to Theorem 1.1, so we omit it and refer to [11]. As we have seen in the proof of Theorem 1.1, with the help of estimate (4.6), the proof of Theorem 1.1 can also be done in the same way as that for quasilinear equations with definite nonlinearity.

Next, we consider the perturbed functional H_{μ} with $\mu \in (0, 1]$.

Lemma 4.5. *Let $\{u_n\}$ be a Palais–Smale sequence of the functional H_{μ} with $\mu > 0$. Then u_n is bounded in $W_0^{1,p}(\Omega)$.*

Proof. By Lemma 4.1, we only need to prove

$$\mu \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |u_n|^p \, dx \right)^{\frac{2}{p}} + \int_{\Omega} (1 + u_n^2) |u_n|^2 \, dx \leq C.$$

Otherwise, we have

$$\rho_n^4 = \mu \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |u_n|^p \, dx \right)^{\frac{2}{p}} + \int_{\Omega} (1 + u_n^2) |u_n|^2 \, dx \rightarrow +\infty.$$

Let

$$v_n = \rho_n^{-1} u_n \text{ and } \left(\int_{\Omega} |v_n|^p |\nabla v_n|^p \, dx \right)^{\frac{2}{p}} = \rho_n^{-4} \left(\int_{\Omega} |u_n|^p |\nabla u_n|^p \, dx \right)^{\frac{2}{p}} \leq C.$$

Then $v_n \rightarrow v$ in $L^q(\Omega)$, $1 \leq q < \frac{4N}{N-2}$, $v_n^2 \rightarrow v^2$ in $C^{\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$, and $v_n^2 \rightharpoonup v^2$ in $W_0^{1,p}(\Omega)$ (and $H_0^1(\Omega)$). As in the proof of Lemma 2.2, we know that $v(x) = 0$ for $x \in \overline{\Omega}_+ \cap \overline{\Omega}_-$, and $v^2 \in W_0^{1,p}(\Omega_0) \subset W_0^{1,p}(\Omega)$. For $\psi \geq 0$ and $\psi \in W_0^{1,p}(\Omega_0)$, we let $\varphi_n = \psi u_n / (M + u_n^2)$, $\varphi_n \in W_0^{1,p}(\Omega)$ and

$$\nabla \varphi_n = \nabla \psi \frac{u_n}{M + u_n^2} + \psi \frac{M - u_n^2}{(M + u_n^2)^2} \nabla u_n, \quad \|\varphi_n\| \leq C \rho_n^2 \|\psi\|.$$

Then we deduce that

$$\begin{aligned} o(1) \|\psi\| &= \rho_n^{-2} \langle DH_{\mu}(u_n), \varphi_n \rangle \\ &= \rho_n^{-2} \mu \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p \, dx \right)^{\frac{2}{p}-1} \int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^{p-2} \nabla u_n \nabla \psi \frac{u_n}{M + u_n^2} \, dx \\ &\quad + \rho_n^{-2} \mu \left(\int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p \, dx \right)^{\frac{2}{p}-1} \\ &\quad \cdot \int_{\Omega} (1 + u_n^2)^p |\nabla u_n|^p \psi \left(\frac{M}{(M + u_n^2)^2} + \frac{(M-1)u_n^2}{(1 + u_n^2)(M + u_n^2)^2} \right) \, dx \\ &\quad + \rho_n^{-2} \int_{\Omega} \sum_{i,j=1}^N \frac{a_{ij}(x, u_n)}{M + u_n^2} u_n \partial_i u_n \partial_j \psi \, dx \\ &\quad + \rho_n^{-2} \int_{\Omega} \sum_{i,j=1}^N \left(a_{ij}(x, u_n) \frac{M - u_n^2}{(M + u_n^2)^2} + \frac{1}{2} D_s a_{ij}(x, u_n) \frac{u_n}{M + u_n^2} \right) D_i u_n D_j u_n \psi \, dx \\ &\geq \mu \left(\int_{\Omega} (\rho_n^{-2} + v_n^2)^{\frac{p}{2}} |\nabla v_n|^p \, dx \right)^{\frac{2}{p}-1} \int_{\Omega} (\rho_n^{-2} + v_n^2)^{\frac{p}{2}} |\nabla v_n|^{p-2} \nabla v_n \nabla \psi \frac{v_n}{M + u_n^2} \, dx \\ &\quad + \int_{\Omega} \sum_{i,j=1}^N \frac{a_{ij}(x, u_n)}{M + v_n^2} v_n D_i v_n D_j \psi \, dx. \end{aligned}$$

Choose $\psi = \frac{1}{2}(v_n^2 - C_n)_+ \in W_0^{1,p}(\Omega_0)$ and $C_n = \max_{\partial\Omega_0} v_n^2 \rightarrow 0$. By the lower semi-continuity, we obtain

$$0 \geq \mu \left(\int_{\Omega} |v|^p |\nabla v|^p dx \right)^{\frac{2}{p}} + c \int_{\Omega} v^2 |\nabla v|^2 dx.$$

Hence, $v^2 = 0$. This yields a contradiction according to Lemma 3.1. $\square$

Proposition 4.6. *The functional H_{μ} with $\mu > 0$ satisfies the Palais–Smale condition.*

Proof. Let $\{u_n\}$ be a Palais–Smale sequence of H_{μ} . By Lemma 4.5, u_n is bounded in $W_0^{1,p}(\Omega)$. Assume that $u_n \rightarrow u$ in $C^{\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$. Set

$$C_0 = \lim_{n \rightarrow \infty} \int_{\Omega} (1 + u_n^2)^{\frac{p}{2}} |\nabla u_n|^p dx.$$

If $C_0 = 0$, we are done. Otherwise, there holds

$$\begin{aligned} o(1) &= \langle DH_{\mu}(u_n) - DH_{\mu}(u_m), u_n - u_m \rangle \\ &= \mu C_0^{\frac{2}{p}-1} \int_{\Omega} (1 + u^2)^{\frac{p}{2}} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_m|^{p-2} \nabla u_m, \nabla u_n - \nabla u_m) dx \\ &\quad + \int_{\Omega} \sum_{i,j=1}^N a_{ij}(x, u) D_i(u_n - u_m) D_j(u_n - u_m) dx + o(1) \\ &\geq c \int_{\Omega} |\nabla u_n - \nabla u_m|^p dx + o(1). \end{aligned}$$

So, $\{u_n\}$ is a Cauchy sequence of $W_0^{1,p}(\Omega)$. $\square$

Proof of Theorem 1.3. Define

$$c_k(\mu) = \inf_{\varphi \in \Gamma_k} \sup_{t \in B_k} H_{\mu}(\varphi(t)), \quad k = 1, 2, \dots,$$

where

$$\Gamma_k = \{\varphi \mid \varphi \in C(B_k, W_0^{1,p}(\Omega)), \varphi \text{ is odd}, H_1(\varphi(t)) < 0 \text{ for } t \in \partial B_k\}.$$

A straightforward estimate on $H_k(u)$ gives

$$\begin{aligned} H_k(u) &= \frac{1}{2} \mu \left(\int_{\Omega} (1 + u^2)^{\frac{p}{2}} |\nabla u|^p dx \right)^{\frac{2}{p}} + H(u) \\ &\geq \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^N a_{ij}(x, u) D_i u D_j u dx + \frac{1}{r} \int_{\Omega} a_- |u|^r dx - \frac{1}{s} a_+ |u|^s dx \\ &\geq \frac{c_1}{2} \int_{\Omega} (1 + u^2) |\nabla u|^2 dx - \frac{1}{s} \int_{\Omega} a_+ |u|^s dx \\ &\geq \frac{c_1}{2} \int_{\Omega} |\nabla w|^2 dx - c \int_{\Omega} |w|^{\frac{s}{2}} dx \\ &:= J(w). \end{aligned}$$

One can find $\alpha_k \leq \beta_k$ such that $\alpha_k \rightarrow \infty$ as $k \rightarrow \infty$, and $\alpha_k \leq c_k(\mu) \leq \beta_k$.

Let

$$c_k = \lim_{\mu \rightarrow 0} c_k(\mu).$$

According to Proposition 4.4, c_k , $k = 1, 2, \dots$, are critical values of H and $c_k \rightarrow +\infty$ as $k \rightarrow \infty$. Consequently, the proof is completed. $\square$

Funding: Junfang Zhao is supported by National Science Foundation of China under No. 11601493 and Beijing Higher Education Young Elite Faculty Project. Xiangqing Liu is supported by National Science Foundation of China under No. 11761082 and Yunnan Young Academic and Technical Leaders' Program (2015HB028). This work is also partially supported by UTRGV Faculty Research Council Award 110000327.

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