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Concentrating solutions for a planar elliptic problem with large nonlinear exponent and Robin boundary condition

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Abstract: Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and $b(x) > 0$ a smooth function defined on $\partial\Omega$. We study the following Robin boundary value problem:

$$\begin{cases} \Delta u + u^p = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \lambda b(x)u = 0 & \text{on } \partial\Omega, \end{cases}$$

where ν denotes the exterior unit vector normal to $\partial\Omega$, $0 < \lambda < +\infty$ and $p > 1$ is a large exponent. We construct solutions of this problem which exhibit concentration as $p \rightarrow +\infty$ and simultaneously as $\lambda \rightarrow +\infty$ at points that get close to the boundary, and show that in general the set of solutions of this problem exhibits a richer structure than the problem with Dirichlet boundary condition.

Keywords: Concentrating solutions, large exponent, Robin boundary condition, finite-dimensional reduction

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1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and $b(x) > 0$ a smooth function defined on $\partial\Omega$. This paper deals with the analysis of solutions of the Robin boundary value problem

$$\begin{cases} \Delta u + u^p = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \lambda b(x)u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where ν denotes the exterior unit vector normal to $\partial\Omega$, $0 < \lambda < +\infty$ and $p > 1$ is a large exponent. Some solutions of (1.1) can be obtained as appropriately scaled extremals of

$$S_p^\lambda = \inf_{u \in H^1(\Omega) \setminus \{0\}} I_p^\lambda(u) \quad \text{where} \quad I_p^\lambda(u) = \frac{\int_\Omega |\nabla u|^2 + \lambda \int_{\partial\Omega} b(x)u^2}{(\int_\Omega |u|^{p+1})^{2/(p+1)}},$$

which are guaranteed to exist thanks to the compactness of the Sobolev embedding $H^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$. They are known as least energy solutions of (1.1).

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Problem (1.1) has different names depending on the different values of the parameter λ . It is called Dirichlet if $\lambda = +\infty$, Neumann if $\lambda = 0$ and Robin if $0 < \lambda < +\infty$. From integration by parts it is trivial to observe that problem (1.1) has no solution when $\lambda = 0$. It is also worth pointing out that the Robin boundary condition in (1.1) is particularly interesting in various branches of biological models (see [10, 16]). Formally, as $\lambda \rightarrow +\infty$, the Robin boundary condition in (1.1) tends to the Dirichlet boundary condition $u|_{\partial\Omega} = 0$, and problem (1.1) becomes

$$\begin{cases} \Delta u + u^p = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.2)$$

Standard variational methods show that problem (1.2) always has a least energy solution u_p . In [20, 21] it is proved that as $p \rightarrow +\infty$, the solution u_p develops one interior peak, namely u_p approaches zero except one interior point where it stays bounded and bounded away from zero. Furthermore, up to a subsequence, the renormalized energy pu_p^{p+1} concentrates as a Dirac mass around a critical point of the Robin function $H_\infty(x, x)$, where H_∞ is the regular part of Green's function G_∞ of the Dirichlet Laplacian in Ω . Namely, Green's function $G_\infty(x, y)$ is the solution of the problem

$$\begin{cases} -\Delta_x G_\infty(x, y) = \delta_y(x) & \text{in } \Omega, \\ G_\infty(x, y) = 0 & \text{on } \partial\Omega, \end{cases}$$

and $H_\infty(x, y)$ is the regular part defined as

$$H_\infty(x, y) = G_\infty(x, y) - \frac{1}{2\pi} \log \frac{1}{|x - y|}.$$

In [1] and [11] Adimurthi and Grossi, and El Mehdi and Grossi, respectively, give a further description of the asymptotic behavior of u_p as $p \rightarrow +\infty$, by identifying a limit profile problem of Liouville-type, that is,

$$\begin{cases} \Delta u + e^u = 0 & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^u < +\infty, \end{cases}$$

and show that $\|u_p\|_\infty \rightarrow \sqrt{e}$ as $p \rightarrow +\infty$. On the other hand, problem (1.2) may have solutions with an arbitrarily large number of interior peaks, as shown by Esposito, Musso and Pistoia in [13]. Indeed, they proved that if Ω is not simply connected, then given any $m \geq 1$ and for p large enough, problem (1.2) has a family of solutions u_p which concentrates at m different points of Ω , namely

$$pu_p^{p+1} \rightharpoonup 8\pi e \sum_{i=1}^m \delta_{\xi_i} \quad \text{weakly in the sense of measure in } \overline{\Omega} \quad (1.3)$$

as $p \rightarrow +\infty$, and the peaks of these solutions are located near the points $\xi = (\xi_1, \dots, \xi_m) \in \Omega^m$ corresponding to a critical point of the function

$$\varphi_m^\infty(\xi_1, \dots, \xi_m) = \sum_{j=1}^m H_\infty(\xi_j, \xi_j) + \sum_{j=k} G_\infty(\xi_j, \xi_k).$$

More precisely, there is an m -tuple $\xi^p = (\xi_{1,p}, \dots, \xi_{m,p}) \in \Omega^m$ converging to ξ such that for any $d > 0$, as $p \rightarrow +\infty$,

$$u_p \rightarrow 0 \quad \text{uniformly in } \Omega \setminus \bigcup_{i=1}^m B_d(\xi_i^p) \quad \text{and} \quad \sup_{x \in \Omega \cap B_d(\xi_i^p)} u_p(x) \rightarrow \sqrt{e}.$$

Additionally, Grossi and Takahashi in [15] prove that when Ω is convex, problem (1.2) has no solutions that have multiple interior peaks and satisfy property (1.3). Thus, the topological assumption on the domain in [13] is sharp for the construction of multi-peak solutions to problem (1.2).

In contrast, we will see that for any bounded smooth domain, when λ and p are large enough, the set of solutions of the so called Robin problem (1.1) is much richer.

Given $\lambda > 0$, let $G_\lambda(x, y)$ be Green's function with Robin boundary condition, i.e.,

$$\begin{cases} -\Delta_x G_\lambda(x, y) = \delta_y(x) & \text{in } \Omega, \\ \frac{\partial G_\lambda}{\partial \nu}(x, y) + \lambda b(x) G_\lambda(x, y) = 0 & \text{on } \partial\Omega, \end{cases}$$

and let $H_\lambda(x, y)$ be its regular part defined as

$$H_\lambda(x, y) = G_\lambda(x, y) - \frac{1}{2\pi} \log \frac{1}{|x - y|}. \quad (1.4)$$

As in the Dirichlet problem (1.2), critical points of the Robin function $H_\lambda(x, x)$ may play a similar role in determining the location where solutions of problem (1.1) exhibit single-point concentration phenomena. From [7] the Robin function $H_\lambda(x, x)$ admits two precise asymptotic formulas, that is, if we set $d(x) := \text{dist}(x, \partial\Omega)$ and introduce the notation $\hat{x}$ as the point on $\partial\Omega$ closest to x , then for any $\alpha > 1$, there exists λ_α such that for each $\lambda \geq \lambda_\alpha$ and for each $x \in \Omega$ satisfying $\alpha^{-1} < \lambda d(x) < \alpha$, one has that if b is not a constant function,

$$H_\lambda(x, x) = \frac{1}{2\pi} h_\lambda(\lambda d(x), b(\hat{x})) + O\left(\frac{1}{\lambda}\right), \quad (1.5)$$

while if $b \equiv 1$,

$$H_\lambda(x, x) = \frac{1}{2\pi} h_\lambda(\lambda d(x), 1) + \frac{1}{2\pi\lambda} \kappa(\hat{x}) v(\lambda d(x)) + O\left(\frac{1}{\lambda^{1+\alpha}}\right), \quad (1.6)$$

where $0 < \alpha < 1$, $\kappa(\hat{x})$ is the mean curvature of $\partial\Omega$ at $\hat{x}$

$$h_\lambda(\theta, \tilde{b}) = -\log \lambda - \log 2\theta + 2 \int_0^\infty e^{-t} \log\left(2\theta + \frac{t}{\tilde{b}}\right) dt \quad (1.7)$$

and

$$v(\theta) = -\frac{\theta}{2} - \theta \int_0^\infty e^{-2\theta s} \frac{1}{(1+s)^2} ds. \quad (1.8)$$

Moreover, for any $\tilde{b} > 0$ the function $h_\lambda(\cdot, \tilde{b})$ has a unique minimum $\theta_{\tilde{b}} \in (0, +\infty)$, which is non-degenerate and independent of λ . Thus, formulas (1.5)–(1.6) suggest that for λ and p large enough, problem (1.1) may have solutions which exhibit one or multiple points concentration phenomena in the intermediate region $\lambda d(x) = O(1)$. Our interest here is to find whether these solutions of problem (1.1) exist as $p \rightarrow +\infty$ and simultaneously as $\lambda \rightarrow +\infty$.

The first result we will establish is that when p and λ satisfy some restrained growth conditions, for any $m \geq 1$, problem (1.1) possesses at least two solutions which concentrate at m different points with distance to the boundary uniformly approaching zero. Meanwhile, the location of such m points is related to critical points of the function

$$\varphi_m^\lambda(\xi_1, \dots, \xi_m) = \sum_{j=1}^m H_\lambda(\xi_j, \xi_j) + \sum_{j=k}^m G_\lambda(\xi_j, \xi_k). \quad (1.9)$$

Theorem 1.1. *Let $m \geq 1$ be an integer. There exist $p_m > 0$ and $\lambda_m > 0$ such that for any $p > p_m$ and $\lambda > \lambda_m$ satisfying $\frac{\log \lambda}{p} = o(1)$ and $\frac{\lambda}{p^2 \log \lambda} = O(1)$, there are at least two solutions $u_{p,\lambda}$ for problem (1.1) with m different concentration points $\xi_{1,p,\lambda}, \dots, \xi_{m,p,\lambda}$ located at distance $O(1/\lambda)$ from $\partial\Omega$. More precisely,*

$$u_{p,\lambda}(x) = \sum_{j=1}^m \frac{1}{y \mu_j^{2/(p-1)}} \left[\log \frac{1}{(\delta_j^2 + |x - \xi_{j,p,\lambda}|^2)^2} + 8\pi H_\lambda(x, \xi_{j,p,\lambda}) \right] + O\left(\frac{1}{p}\right),$$

where the parameters y , δ_j and μ_j satisfy

$$y = p^{p/(p-1)} \rho^{2/(p-1)}, \quad \delta_j = \mu_j \rho, \quad \rho = \frac{1}{\lambda} e^{-p/4}, \quad \frac{1}{C} < \mu_j < C$$

for some $C > 0$, and $\xi_{j,p,\lambda} \in \Omega$ satisfies

$$|\lambda d(\xi_{j,p,\lambda}) - \theta_{b(\xi_{j,p,\lambda})}| = O(\lambda^{-1/2}) \quad \text{as } \lambda \rightarrow +\infty$$

and

$$\lim_{p \rightarrow +\infty} \nabla \varphi_m^\lambda(\xi_{1,p,\lambda}, \dots, \xi_{m,p,\lambda}) = 0 \quad \text{uniformly for all large } \lambda,$$

where φ_m^λ is defined in (1.9). In particular, for any $\varepsilon \in (0, 1)$, as $\lambda \rightarrow +\infty$,

$$\begin{aligned} p u_{p,\lambda}^{p+1} - 8\pi e \sum_{j=1}^m \delta_{\xi_{j,p,\lambda}} &\rightharpoonup 0 \quad \text{weakly in the sense of measure in } \bar{\Omega}, \\ u_{p,\lambda} &\rightarrow 0 \quad \text{uniformly in } \bar{\Omega} \setminus \bigcup_{j=1}^m B_{\varepsilon d(\xi_{j,p,\lambda})}(\xi_{j,p,\lambda}), \end{aligned}$$

and

$$\sup_{B_{\varepsilon d(\xi_{j,p,\lambda})}(\xi_{j,p,\lambda})} u_{p,\lambda} \rightarrow \sqrt{e}.$$

Note that as in the Dirichlet problem (1.2), when Ω is not simple connected, one can find that given any $m \geq 1$, for any fixed λ large, problem (1.1) has a third solution $u_{p,\lambda}$ with m different peak points $\xi_{1,p,\lambda}, \dots, \xi_{m,p,\lambda}$ uniformly away from the boundary and each other as $p \rightarrow +\infty$. We will not provide the construction of this solution because it can be easily realized using the same approach as in [13]. Besides, under a faster restrained growth condition of p and λ , we also obtain the case $m = 1$ of Theorem 1.1 and find solutions with a concentration point near the boundary.

Theorem 1.2. *There exist $p_0 > 0$ and $\lambda_0 > 0$ such that for any $p > p_0$ and $\lambda > \lambda_0$ satisfying $\frac{1}{p} \log \lambda = o(1)$, problem (1.1) has at least two solutions $u_{p,\lambda}$ with a concentration point $\xi_{p,\lambda}$ such that*

$$|\lambda d(\xi_{p,\lambda}) - \theta_{b(\xi_{p,\lambda})}| = O(\lambda^{-1/2}) \quad \text{as } \lambda \rightarrow +\infty.$$

The following theorems deal with the case $b \equiv 1$ and the case where b is not a constant function, respectively.

Theorem 1.3. *Assume $b \equiv 1$. If $x_0 \in \partial\Omega$ is a non-degenerate critical point of κ , then for each $\beta \in (0, 1)$, there exist $p_0 > 0$ and $\lambda_0 > 0$ such that for any $p > p_0$ and $\lambda > \lambda_0$ satisfying $\frac{1}{p} \lambda^{1+\beta} = o(1)$, problem (1.1) has a solution $u_{p,\lambda}$ with a concentration point $\xi_{p,\lambda}$ such that*

$$|\lambda d(\xi_{p,\lambda}) - \theta_b| = O(\lambda^{-1/2}) \quad \text{and} \quad |\hat{\xi}_{p,\lambda} - x_0| = O(\lambda^{-\beta}) \quad \text{as } \lambda \rightarrow +\infty.$$

Theorem 1.4. *Let $x_0 \in \partial\Omega$ be a non-degenerate critical point of b . Then, for each $\beta \in (0, 1)$, there exist $p_0 > 0$ and $\lambda_0 > 0$ such that for any $p > p_0$ and $\lambda > \lambda_0$ satisfying $\frac{1}{p} (\lambda^{(1+\beta)/3} + \lambda^\beta) = o(1)$, problem (1.1) has a solution $u_{p,\lambda}$ with a concentration point $\xi_{p,\lambda}$ such that*

$$|\lambda d(\xi_{p,\lambda}) - \theta_{b(\xi_{p,\lambda})}| = O(\lambda^{-1/2}) \quad \text{and} \quad |\hat{\xi}_{p,\lambda} - x_0| = O(\lambda^{-\beta}) \quad \text{as } \lambda \rightarrow +\infty.$$

Let us remark the interesting analogy between all above results and those known for the Liouville equation with Robin boundary condition

$$\begin{cases} \Delta u + \varepsilon^2 e^u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \lambda b(x)u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.10)$$

where $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain, $\varepsilon > 0$ is small but $\lambda > 0$ is large, and $b(x) > 0$ is a smooth function defined on $\partial\Omega$. For $b(x) \equiv 1$, Davila and Topp in [8] construct solutions of problem (1.10) which exhibit concentration phenomena as $\varepsilon \rightarrow 0$ and simultaneously as $\lambda \rightarrow +\infty$ at one or multiple points that get close to

the boundary. More precisely, they prove that if ε and λ satisfy some restrained condition, then for any $m \geq 1$, problem (1.10) has at least two solutions with m different concentration points located at distance $O(1/\lambda)$ from $\partial\Omega$, while if ε and λ satisfy a weaker restrained condition so that λ has a faster growth, problem (1.10) has at least two solutions with a concentration point near the boundary. Additionally, when x_0 is a non-degenerate critical point of the mean curvature function κ on $\partial\Omega$, they find that under a certain restrained condition of p and λ , problem (1.10) has a solution with a concentration point located at distance $O(1/\lambda)$ from x_0 .

Finally, it is necessary to compare Theorem 1.2 with the results given in [3], which is concerned with the following singularly perturbed elliptic problem with Robin boundary condition:

$$\begin{cases} \varepsilon^2 \Delta u - u + f(u) = 0, & u > 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \lambda u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.11)$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded smooth domain, $\varepsilon > 0$ is a small constant, $0 < \lambda < +\infty$, $f(u) = u^p - au^q$ with $a \geq 0$, and $1 < q < p < \frac{N+2}{N-2}$ if $N \geq 3$ whereas $1 < q < p < +\infty$ if $N = 2$. The cases $\lambda = 0$ (Neumann problem) and $\lambda = +\infty$ (Dirichlet problem) have been studied, respectively, in [17, 18] and [19], proving that the least energy solution of problem (1.11) with Dirichlet boundary condition concentrates a point that maximizes the distance to the boundary, while Neumann concentration takes place at a point on the boundary. In [3] it is proved that for fixed small ε , as λ moves from 0 to $+\infty$, the asymptotic behavior of the least energy solution of problem (1.11) changes dramatically. It will be like in the case of Neumann boundary condition if $\lambda < \lambda_*/\varepsilon$ and like in the case of Dirichlet boundary condition if $\lambda > \lambda_*/\varepsilon$, where λ_* is the borderline number associated to the corresponding problem on the half space. Thus, $\lambda \sim 1/\varepsilon$ represents a dramatic change in asymptotic behavior of the least energy solution of problem (1.11). Our results suggest that for least energy solutions of problem (1.1), the critical range of λ may be $\log \lambda \sim p$.

The general strategy for proving our main results relies on the very well-known Lyapunov–Schmidt reduction, which has been used in many papers, for example, in [8, 9, 12, 13]. The paper is organized as follows. In Section 2 we exactly describe the ansatz for the solution of problem (1.1) and estimate the error. Then, we rewrite problem (1.1) in terms of a linearized operator for which a solvability theory, subject to suitable orthogonality conditions, is performed through solving a linearized problem in Section 3. In Section 4 we solve an auxiliary nonlinear problem. In Section 5 we reduce (1.1) to a finite system. In the last section, we give the proof of Theorems 1.1–1.4.

In this paper, the symbol C denotes always a generic positive constant independent of p and λ , which could be changed from one line to another. The symbols $O(t)$ (respectively $o(t)$) denote quantities for which $\frac{O(t)}{|t|}$ stays bounded (respectively, $\frac{o(t)}{|t|}$ tends to zero) as the parameter t goes to zero. In particular, we will often use the notations $O(1)$ (respectively, $o(1)$), which represents a quantity that remains uniformly bounded (respectively, that tends to zero) as $p \rightarrow +\infty$ and simultaneously as $\lambda \rightarrow +\infty$.

2 A first approximation of the solution

In this section we will provide an ansatz for solutions of problem (1.1). A key ingredient to describe an approximate solution of problem (1.1) is given by the standard bubble:

$$U_{\delta, \xi}(x) = \log \frac{8\delta^2}{(\delta^2 + |x - \xi|^2)^2}, \quad \delta > 0, \xi \in \mathbb{R}^2. \quad (2.1)$$

It is well known (see [5]) that those are all the solutions of the following Liouville-type equation:

$$\begin{cases} \Delta u + e^u = 0 & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^u < +\infty. \end{cases}$$

The configuration space we try to look for m concentration points $\xi = (\xi_1, \dots, \xi_m)$ is the following:

$$\mathcal{O}_\varepsilon := \left\{ \xi = (\xi_1, \dots, \xi_m) \in \Omega^m : |\xi_j - \xi_k| \geq \varepsilon, |\lambda d(\xi_j) - \theta_{b(\xi_j)}| \leq \frac{1}{\varepsilon \lambda^{1/2}}, j, k = 1, \dots, m, j \neq k \right\}, \quad (2.2)$$

where $\varepsilon > 0$ is a sufficiently small but fixed number. Note that by the choice of ξ_j , if $\xi = (\xi_1, \dots, \xi_m) \in \mathcal{O}_\varepsilon$, then we have

$$d(\xi_j) = \text{dist}(\xi_j, \partial\Omega) = O\left(\frac{1}{\lambda}\right) \text{ and } \theta_{b(\xi_j)} = O(1) \text{ for all } j = 1, \dots, m. \quad (2.3)$$

In what remains of this paper, we will always assume the following restrained growth condition:

$$\frac{1}{p} \log \lambda \rightarrow 0 \text{ as } p \rightarrow +\infty \text{ and simultaneously as } \lambda \rightarrow +\infty. \quad (2.4)$$

Also, for each $j = 1, \dots, m$, we set

$$y = p^{p/(p-1)} \rho^{2/(p-1)}, \quad \delta_j = \mu_j \rho, \quad \rho = \frac{1}{\lambda} e^{-p/4}, \quad (2.5)$$

where the choice of μ_j will be determined later. Define now

$$U_j(x) = \frac{1}{y \mu_j^{2/(p-1)}} \left[U_{\delta_j, \xi_j}(x) + \frac{1}{p} \omega_1 \left(\frac{x - \xi_j}{\delta_j} \right) + \frac{1}{p^2} \omega_2 \left(\frac{x - \xi_j}{\delta_j} \right) \right]. \quad (2.6)$$

Here, ω_i , $i = 1, 2$, are radial solutions of

$$\Delta \omega_i + \frac{8}{(1 + |y|^2)^2} \omega_i = \frac{8}{(1 + |y|^2)^2} f_i(y) \quad \text{in } \mathbb{R}^2 \quad (2.7)$$

with

$$f_1 = \frac{1}{2} U_{1,0}^2, \quad f_2 = \omega_1 U_{1,0} - \frac{1}{3} U_{1,0}^3 - \frac{1}{2} \omega_1^2 - \frac{1}{8} U_{1,0}^4 + \frac{1}{2} \omega_1 U_{1,0}^2, \quad (2.8)$$

having the asymptotic properties

$$\begin{cases} \omega_i(y) = C_i \log |y| + O\left(\frac{1}{|y|}\right) & \text{as } |y| \rightarrow +\infty, \\ \nabla \omega_i(y) = C_i \cdot \frac{y}{1 + |y|^2} + O\left(\frac{1}{1 + |y|^2}\right) & \text{for all } y \in \mathbb{R}^2, \end{cases} \quad (2.9)$$

where

$$C_i = 8 \int_0^\infty t \frac{t^2 - 1}{(t^2 + 1)^3} f_i(t) dt.$$

In particular,

$$\begin{aligned} \omega_1(y) &= \frac{1}{2} U_{1,0}^2(y) + 6 \log(|y|^2 + 1) + \frac{2 \log 8 - 10}{|y|^2 + 1} + \frac{|y|^2 - 1}{|y|^2 + 1} \\ &\quad \times \left\{ -\frac{1}{2} \log^2 8 + 2 \log^2(|y|^2 + 1) + 4 \int_{|y|^2}^\infty \frac{ds}{s+1} \log \frac{s+1}{s} - 8 \log |y| \log(|y|^2 + 1) \right\} \end{aligned} \quad (2.10)$$

and (see [4, 13])

$$C_1 = 12 - 4 \log 8. \quad (2.11)$$

Our ansatz for a solution of problem (1.1) is

$$U_\xi(x) = \sum_{j=1}^m [U_j(x) + H_j(x)], \quad (2.12)$$

where H_j is a correction term defined as the solution of the following problem:

$$\begin{cases} -\Delta H_j = 0 & \text{in } \Omega, \\ \frac{\partial H_j}{\partial \nu} + \lambda b(x) H_j = -\frac{\partial U_j}{\partial \nu} - \lambda b(x) U_j & \text{on } \partial\Omega. \end{cases} \quad (2.13)$$

Lemma 2.1. For any points $\xi = (\xi_1, \dots, \xi_m) \in \mathcal{O}_\varepsilon$,

$$H_j(x) = \frac{1}{y\mu_j^{2/(p-1)}} \left[\left(1 - \frac{C_1}{4p} - \frac{C_2}{4p^2} \right) 8\pi H_\lambda(x, \xi_j) - \log(8\mu_j^2 \rho^2) + \left(\frac{C_1}{p} + \frac{C_2}{p^2} \right) \log(\mu_j \rho) + O\left(\frac{\mu_j \lambda \rho}{p} \right) \right] \quad (2.14)$$

uniformly in $C(\overline{\Omega})$ and in $C_{\text{loc}}^2(\Omega)$ as p and λ go to $+\infty$ and satisfy (2.4), where H_λ is the regular part of Green's function defined in (1.4).

Proof. First, on the boundary, by (2.1) and (2.9), we have

$$\begin{aligned} \frac{\partial H_j}{\partial \nu} + \lambda b(x) H_j = & -\frac{1}{y\mu_j^{2/(p-1)}} \left\{ \left(-4 + \frac{C_1}{p} + \frac{C_2}{p^2} \right) \frac{(x - \xi_j) \cdot \nu(x)}{\mu_j^2 \rho^2 + |x - \xi_j|^2} \right. \\ & + \lambda b(x) \left[\log \frac{1}{(\mu_j^2 \rho^2 + |x - \xi_j|^2)^2} + \left(\frac{C_1}{p} + \frac{C_2}{p^2} \right) \log |x - \xi_j| \right] \\ & \left. + \lambda b(x) \left[\log(8\mu_j^2 \rho^2) - \left(\frac{C_1}{p} + \frac{C_2}{p^2} \right) \log(\mu_j \rho) \right] + \frac{1}{p} O\left(\frac{\mu_j \rho}{|x - \xi_j|^2} + \frac{\lambda \mu_j \rho}{|x - \xi_j|} \right) \right\}. \end{aligned}$$

The regular part of Green's function with Robin boundary condition $H_\lambda(x, \xi_j)$ satisfies

$$\begin{cases} -\Delta H_\lambda(x, \xi_j) = 0 & \text{in } \Omega, \\ \frac{\partial H_\lambda(x, \xi_j)}{\partial \nu} + \lambda b(x) H_\lambda(x, \xi_j) = \frac{1}{2\pi} \frac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j|^2} - \frac{1}{2\pi} \lambda b(x) \log \frac{1}{|x - \xi_j|} & \text{on } \partial\Omega. \end{cases} \quad (2.15)$$

So, if we set

$$\tilde{H}_j(x) = y\mu_j^{2/(p-1)} H_j(x) - \left[\left(1 - \frac{C_1}{4p} - \frac{C_2}{4p^2} \right) 8\pi H_\lambda(x, \xi_j) - \log(8\mu_j^2 \rho^2) + \left(\frac{C_1}{p} + \frac{C_2}{p^2} \right) \log(\mu_j \rho) \right],$$

then $\tilde{H}_j(x)$ satisfies $-\Delta \tilde{H}_j = 0$ in Ω , and

$$\frac{\partial \tilde{H}_j}{\partial \nu} + \lambda b(x) \tilde{H}_j = -\frac{4\mu_j^2 \rho^2}{\mu_j^2 \rho^2 + |x - \xi_j|^2} \frac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j|^2} - \lambda b(x) \log \frac{|x - \xi_j|^4}{(\mu_j^2 \rho^2 + |x - \xi_j|^2)^2} + \frac{1}{p} O\left(\frac{\mu_j \rho}{|x - \xi_j|^2} + \frac{\lambda \mu_j \rho}{|x - \xi_j|} \right)$$

on $\partial\Omega$. From the maximum principle with Robin boundary condition (see [7, Lemma 2.6]), we deduce

$$\max_{\overline{\Omega}} |\tilde{H}_j(x)| + \max_{\overline{\Omega}} |d(x) \nabla \tilde{H}_j(x)| \leq \frac{C}{\lambda} \left\| \frac{\partial \tilde{H}_j}{\partial \nu} + \lambda \tilde{H}_j \right\|_{L^\infty(\partial\Omega)} = \frac{1}{p} O\left(\frac{\mu_j \rho}{\lambda d(\xi_j)^2} + \frac{\mu_j \rho}{d(\xi_j)} \right) = O\left(\frac{\mu_j \lambda \rho}{p} \right).$$

Thus, using the interior estimate of the derivative of a harmonic function (see [14, Theorem 2.10]), we get

$$\max_K |D^\alpha \tilde{H}_j(x)| \leq \left(\frac{2|\alpha|}{\text{dist}(K, \partial\Omega)} \right)^{|\alpha|} \max_{\overline{\Omega}} |\tilde{H}_j(x)| = O\left(\frac{\mu_j \lambda \rho}{p} \right)$$

for any compact subset K of Ω and any multi-index α with $|\alpha| \leq 2$. This derives estimate (2.14) uniformly in $C(\overline{\Omega})$ and in $C_{\text{loc}}^2(\Omega)$ for any sufficiently large p and λ satisfying (2.4). $\square$

From Lemma 2.1, away from the points ξ_j , namely $|x - \xi_j| \geq \varepsilon d(\xi_j)$ for any $j = 1, \dots, m$, one has

$$U_\xi(x) = \sum_{j=1}^m \frac{1}{y\mu_j^{2/(p-1)}} \left[\left(1 - \frac{C_1}{4p} - \frac{C_2}{4p^2} \right) 8\pi G_\lambda(x, \xi_j) + \frac{1}{p} O(\mu_j \lambda \rho) \right]. \quad (2.16)$$

While for $|x - \xi_j| \leq \varepsilon d(\xi_j)$ with some j , if we write $x = \xi_j + \delta_j y$, then, using (2.2), (2.5), (2.6), (2.12), (2.14), (2.16) and the fact that $|\nabla_x H_\lambda(x, \xi_j)| = O(\lambda)$ uniformly holds in the intermediate region $\lambda d(x) = O(1)$ (see [7]), we

deduce

$$\begin{aligned}
 U_{\xi}(x) &= \frac{1}{y\mu_j^{2/(p-1)}} \left\{ U_{\delta_j, \xi_j}(x) + \frac{1}{p} \omega_1 \left(\frac{x - \xi_j}{\delta_j} \right) + \frac{1}{p^2} \omega_2 \left(\frac{x - \xi_j}{\delta_j} \right) + \left(1 - \frac{C_1}{4p} - \frac{C_2}{4p^2} \right) 8\pi H_{\lambda}(x, \xi_j) \right. \\
 &\quad \left. - \log(8\mu_j^2 \rho^2) + \left(\frac{C_1}{p} + \frac{C_2}{p^2} \right) \log(\mu_j \rho) + \frac{1}{p} O(\mu_j \lambda \rho) \right\} \\
 &\quad + \sum_{k=j} \frac{1}{y\mu_k^{2/(p-1)}} \left[\left(1 - \frac{C_1}{4p} - \frac{C_2}{4p^2} \right) 8\pi G_{\lambda}(x, \xi_k) + \frac{1}{p} O(\mu_k \lambda \rho) \right] \\
 &= \frac{1}{y\mu_j^{2/(p-1)}} \left\{ U_{1,0}(y) - \log(\mu_j^2 \rho^2) + \frac{1}{p} \omega_1(y) + \frac{1}{p^2} \omega_2(y) + \left(1 - \frac{C_1}{4p} - \frac{C_2}{4p^2} \right) 8\pi H_{\lambda}(\xi_j, \xi_j) \right. \\
 &\quad \left. - \log(8\mu_j^2 \rho^2) + \left(\frac{C_1}{p} + \frac{C_2}{p^2} \right) \log(\mu_j \rho) + O(\lambda \delta_j |y|) + \frac{1}{p} O(\mu_j \lambda \rho) \right. \\
 &\quad \left. + \sum_{k=j} \frac{\mu_j^{2/(p-1)}}{\mu_k^{2/(p-1)}} \left[\left(1 - \frac{C_1}{4p} - \frac{C_2}{4p^2} \right) 8\pi G_{\lambda}(\xi_j, \xi_k) + O(\lambda \delta_j |y|) + \frac{1}{p} O(\mu_k \lambda \rho) \right] \right\} \\
 &= \frac{1}{y\mu_j^{2/(p-1)}} \left\{ p + U_{1,0}(y) + \frac{1}{p} \omega_1(y) + \frac{1}{p^2} \omega_2(y) + \left(1 - \frac{C_1}{4p} - \frac{C_2}{4p^2} \right) 8\pi H_{\lambda}(\xi_j, \xi_j) \right. \\
 &\quad \left. - \log(8\mu_j^4) + 4 \log \lambda + \left(\frac{C_1}{p} + \frac{C_2}{p^2} \right) \log(\mu_j \rho) + O(\lambda \mu_j \rho |y|) + \frac{1}{p} O(\mu_j \lambda \rho) \right. \\
 &\quad \left. + \sum_{k=j} \frac{\mu_j^{2/(p-1)}}{\mu_k^{2/(p-1)}} \left[\left(1 - \frac{C_1}{4p} - \frac{C_2}{4p^2} \right) 8\pi G_{\lambda}(\xi_j, \xi_k) + O(\lambda \mu_j \rho |y|) + \frac{1}{p} O(\mu_k \lambda \rho) \right] \right\}.
 \end{aligned}$$

We now choose the parameters μ_j . We assume they are defined by the relation

$$\begin{aligned}
 \log(8\mu_j^4) &= 4 \log \lambda + \left(1 - \frac{C_1}{4p} - \frac{C_2}{4p^2} \right) 8\pi H_{\lambda}(\xi_j, \xi_j) + \left(\frac{C_1}{p} + \frac{C_2}{p^2} \right) \log \left(\mu_j \frac{1}{\lambda} e^{-p/4} \right) \\
 &\quad + \sum_{k=j} \frac{\mu_j^{2/(p-1)}}{\mu_k^{2/(p-1)}} \left(1 - \frac{C_1}{4p} - \frac{C_2}{4p^2} \right) 8\pi G_{\lambda}(\xi_j, \xi_k).
 \end{aligned}$$

Taking into account the explicit expression (2.11) of the constant C_1 , we observe that as p tends to $+\infty$, μ_j bifurcates from the value

$$\bar{\mu}_j = \lambda e^{-3/4} e^{2\pi H_{\lambda}(\xi_j, \xi_j) + 2\pi \sum_{k=j} G_{\lambda}(\xi_j, \xi_k)},$$

solution of the equation

$$\log(8\mu_j^4) = 4 \log \lambda + 8\pi H_{\lambda}(\xi_j, \xi_j) - \frac{1}{4} C_1 + 8\pi \sum_{k=j} G_{\lambda}(\xi_j, \xi_k).$$

Thus, by (2.4), μ_j is a perturbation of order $\frac{\log \lambda}{p}$ of the value $\bar{\mu}_j$, namely

$$\mu_j = \lambda e^{-3/4} e^{2\pi H_{\lambda}(\xi_j, \xi_j) + 2\pi \sum_{k=j} G_{\lambda}(\xi_j, \xi_k)} \left[1 + O\left(\frac{\log \lambda}{p}\right) \right]. \quad (2.17)$$

Moreover, from (1.5)–(1.8) and (2.2), there exists a constant $C > 0$ independent of p , λ and j such that for any sufficiently large p and λ satisfying (2.4),

$$\frac{1}{C} < \mu_j < C \quad (2.18)$$

and

$$|D_{\xi} \log \mu_j| < C\sqrt{\lambda}, \quad (2.19)$$

where the second inequality is due to the fact that if $x \in \Omega$ satisfies $|\lambda \text{dist}(x, \partial\Omega) - \theta_{b(\hat{x})}| = O(\lambda^{-1/2})$, then

$$|\nabla_x H_{\lambda}(x, x)| = O(\sqrt{\lambda}).$$

From this choice of the parameters μ_j , we deduce that for $|x - \xi_j| = \delta_j|y| \leq \varepsilon d(\xi_j)$,

$$U_\xi(x) = \frac{1}{y\mu_j^{2/(p-1)}} \left[p + U_{1,0}(y) + \frac{1}{p}\omega_1(y) + \frac{1}{p^2}\omega_2(y) + O(\lambda\rho|y|) + \frac{1}{p}O(\lambda\rho) \right]. \quad (2.20)$$

Remark 2.2. Let us remark that U_ξ is a positive uniformly bounded function. Observe that for $|y| \leq \varepsilon d(\xi_j)/\delta_j$,

$$\begin{aligned} p + U_{1,0}(y) + \frac{1}{p}\omega_1(y) + \frac{1}{p^2}\omega_2(y) &\geq p + \log 8 - 2 \log \left(1 + \frac{\varepsilon^2 d(\xi_j)^2}{\delta_j^2} \right) + \frac{1}{4} + O\left(\frac{1}{p}\right) \\ &= p + \log 8 - 4 \log \frac{\varepsilon d(\xi_j)}{\delta_j} + \frac{1}{4} + O\left(\frac{\delta_j^2}{\varepsilon^2 d(\xi_j)^2}\right) + O\left(\frac{1}{p}\right) \\ &= 4 \log \frac{1}{\varepsilon} + \log(8\mu_j^4) - 4 \log(\lambda d(\xi_j)) + \frac{1}{4} + O\left(\frac{1}{p}\right). \end{aligned}$$

By choosing $\varepsilon > 0$ small enough, it is easy to check that $U_\xi > 0$ in $B(\xi_j, \varepsilon d(\xi_j))$, and $\sup_{B(\xi_j, \varepsilon d(\xi_j))} U_\xi \rightarrow \sqrt{e}$ as p and λ go to $+\infty$ and satisfy (2.4). Moreover, by the maximum principle, we see that $G_\lambda(x, \xi_j) > 0$ in $\bar{\Omega}$ and thus, by (2.16), U_ξ is a positive uniformly bounded function in $\bar{\Omega}$. In conclusion, $0 < U_\xi \leq 2\sqrt{e}$ in $\bar{\Omega}$.

Let us define

$$S_p(u) = \Delta u + u_+^p, \quad \text{where } u_+ = \max\{u, 0\},$$

and introduce the following functional:

$$J_p^\lambda(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{1}{p+1} \int_{\Omega} u_+^{p+1} + \frac{\lambda}{2} \int_{\partial\Omega} b(x)u^2, \quad u \in H^1(\Omega), \quad (2.21)$$

whose nontrivial critical points are solutions of problem (1.1). Obviously, by the maximum principle, problem (1.1) is equivalent to

$$S_p(u) = 0, \quad u_+ \not\equiv 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} + \lambda b(x)u = 0 \quad \text{on } \partial\Omega.$$

We will seek solutions of problem (1.1) in the form $u = U_\xi + \phi$, where ϕ will represent a higher order correction. Observe that

$$S_p(U_\xi + \phi) = L(\phi) + R_\xi + N(\phi) = 0,$$

where

$$\begin{aligned} L(\phi) &= \Delta\phi + W_\xi\phi \quad \text{with } W_\xi = pU_\xi^{p-1}, \\ R_\xi &= \Delta U_\xi + U_\xi^p \quad \text{and } N(\phi) = (U_\xi + \phi)_+^p - U_\xi^p - pU_\xi^{p-1}\phi. \end{aligned}$$

In terms of ϕ , problem (1.1) becomes

$$\begin{cases} L(\phi) = -[R_\xi + N(\phi)] & \text{in } \Omega, \\ \frac{\partial \phi}{\partial \nu} + \lambda b(x)\phi = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.22)$$

The main step in solving problem (2.22) is a solvability theory for the linear operator L under a suitable choice of the points ξ_j . In developing this theory, we will take into account the invariance, under translations and dilations, of the problem $\Delta e^v + e^v = 0$ in $\mathbb{R}^2$. We will apply the solvability theory for the linear operator L in a weighted L^∞ -norm space, following [9, 13]. For any $\xi = (\xi_1, \dots, \xi_m) \in \mathcal{O}_\varepsilon$ and $h \in L^\infty(\Omega)$, define

$$\|h\|_* = \sup_{x \in \Omega} \left| \left(\sum_{j=1}^m \frac{\delta_j}{(\delta_j^2 + |x - \xi_j|^2)^{3/2}} \right)^{-1} h(x) \right|. \quad (2.23)$$

We conclude this section by showing an estimate of R_ξ in $\|\cdot\|_*$.

Proposition 2.3. Let $\varepsilon > 0$ be fixed and small. There exist $C > 0$, $p_0 > 0$ and $\lambda_0 > 0$ such that for any points $\xi = (\xi_1, \dots, \xi_m) \in \odot_\varepsilon$, and any $p \geq p_0$ and $\lambda \geq \lambda_0$ satisfying (2.4),

$$\|R_\xi\|_* \leq \frac{C}{p^4}. \quad (2.24)$$

Proof. Observe that by (2.6)–(2.7) and (2.12)–(2.13),

$$\begin{aligned} \Delta U_\xi &= \sum_{j=1}^m \frac{1}{y\mu_j^{2/(p-1)}} \left[-e^{U_{\delta_j, \xi_j}} + \frac{1}{p\delta_j^2} \Delta \omega_1 \left(\frac{x - \xi_j}{\delta_j} \right) + \frac{1}{p^2 \delta_j^2} \Delta \omega_2 \left(\frac{x - \xi_j}{\delta_j} \right) \right] \\ &= \sum_{j=1}^m \frac{e^{U_{\delta_j, \xi_j}}}{y\mu_j^{2/(p-1)}} \left[-1 + \frac{1}{p} f_1 \left(\frac{x - \xi_j}{\delta_j} \right) + \frac{1}{p^2} f_2 \left(\frac{x - \xi_j}{\delta_j} \right) - \frac{1}{p} \omega_1 \left(\frac{x - \xi_j}{\delta_j} \right) - \frac{1}{p^2} \omega_2 \left(\frac{x - \xi_j}{\delta_j} \right) \right]. \end{aligned} \quad (2.25)$$

Far away from the points ξ_j , namely $|x - \xi_j| \geq \varepsilon d(\xi_j)$ for any $j = 1, \dots, m$, by (2.3)–(2.5), (2.16) and (2.25), we get

$$\left(\sum_{k=1}^m \frac{\delta_k}{(\delta_k^2 + |x - \xi_k|^2)^{3/2}} \right)^{-1} |(\Delta U_\xi + U_\xi^p)(x)| \leq \sum_{j=1}^m \frac{C}{\delta_j} \left(\frac{\delta_j^2 \log^4 \delta_j}{y p^2 d(\xi_j)^4} + \frac{1}{y^p} \right) = O(p \lambda^3 e^{-p/4}). \quad (2.26)$$

Let us now fix the index j in $\{1, \dots, m\}$ and the region $|x - \xi_j| \leq \varepsilon d(\xi_j)$. Taking into account (2.20) and the relation

$$\left(\frac{p}{y\mu_j^{2/(p-1)}} \right)^p = \frac{1}{y\delta_j^2 \mu_j^{2/(p-1)}}, \quad (2.27)$$

for $|x - \xi_j| = \delta_j |y| \leq \varepsilon d(\xi_j)$, we get

$$U_\xi^p(x) = \frac{1}{y\delta_j^2 \mu_j^{2/(p-1)}} \left[1 + \frac{1}{p} U_{1,0}(y) + \frac{1}{p^2} \omega_1(y) + \frac{1}{p^3} [\omega_2(y) + O(p^2 \lambda \rho |y|) + O(p \lambda \rho)] \right]^p. \quad (2.28)$$

From the Taylor expansions of the exponential and logarithmic function, we have

$$\left(1 + \frac{a}{p} + \frac{b}{p^2} + \frac{c}{p^3} \right)^p = e^a \left[1 + \frac{1}{p} \left(b - \frac{a^2}{2} \right) + \frac{1}{p^2} \left(c - ab + \frac{a^3}{3} + \frac{b^2}{2} - \frac{a^2 b}{2} + \frac{a^4}{8} \right) + O\left(\frac{\log^6(|y| + 2)}{p^3} \right) \right], \quad (2.29)$$

which holds for $|y| \leq C e^{p/8}$, provided that $-4 \log(|y| + 2) \leq a(y) \leq C$ and $|b(y)| + |c(y)| \leq C \log(|y| + 2)$. For $|x - \xi_j| = \delta_j |y| \leq \varepsilon d(\xi_j) \sqrt{\delta_j}$, we deduce that

$$\begin{aligned} U_\xi^p(x) &= \frac{1}{y\delta_j^2 \mu_j^{2/(p-1)}} e^{U_{1,0}(y)} \left[1 + \frac{1}{p} \left(\omega_1 - \frac{1}{2} U_{1,0}^2 \right) + \frac{1}{p^2} \left(\omega_2 - \omega_1 U_{1,0} + \frac{1}{3} U_{1,0}^3 + \frac{1}{2} \omega_1^2 - \frac{1}{2} \omega_1 U_{1,0}^2 + \frac{1}{8} U_{1,0}^4 \right) \right. \\ &\quad \left. + O\left(\frac{\log^6(|y| + 2)}{p^3} \right) + O(\lambda \rho |y|) + O\left(\frac{\lambda \rho}{p} \right) \right]. \end{aligned} \quad (2.30)$$

Combining (2.1), (2.8), (2.25) and (2.30), for $|x - \xi_j| = \delta_j |y| \leq \varepsilon d(\xi_j) \sqrt{\delta_j}$, we have that

$$(\Delta U_\xi + U_\xi^p)(x) = \frac{1}{y\delta_j^2 \mu_j^{2/(p-1)}} \frac{8}{(1 + |y|^2)^2} O\left(\frac{\log^6(|y| + 2)}{p^3} \right),$$

and therefore

$$\begin{aligned} \left(\sum_{k=1}^m \frac{\delta_k}{(\delta_k^2 + |x - \xi_k|^2)^{3/2}} \right)^{-1} |(\Delta U_\xi + U_\xi^p)(x)| &\leq \frac{(\delta_j^2 + |x - \xi_j|^2)^{3/2}}{\delta_j} |(\Delta U_\xi + U_\xi^p)(x)| \\ &\leq \frac{C}{y} \frac{1}{(1 + |y|^2)^{1/2}} O\left(\frac{\log^6(|y| + 2)}{p^3} \right) \\ &\leq \frac{C}{p^4}. \end{aligned} \quad (2.31)$$

On the other hand, if $\varepsilon d(\xi_j)\sqrt{\delta_j} \leq |x - \xi_j| = \delta_j|y| \leq \varepsilon d(\xi_j)$, then by (2.28) we have

$$U_\xi^p(x) = O\left(\frac{1}{y\delta_j^2} \frac{1}{(1+|y|^2)^2}\right),$$

since $(1 + \frac{s}{p})^p \leq e^s$. Thus, in this region,

$$\left(\sum_{k=1}^m \frac{\delta_k}{(\delta_k^2 + |x - \xi_k|^2)^{3/2}}\right)^{-1} |(\Delta U_\xi + U_\xi^p)(x)| \leq \frac{1}{y} \frac{1}{(1+|y|^2)^{1/2}} O\left(\frac{1}{p^2} \log^4|y|\right) \leq p\sqrt{\lambda}e^{-p/8}. \quad (2.32)$$

Joining together (2.26), (2.31) and (2.32), we obtain estimate (2.24). $\square$

3 Analysis of the linearized operator

In this section we prove the bounded invertibility of the operator L , uniformly on $\xi \in \mathcal{O}_\varepsilon$, by using the weighted L^∞ -norms introduced in (2.23). Let us recall that $L(\phi) = \Delta\phi + W_\xi\phi$, where $W_\xi(x) = pU_\xi^{p-1}(x)$. As in Proposition 2.3, for the potential $W_\xi(x)$ we have the following expansions.

Lemma 3.1. *Let $\varepsilon > 0$ be fixed and small. There exist $D_0 > 0$, $p_0 > 0$ and $\lambda_0 > 0$ such that*

$$W_\xi(x) \leq D_0 \sum_{j=1}^m e^{U_{\delta_j, \xi_j}(x)} \quad (3.1)$$

for any points $\xi = (\xi_1, \dots, \xi_m) \in \mathcal{O}_\varepsilon$, and any $p \geq p_0$ and $\lambda \geq \lambda_0$ satisfying (2.4). Furthermore,

$$W_\xi(x) = \frac{8}{\delta_j^2(1+|y|^2)^2} \left[1 + \frac{1}{p} \left(\omega_1 - U_{1,0} - \frac{1}{2} U_{1,0}^2 \right)(y) + O\left(\frac{\log^4(|y|+2)}{p^2}\right) \right]$$

for any $|x - \xi_j| \leq \varepsilon d(\xi_j)\sqrt{\delta_j}$, where $y = \frac{x - \xi_j}{\delta_j}$.

Proof. If $|x - \xi_j| = \delta_j|y| \leq \varepsilon d(\xi_j)$ for some $j = 1, \dots, m$, then by (2.20) and (2.27),

$$\begin{aligned} W_\xi(x) &= p \left(\frac{p}{y\mu_j^{2/(p-1)}} \right)^{p-1} \left[1 + \frac{1}{p} U_{1,0}(y) + \frac{1}{p^2} \omega_1(y) + \frac{1}{p^3} \omega_2(y) + \frac{1}{p} O(\lambda\rho|y|) + \frac{1}{p^2} O(\lambda\rho) \right]^{p-1} \\ &= \frac{1}{\delta_j^2} \left[1 + \frac{1}{p} U_{1,0}(y) + \frac{1}{p^2} \omega_1(y) + \frac{1}{p^3} \omega_2(y) + \frac{1}{p} O(\lambda\rho|y|) + \frac{1}{p^2} O(\lambda\rho) \right]^{p-1}, \end{aligned}$$

where again we use the notation $y = \delta_j^{-1}(x - \xi_j)$. In this region, we have that

$$W_\xi(x) \leq \frac{C}{\delta_j^2} e^{U_{1,0}(y) + O(\lambda\rho|y|)} e^{-\frac{1}{p}[U_{1,0}(y) + O(\lambda\rho|y|)]} = O(e^{U_{\delta_j, \xi_j}(x)}),$$

since $(1 + \frac{a}{p})^{p-1} \leq e^{(p-1)a/p}$ and $U_{1,0}(y) \geq -p + O(1)$. In particular, from a slight modification of formula (2.29), that is,

$$\left(1 + \frac{a}{p} + \frac{b}{p^2} + \frac{c}{p^3} \right)^{p-1} = e^a \left[1 + \frac{1}{p} \left(b - a - \frac{a^2}{2} \right) + O\left(\frac{\log^4(|y|+2)}{p^2}\right) \right],$$

we obtain that if $|x - \xi_j| = \delta_j|y| \leq \varepsilon d(\xi_j)\sqrt{\delta_j}$, then

$$W_\xi(x) = \frac{8}{\delta_j^2(1+|y|^2)^2} \left[1 + \frac{1}{p} \left(\omega_1 - U_{1,0} - \frac{1}{2} U_{1,0}^2 \right)(y) + O\left(\frac{\log^4(|y|+2)}{p^2}\right) \right].$$

In addition, if $|x - \xi_j| \geq \varepsilon d(\xi_j)$ for all j , then by (2.16) we deduce

$$W_\xi(x) = O\left(p \left(\frac{C}{y}\right)^{p-1}\right),$$

which completes the proof. $\square$

Remark 3.2. As for W_ξ , let us point out that if $|x - \xi_j| \leq \varepsilon d(\xi_j)$ for some $j = 1, \dots, m$, then

$$p \left(U_\xi(x) + O\left(\frac{1}{p^3}\right) \right)^{p-2} \leq Cp \left(\frac{p}{y\mu_j^{2/(p-2)}} \right)^{p-2} e^{\frac{p-2}{p} U_{1,0}\left(\frac{x-\xi_j}{\delta_j}\right)} = O(e^{U_{\delta_j, \xi_j}(x)}).$$

Since this estimate is true if $|x - \xi_j| \geq \varepsilon d(\xi_j)$ for any $j = 1, \dots, m$, we have

$$p \left(U_\xi(x) + O\left(\frac{1}{p^3}\right) \right)^{p-2} \leq C \sum_{j=1}^m e^{U_{\delta_j, \xi_j}(x)}.$$

Let

$$z_0(y) = \frac{|y|^2 - 1}{|y|^2 + 1}, \quad z_i(y) = 4 \frac{y_i}{|y|^2 + 1}, \quad i = 1, 2.$$

It is well known that any bounded solution to

$$\Delta \phi + \frac{8}{(1 + |y|^2)^2} \phi = 0 \quad \text{in } \mathbb{R}^2$$

is a linear combination of z_i , $i = 0, 1, 2$ (see [2]). Now we consider the following linear problem: given $h \in C(\bar{\Omega})$ and points $\xi = (\xi_1, \dots, \xi_m) \in \mathcal{O}_\varepsilon$, we find a function $\phi \in H^2(\Omega)$ such that

$$\begin{cases} L(\phi) = \Delta \phi + W_\xi \phi = h + \sum_{i=1}^2 \sum_{j=1}^m c_{ij} e^{U_{\delta_j, \xi_j}} Z_{ij} & \text{in } \Omega, \\ \frac{\partial \phi}{\partial \nu} + \lambda b(x) \phi = 0 & \text{on } \partial \Omega, \\ \int_{\Omega} e^{U_{\delta_j, \xi_j}} Z_{ij} \phi = 0 & \text{for } i = 1, 2, j = 1, \dots, m, \end{cases} \quad (3.2)$$

for some coefficients c_{ij} , $i = 1, 2$ and $j = 1, \dots, m$. Here and in the sequel, for any $i = 0, 1, 2$ and $j = 1, \dots, m$, we set

$$Z_{ij}(x) := z_i \left(\frac{x - \xi_j}{\delta_j} \right) = \begin{cases} \frac{|x - \xi_j|^2 - \delta_j^2}{|x - \xi_j|^2 + \delta_j^2} & \text{if } i = 0, \\ \frac{4\delta_j(x - \xi_j)_i}{|x - \xi_j|^2 + \delta_j^2} & \text{if } i = 1, 2. \end{cases} \quad (3.3)$$

The main result of this section is the following proposition.

Proposition 3.3. Let $\varepsilon > 0$ be fixed and small. There exist $p_0 > 0$, $\lambda_0 > 0$ and $C > 0$ such that for any $h \in C(\bar{\Omega})$, any points $\xi = (\xi_1, \dots, \xi_m) \in \mathcal{O}_\varepsilon$, and any $p > p_0$ and $\lambda > \lambda_0$ satisfying (2.4), there exists a unique solution ϕ to problem (3.2) for some scalars c_{ij} , $i = 1, 2$, $j = 1, \dots, m$, which satisfies

$$\|\phi\|_\infty \leq Cp \|h\|_{*}. \quad (3.4)$$

Proof. The proof of this result will be divided into six steps.

Step 1:

The operator L satisfies the maximum principle in $\tilde{\Omega} := \Omega \setminus \bigcup_{j=1}^m B(\xi_j, R\delta_j)$ for R large, independent of p and λ . Specifically, if ψ satisfies

$$L(\psi) = \Delta \psi + W_\xi \psi \leq 0 \quad \text{in } \tilde{\Omega}, \quad \psi \geq 0 \quad \text{on } \bigcup_{j=1}^m \partial B(\xi_j, R\delta_j) \quad \text{and} \quad \frac{\partial \psi}{\partial \nu} + \lambda b(x) \psi \geq 0 \quad \text{on } \partial \Omega,$$

then $\psi \geq 0$ in $\tilde{\Omega}$. To prove this, it suffices to construct a positive function Z on $\tilde{\Omega}$ such that

$$L(Z) = \Delta Z + W_\xi Z < 0 \quad \text{in } \tilde{\Omega}, \quad Z > 0 \quad \text{on } \bigcup_{j=1}^m \partial B(\xi_j, R\delta_j) \quad \text{and} \quad \frac{\partial Z}{\partial \nu} + \lambda b(x) Z > 0 \quad \text{on } \partial \Omega.$$

Indeed, let

$$Z(x) = \sum_{j=1}^m z_0 \left(\frac{a(x - \xi_j)}{\delta_j} \right), \quad a > 0.$$

First, observe that if $|x - \xi_j| \geq R\delta_j$ for $R > \frac{1}{a}$, then $Z(x) > 0$. On the other hand, since $Z(x) \leq m$,

$$W_\xi(x)Z(x) \leq D_0 \left(\sum_{j=1}^m e^{U_{\delta_j, \xi_j}(x)} \right) Z(x) \leq D_0 Z(x) \sum_{j=1}^m \frac{8\delta_j^2}{|x - \xi_j|^4} \leq mD_0 \sum_{j=1}^m \frac{8\delta_j^2}{|x - \xi_j|^4},$$

where D_0 is the constant in Lemma 3.1. Further, by the definition of z_0 ,

$$-\Delta Z(x) = \sum_{j=1}^m a^2 \frac{8\delta_j^2(a^2|x - \xi_j|^2 - \delta_j^2)}{(a^2|x - \xi_j|^2 + \delta_j^2)^3} \geq \frac{1}{3} \sum_{j=1}^m \frac{8a^2\delta_j^2}{(a^2|x - \xi_j|^2 + \delta_j^2)^2} \geq \frac{4}{27} \sum_{j=1}^m \frac{8\delta_j^2}{a^2|x - \xi_j|^4},$$

provided that $R > \frac{\sqrt{3}}{a}$. Thus, if a is taken small and fixed but independent of p and λ , and R is chosen sufficiently large depending on this a , then we have that

$$L(Z) = \Delta Z + W_\xi Z \leq \left(-\frac{4}{27} \frac{1}{a^2} + mD_0 \right) \sum_{j=1}^m \frac{8\delta_j^2}{|x - \xi_j|^4} < 0.$$

Moreover,

$$\left| \frac{\partial}{\partial \nu} Z(x) \right| \leq \sum_{j=1}^m \frac{C\delta_j^2}{a^2|x - \xi_j|^3} = O \left(\frac{\rho^2}{a^2} \sum_{j=1}^m \frac{1}{d(\xi_j)^3} \right) \quad \text{on } \partial\Omega$$

and

$$Z(x) \geq \frac{1}{2} \quad \text{on } \partial\Omega \cup \left(\bigcup_{j=1}^m \partial B(\xi_j, R\delta_j) \right),$$

which, together with (2.3)–(2.5), deduce that on $\partial\Omega$,

$$\frac{\partial Z}{\partial \nu} + \lambda b(x)Z \geq O \left(\frac{1}{a^2} \lambda^3 \rho^2 \right) + \frac{1}{2} \lambda b(x) \geq O(\lambda e^{-p/2}) + \frac{1}{2} \lambda \min_{x \in \partial\Omega} b(x) > 0, \quad (3.5)$$

provided that p and λ are chosen sufficiently large. The function $Z(x)$ is what we are looking for.

Step 2:

Let R and ε_0 be as before. We define the “inner norm” of ϕ as

$$\|\phi\|_i = \sup_{x \in \bigcup_{j=1}^m B(\xi_j, R\delta_j)} |\phi|(x),$$

and claim that there exists a constant $C > 0$ such that if $L(\phi) = h$ in Ω and $\frac{\partial \phi}{\partial \nu} + \lambda b(x)\phi = g$ on $\partial\Omega$, then

$$\|\phi\|_{L^\infty(\Omega)} \leq C \left(\|\phi\|_i + \|h\|_* + \frac{1}{\lambda} \|g\|_{L^\infty(\partial\Omega)} \right)$$

for any $h \in C^{0,\alpha}(\overline{\Omega})$ and $g \in C^{0,\alpha}(\partial\Omega)$. We will establish this estimate with the use of suitable barriers. Let $M = 2\text{diam}\Omega$. Consider the solution $\psi_j(x)$ of the problem

$$\begin{cases} -\Delta \psi_j = \frac{2\delta_j}{|x - \xi_j|^3} & \text{in } R\delta_j < |x - \xi_j| < M, \\ \psi_j(x) = 0 & \text{on } |x - \xi_j| = R\delta_j \text{ and } |x - \xi_j| = M. \end{cases}$$

The function $\psi_j(x)$ is the positive function given by

$$\psi_j = -\frac{2\delta_j}{|x - \xi_j|} + A + B \log|x - \xi_j|,$$

where

$$B = 2 \left(\frac{\delta_j}{M} - \frac{1}{R} \right) \frac{1}{\log \left(\frac{M}{R\delta_j} \right)} < 0 \text{ and } A = \frac{2\delta_j}{M} - B \log M.$$

Observe that for $R\delta_j \leq |x - \xi_j| \leq M$,

$$\psi_j(x) \leq A + B \log(R\delta_j) = \frac{2\delta_j}{M} - B \log \frac{M}{R\delta_j} = \frac{2}{R}. \quad (3.6)$$

Thus, $\psi_j(x)$ is uniformly bounded from above by a constant independent of p and λ . In view of

$$|\nabla \psi_j(x)| = O \left(\frac{\rho}{|x - \xi_j|^2} + \frac{1}{|\log \rho|} \frac{1}{|x - \xi_j|} \right),$$

and by (2.3)–(2.5), we get

$$\left| \frac{\partial}{\partial \nu} \psi_j(x) \right| = O \left(\lambda e^{-p/4} + \frac{\lambda}{p} \right) \text{ on } \partial\Omega. \quad (3.7)$$

Now let

$$\tilde{\phi}(x) = C_0 \left(2Z(x) + \sum_{j=1}^m \psi_j(x) \right) \left(\|\phi\|_i + \|h\|_* + \frac{1}{\lambda} \|g\|_{L^\infty(\partial\Omega)} \right),$$

where Z was defined in the previous step, and $C_0 > 2$ is chosen larger if necessary. First of all, observe that for $x \in \bigcup_{j=1}^m \partial B(\xi_j, R\delta_j)$, by the definition of Z ,

$$\tilde{\phi}(x) \geq 2C_0 \|\phi\|_i Z(x) \geq \|\phi\|_i \geq |\phi|(x).$$

For $x \in \partial\Omega$, by (3.5), (3.7) and the positivity of $Z(x)$ and $\psi_j(x)$,

$$\begin{aligned} \frac{\partial \tilde{\phi}}{\partial \nu}(x) + \lambda b(x) \tilde{\phi}(x) &\geq \left[O \left(\lambda e^{-p/2} + \lambda e^{-p/4} + \frac{\lambda}{p} \right) + C_0 \lambda \min_{x \in \partial\Omega} b(x) \right] \left(\|\phi\|_i + \|h\|_* + \frac{1}{\lambda} \|g\|_{L^\infty(\partial\Omega)} \right) \\ &\geq \frac{1}{2} C_0 \lambda \left(\min_{x \in \partial\Omega} b(x) \right) \left(\|\phi\|_i + \|h\|_* + \frac{1}{\lambda} \|g\|_{L^\infty(\partial\Omega)} \right) \\ &\geq |g(x)|. \end{aligned}$$

Finally, for $x \in \Omega \setminus \bigcup_{j=1}^m B(\xi_j, R\delta_j)$, by (3.1), (3.6) and the definition of $\|\cdot\|_*$ in (2.23),

$$\begin{aligned} L(\tilde{\phi}) &\leq C_0 \left(\|\phi\|_i + \|h\|_* + \frac{1}{\lambda} \|g\|_{L^\infty(\partial\Omega)} \right) \sum_{j=1}^m L(\psi_j)(x) \\ &= C_0 \left(\|\phi\|_i + \|h\|_* + \frac{1}{\lambda} \|g\|_{L^\infty(\partial\Omega)} \right) \sum_{j=1}^m \left(-\frac{2\delta_j}{|x - \xi_j|^3} + W_{\xi_j}(x) \psi_j(x) \right) \\ &\leq C_0 \left(\|\phi\|_i + \|h\|_* + \frac{1}{\lambda} \|g\|_{L^\infty(\partial\Omega)} \right) \sum_{j=1}^m \left(-\frac{2\delta_j}{|x - \xi_j|^3} + \frac{2mD_0}{R} e^{U_{\delta_j, \xi_j}(x)} \right) \\ &\leq -C_0 \|h\|_* \sum_{j=1}^m \frac{\delta_j}{(\delta_j^2 + |x - \xi_j|^2)^{3/2}} \\ &\leq -|h(x)| \\ &\leq -|L(\phi)|(x), \end{aligned}$$

provided that $R > 16mD_0$, and p and λ are large enough. Hence, by the maximum principle in Step 1, we obtain

$$|\phi(x)| \leq \tilde{\phi}(x) \text{ for } x \in \tilde{\Omega},$$

and therefore, since $Z(x) \leq m$ and $\psi_j(x) \leq \frac{2}{R}$,

$$\|\phi\|_{L^\infty(\Omega)} \leq C \left(\|\phi\|_i + \|h\|_* + \frac{1}{\lambda} \|g\|_{L^\infty(\partial\Omega)} \right).$$

Step 3:

We prove uniform a priori estimates for the solutions ϕ of the problem

$$\begin{cases} L(\phi) = h & \text{in } \Omega, \\ \frac{\partial \phi}{\partial \nu} + \lambda b(x)\phi = g & \text{on } \partial\Omega, \end{cases}$$

where $h \in C^{0,\alpha}(\overline{\Omega})$ and $g \in C^{0,\alpha}(\partial\Omega)$, and in addition ϕ satisfy the orthogonality conditions

$$\int_{\Omega} e^{U_{\delta_j, \xi_j}} Z_{ij} \phi = 0 \quad \text{for } i = 0, 1, 2, j = 1, \dots, m.$$

Namely, we prove that there exists a positive constant C such that for any points $\xi = (\xi_1, \dots, \xi_m) \in \mathcal{O}_\varepsilon$, $h \in C^{0,\alpha}(\overline{\Omega})$ and $g \in C^{0,\alpha}(\partial\Omega)$,

$$\|\phi\|_{L^\infty(\Omega)} \leq C \left(\|h\|_* + \frac{1}{\lambda} \|g\|_{L^\infty(\partial\Omega)} \right)$$

for p and λ sufficiently large. By contradiction, assume the existence of sequences $p_n \rightarrow +\infty$, $\lambda_n \rightarrow +\infty$, of points $\xi^n = (\xi_1^n, \dots, \xi_m^n) \in \mathcal{O}_\varepsilon$, and of functions h_n , g_n and associated solutions ϕ_n such that $\|h_n\|_* \rightarrow 0$, $\frac{1}{\lambda_n} \|g_n\|_{L^\infty(\partial\Omega)} \rightarrow 0$ and $\|\phi_n\|_{L^\infty(\Omega)} = 1$.

Since $\|\phi_n\|_{L^\infty(\Omega)} = 1$, Step 2 shows that $\liminf_{n \rightarrow +\infty} \|\phi_n\|_i > 0$. Set $\widehat{\phi}_j^n(y) = \phi_n(\delta_j^n y + \xi_j^n)$ for $j = 1, \dots, m$. Observe that by (2.3), (2.5) and (2.18),

$$\min_{j=1, \dots, m} \frac{\text{dist}(\xi_j^n, \partial\Omega)}{\delta_j^n} \rightarrow +\infty,$$

which means that the domain of definition of $\widehat{\phi}_j^n$ approaches $\mathbb{R}^2$ as $n \rightarrow +\infty$. By Lemma 3.1, elliptic estimates readily imply that $\widehat{\phi}_j^n$ converges uniformly over compact sets to a bounded solution $\widehat{\phi}_j^\infty$ of

$$\Delta \phi + \frac{8}{(1 + |y|^2)^2} \phi = 0 \quad \text{in } \mathbb{R}^2.$$

This implies that $\widehat{\phi}_j^\infty$ is a linear combination of the functions z_i , $i = 0, 1, 2$. Since $\|\widehat{\phi}_j^n\|_{L^\infty(\Omega)} \leq 1$, by Lebesgue's theorem, the orthogonality conditions on $\widehat{\phi}_j^n$ pass to the limit and give

$$\int_{\mathbb{R}^2} \frac{8}{(1 + |y|^2)^2} z_i(y) \widehat{\phi}_j^\infty dy = 0 \quad \text{for } i = 0, 1, 2.$$

Hence, $\widehat{\phi}_j^\infty \equiv 0$ for any $j = 1, \dots, m$ contradicting $\liminf_{n \rightarrow +\infty} \|\phi_n\|_i > 0$.

Step 4:

We prove that for any solution ϕ of the problem

$$\begin{cases} L(\phi) = h & \text{in } \Omega, \\ \frac{\partial \phi}{\partial \nu} + \lambda b(x)\phi = 0 & \text{on } \partial\Omega, \end{cases}$$

which in addition satisfies the orthogonality conditions

$$\int_{\Omega} e^{U_{\delta_j, \xi_j}} Z_{ij} \phi = 0 \quad \text{for } i = 1, 2, j = 1, \dots, m,$$

there exists a positive constant $C > 0$ such that

$$\|\phi\|_{L^\infty(\Omega)} \leq Cp \|h\|_*$$

for $h \in C^{0,\alpha}(\overline{\Omega})$. Proceeding by contradiction, as in Step 3, we can suppose further that

$$\|\phi_n\|_{L^\infty(\Omega)} = 1 \text{ and } p_n \|h_n\|_* \rightarrow 0 \text{ as } n \rightarrow +\infty \quad (3.8)$$

but we lose the condition

$$\int_{\mathbb{R}^2} \frac{8}{(1+|y|^2)^2} z_0(y) \hat{\phi}_j^\infty = 0$$

in the limit. Hence, we have that

$$\hat{\phi}_j^n \rightarrow \hat{\phi}_j^\infty = C_j \frac{|y|^2 - 1}{|y|^2 + 1} \text{ in } C_{\text{loc}}^0(\mathbb{R}_+^2) \quad (3.9)$$

for some constants C_j . To reach a contradiction, we have to show that $C_j = 0$ for any $j = 1, \dots, m$. We will obtain it from the stronger condition (3.8) on h_n .

To this end, we perform the following construction. According to [4, 13], there exist radial solutions ω and ζ , respectively, of the equations

$$\Delta \omega + \frac{8}{(1+|y|^2)^2} \omega = \frac{8}{(1+|y|^2)^2} z_0(y) \text{ and } \Delta \zeta + \frac{8}{(1+|y|^2)^2} \zeta = \frac{8}{(1+|y|^2)^2} \text{ in } \mathbb{R}^2,$$

such that

$$\omega(y) = \frac{4}{3} \log |y| + O\left(\frac{1}{|y|}\right) \text{ and } \zeta(y) = O\left(\frac{1}{|y|}\right) \text{ as } |y| \rightarrow +\infty,$$

and

$$\nabla \omega(y) = \frac{4}{3} \cdot \frac{y}{1+|y|^2} + O\left(\frac{1}{1+|y|^2}\right) \text{ and } \nabla \zeta(y) = O\left(\frac{1}{1+|y|^2}\right) \text{ for all } y \in \mathbb{R}^2,$$

since

$$8 \int_0^{+\infty} r \frac{(r^2 - 1)^2}{(r^2 + 1)^4} dr = \frac{4}{3} \text{ and } 8 \int_0^{+\infty} r \frac{r^2 - 1}{(r^2 + 1)^3} dr = 0.$$

For simplicity, from now on we omit the dependence on n . For $j = 1, \dots, m$, define now

$$u_j(x) = \omega\left(\frac{x - \xi_j}{\delta_j}\right) + \frac{4}{3} (\log \delta_j) Z_{0j}(x) + \frac{8\pi}{3} H_\lambda(\xi_j, \xi_j) \zeta\left(\frac{x - \xi_j}{\delta_j}\right),$$

and denote its projection $Pu_j = u_j + \tilde{H}_j$, where $\tilde{H}_j$ is a correction term defined as the solution of

$$\begin{cases} -\Delta \tilde{H}_j = 0 & \text{in } \Omega, \\ \frac{\partial \tilde{H}_j}{\partial \nu} + \lambda b(x) \tilde{H}_j = -\frac{\partial u_j}{\partial \nu} - \lambda b(x) u_j & \text{on } \partial \Omega. \end{cases}$$

Observe that on $\partial \Omega$,

$$\left(\frac{\partial}{\partial \nu} + \lambda b(x)\right) \left(\tilde{H}_j + \frac{8\pi}{3} H_\lambda(x, \xi_j)\right) = O(\lambda^2 \rho) + (\log \delta_j) O(\lambda^3 \rho^2) + H_\lambda(\xi_j, \xi_j) O(\lambda^2 \rho).$$

By (1.5)–(1.8) and the maximum principle with Robin boundary condition, we get

$$\begin{cases} Pu_j = u_j - \frac{8\pi}{3} H_\lambda(x, \xi_j) + O(\lambda \rho \log \lambda) & \text{in } C(\overline{\Omega}), \\ Pu_j = -\frac{8\pi}{3} G_\lambda(x, \xi_j) + O\left(\lambda \rho \log \lambda + \frac{\rho \log \lambda}{|x - \xi_j|} + \frac{\rho^2 \log \rho}{|x - \xi_j|^2}\right) & \text{in } C_{\text{loc}}(\overline{\Omega} \setminus \{\xi_j\}). \end{cases} \quad (3.10)$$

The function Pu_j solves

$$\begin{cases} \Delta Pu_j + W_\xi Pu_j = e^{U_{\delta_j, \xi_j}} Z_{0j} + (W_\xi - e^{U_{\delta_j, \xi_j}}) Pu_j + R_j & \text{in } \Omega, \\ \frac{\partial}{\partial \nu} Pu_j + \lambda b(x) Pu_j = 0 & \text{on } \partial \Omega, \end{cases} \quad (3.11)$$

where

$$R_j(x) = \left(Pu_j - u_j + \frac{8\pi}{3} H_\lambda(\xi_j, \xi_j) \right) e^{U_{\delta_j, \xi_j}}. \quad (3.12)$$

Multiply (3.11) by ϕ and integrate by parts to obtain

$$\int_{\Omega} e^{U_{\delta_j, \xi_j}} Z_{0j} \phi + \int_{\Omega} (W_\xi - e^{U_{\delta_j, \xi_j}}) Pu_j \phi = \int_{\Omega} Pu_j h - \int_{\Omega} R_j \phi. \quad (3.13)$$

We estimate each term of (3.13). First of all, by Lebesgue's theorem and (3.9) we get

$$\int_{\Omega} e^{U_{\delta_j, \xi_j}} Z_{0j} \phi \rightarrow C_j \int_{\mathbb{R}^2} \frac{8(|y|^2 - 1)^2}{(|y|^2 + 1)^4} dy = \frac{8\pi}{3} C_j. \quad (3.14)$$

Note that for fixed small $\delta > 0$, there exists a constant $C > 0$ such that for all $|x - \xi_j| = \delta$ and $\lambda d(x) = O(1)$,

$$G_\lambda(x, \xi_j) \geq \frac{C}{\delta \lambda}, \quad (3.15)$$

provided that λ is sufficiently large (see [8, p. 2685]). Thus, by Lemma 3.1, (2.4) and (3.10), we have

$$\begin{aligned} \int_{\Omega} (W_\xi - e^{U_{\delta_j, \xi_j}}) Pu_j \phi &= \int_{B(\xi_j, \varepsilon d(\xi_j)/\sqrt{\delta_j})} (W_\xi - e^{U_{\delta_j, \xi_j}}) Pu_j \phi - \frac{8\pi}{3} \sum_{k=j} G_\lambda(\xi_k, \xi_j) \int_{B(\xi_k, \varepsilon d(\xi_k)/\sqrt{\delta_k})} W_\xi \phi + O(\lambda^2 \sqrt{\rho} + \lambda \rho \log \lambda) \\ &= \int_{B(0, \varepsilon d(\xi_j)/\sqrt{\delta_j})} \frac{8}{(1 + |y|^2)^2} \frac{1}{p} \left(\omega_1 - U_{1,0} - \frac{1}{2} U_{1,0}^2 \right) \frac{4}{3} (\log \delta_j) z_0(y) \hat{\phi}_j \\ &\quad - \frac{8\pi}{3} \sum_{k=j} G_\lambda(\xi_k, \xi_j) \int_{B(0, \varepsilon d(\xi_k)/\sqrt{\delta_k})} \frac{8}{(1 + |y|^2)^2} \hat{\phi}_k + O\left(\frac{1}{p}\right) \\ &= -\frac{C_j}{3} \int_{\mathbb{R}^2} \frac{8(|y|^2 - 1)^2}{(1 + |y|^2)^4} \left(\omega_1 - U_{1,0} - \frac{1}{2} U_{1,0}^2 \right) (y) + o(1), \end{aligned}$$

since Lebesgue's theorem and (3.9) imply

$$\int_{B(0, \varepsilon d(\xi_j)/\sqrt{\delta_j})} \frac{8}{(1 + |y|^2)^2} \left(\omega_1 - U_{1,0} - \frac{1}{2} U_{1,0}^2 \right) z_0(y) \hat{\phi}_j \rightarrow C_j \int_{\mathbb{R}^2} \frac{8(|y|^2 - 1)^2}{(1 + |y|^2)^4} \left(\omega_1 - U_{1,0} - \frac{1}{2} U_{1,0}^2 \right)$$

and

$$\int_{B(0, \varepsilon d(\xi_k)/\sqrt{\delta_k})} \frac{8}{(1 + |y|^2)^2} \hat{\phi}_k \rightarrow C_k \int_{\mathbb{R}^2} \frac{8}{(1 + |y|^2)^2} \frac{|y|^2 - 1}{|y|^2 + 1} = 0.$$

Using (2.10), in a straightforward but tedious way, we can compute

$$\int_{\mathbb{R}^2} \frac{8(|y|^2 - 1)^2}{(1 + |y|^2)^4} \left(\omega_1 - U_{1,0} - \frac{1}{2} U_{1,0}^2 \right) (y) dy = -8\pi$$

(see [13, p. 50]). So we obtain

$$\int_{\Omega} (W_\xi - e^{U_{\delta_j, \xi_j}}) Pu_j \phi = \frac{8\pi}{3} C_j + o(1). \quad (3.16)$$

As for the right-hand side of (3.13), by (2.4), (2.23) and (3.10), we have that

$$\begin{aligned} \left| \int_{\Omega} Pu_j h \right| &\leq C \|h\|_* \int_{\Omega} \sum_{k=1}^m \frac{\delta_k}{(\delta_k^2 + |x - \xi_k|^2)^{3/2}} |Pu_j| dx \\ &= C \|h\|_* \sum_{k=1}^m \int_{\Omega_{\delta_k}} \frac{1}{(1 + |y|^2)^{3/2}} |Pu_j(\delta_k y + \xi_k)| dy \\ &\leq C \|h\|_* \int_{\mathbb{R}^2} \frac{\log(|y| + 2)}{(1 + |y|^2)^{3/2}} dy + Cp \|h\|_* \int_{\mathbb{R}^2} \frac{1}{(1 + |y|^2)^{3/2}} dy \\ &\leq Cp \|h\|_*, \end{aligned} \quad (3.17)$$

where $\Omega_{\delta_k} := \frac{1}{\delta_k}(\Omega - \{\xi_k\})$. Finally, using (2.15) and the maximum principle with Robin boundary condition, we find that $|\nabla H_{\lambda}(x, \xi_j)| = O(\lambda \log \lambda)$ holds uniformly in Ω . By (3.10) and (3.12), we deduce

$$\int_{\Omega} R_j \phi = O \left(\int_{\Omega} e^{U_{\delta_j, \xi_j}} (|x - \xi_j| \lambda \log \lambda + \lambda \rho \log \lambda) dx \right) = O(e^{-p/4} \log \lambda). \quad (3.18)$$

Hence, inserting (3.14), (3.16), (3.17) and (3.18) in (3.13), and taking into account (3.8), we conclude that

$$\frac{16\pi}{3} C_j = o(1) \quad \text{for any } j = 1, \dots, m.$$

Necessarily, $C_j = 0$, a contradiction and the claim is proved.

Step 5:

We establish the validity of the a priori estimate

$$\|\phi\|_{\infty} \leq Cp \|h\|_* \quad (3.19)$$

for solutions of problem (3.2) and $h \in C^{0,\alpha}(\overline{\Omega})$. Step 4 gives

$$\|\phi\|_{L^{\infty}(\Omega)} \leq Cp \left(\|h\|_* + \sum_{i=1}^2 \sum_{j=1}^m |c_{ij}| \cdot \|e^{U_{\delta_j, \xi_j}} Z_{ij}\|_* \right) \leq Cp \left(\|h\|_* + \sum_{i=1}^2 \sum_{j=1}^m |c_{ij}| \right).$$

As before, arguing by contradiction of (3.19), we can proceed, as in Step 3, and suppose further that

$$\|\phi_n\|_{L^{\infty}(\Omega)} = 1, \quad p_n \|h_n\|_* \rightarrow 0 \text{ and } p_n \sum_{i=1}^2 \sum_{j=1}^m |c_{ij}^n| \geq \delta > 0 \quad \text{as } n \rightarrow +\infty. \quad (3.20)$$

We omit the dependence on n . It suffices to estimate the values of the constants c_{ij} . To this end, we define PZ_{ij} as the projection of Z_{ij} under homogeneous Robin boundary condition, namely

$$\begin{cases} \Delta PZ_{ij} = \Delta Z_{ij} = -e^{U_{\delta_j, \xi_j}} Z_{ij} & \text{in } \Omega, \\ \frac{\partial PZ_{ij}}{\partial \nu} + \lambda b(x) PZ_{ij} = 0 & \text{on } \partial\Omega. \end{cases}$$

As in Lemma 2.1, for $i = 1, 2$ and $j = 1, \dots, m$, we have the following expansions:

$$PZ_{ij} = Z_{ij} + 8\pi\delta_j \partial_{(\xi_j)_i} H_{\lambda}(\cdot, \xi_j) + O(\lambda^3 \rho^3) \text{ and } PZ_{0j} = Z_{0j} - 1 + O(\lambda^2 \rho^2) \quad (3.21)$$

in $C(\overline{\Omega})$ and in $C_{\text{loc}}^2(\Omega)$, and

$$PZ_{ij} = 8\pi\delta_j \partial_{(\xi_j)_i} G_{\lambda}(\cdot, \xi_j) + O\left(\lambda^3 \rho^3 + \frac{\rho^3}{|x - \xi_j|^3}\right) \text{ and } PZ_{0j} = O\left(\lambda^2 \rho^2 + \frac{\rho^2}{|x - \xi_j|^2}\right) \quad (3.22)$$

in $C(\overline{\Omega} \setminus \{\xi_j\})$ and in $C_{\text{loc}}^2(\Omega \setminus \{\xi_j\})$. By (3.21) and (3.22), we deduce the following “orthogonality” relations: for each $i, l = 1, 2$ and $j, k = 1, \dots, m$,

$$\int_{\Omega} e^{U_{\delta_j, \xi_j}} Z_{ij} P Z_{lk} = \left(64 \int_{\mathbb{R}^2} \frac{|y|^2}{(1 + |y|^2)^4} \right) \delta_{jk} \delta_{il} + O(\lambda \rho), \quad (3.23)$$

and

$$\int_{\Omega} e^{U_{\delta_j, \xi_j}} Z_{0j} P Z_{lk} = O(\lambda \rho) \quad (3.24)$$

uniformly for any points $\xi = (\xi_1, \dots, \xi_m) \in \mathcal{O}_\varepsilon$, where δ_{jk} and δ_{il} denote Kronecker’s symbols. Indeed, we have

$$\begin{aligned} \int_{\Omega} e^{U_{\delta_j, \xi_j}} Z_{ij} P Z_{lj} &= \int_{B(\xi_j, \varepsilon d(\xi_j))} e^{U_{\delta_j, \xi_j}} Z_{ij} \left[Z_{lj} + 8\pi \delta_j \frac{\partial H_\lambda}{\partial (\xi_j)_l}(x, \xi_j) + O(\lambda^3 \rho^3) \right] dx + O(\lambda^4 \rho^4) \\ &= \int_{B(0, \varepsilon d(\xi_j)/\delta_j)} \frac{8}{(1 + |y|^2)^2} \frac{4y_i}{1 + |y|^2} \left[\frac{4y_l}{1 + |y|^2} + O(\lambda \rho) \right] dy + O(\lambda^4 \rho^4) \\ &= \left(64 \int_{\mathbb{R}^2} \frac{|y|^2}{(1 + |y|^2)^4} \right) \delta_{il} + O(\lambda \rho), \\ \int_{\Omega} e^{U_{\delta_j, \xi_j}} Z_{0j} P Z_{lj} &= \int_{B(\xi_j, \varepsilon d(\xi_j))} e^{U_{\delta_j, \xi_j}} Z_{0j} \left[Z_{lj} + 8\pi \delta_j \frac{\partial H_\lambda}{\partial (\xi_j)_l}(x, \xi_j) + O(\lambda^3 \rho^3) \right] dx + O(\lambda^3 \rho^3) \\ &= \int_{B(0, \varepsilon d(\xi_j)/\delta_j)} \frac{8}{(1 + |y|^2)^2} \frac{|y|^2 - 1}{|y|^2 + 1} \left[\frac{4y_l}{1 + |y|^2} + O(\lambda \rho) \right] dy + O(\lambda^3 \rho^3) \\ &= O(\lambda \rho), \end{aligned}$$

and for $j \neq k$,

$$\begin{aligned} \int_{\Omega} e^{U_{\delta_j, \xi_j}} Z_{ij} P Z_{lk} &= \int_{B(\xi_j, \varepsilon d(\xi_j))} e^{U_{\delta_j, \xi_j}} Z_{ij} \left[8\pi \delta_k \frac{\partial G_\lambda}{\partial (\xi_k)_l}(x, \xi_k) + O\left(\lambda^3 \rho^3 + \frac{\rho^3}{|x - \xi_k|^3}\right) \right] dx + O(\lambda^6 \rho^4) \\ &= O\left(\lambda \rho \int_{B(\xi_j, \varepsilon d(\xi_j))} e^{U_{\delta_j, \xi_j}} Z_{ij} dx\right) + O(\lambda^6 \rho^4) \\ &= O(\lambda \rho), \\ \int_{\Omega} e^{U_{\delta_j, \xi_j}} Z_{0j} P Z_{lk} &= \int_{B(\xi_j, \varepsilon d(\xi_j))} e^{U_{\delta_j, \xi_j}} Z_{0j} \left[8\pi \delta_k \frac{\partial G_\lambda}{\partial (\xi_k)_l}(x, \xi_k) + O\left(\lambda^3 \rho^3 + \frac{\rho^3}{|x - \xi_k|^3}\right) \right] dx + O(\lambda^3 \rho^3) \\ &= O\left(\lambda \rho \int_{B(\xi_j, \varepsilon d(\xi_j))} e^{U_{\delta_j, \xi_j}} Z_{0j} dx\right) + O(\lambda^3 \rho^3) \\ &= O(\lambda \rho), \end{aligned}$$

because $|\partial_{(\xi_j)_l} H_\lambda(x, \xi_j)| = O(\lambda)$ holds uniformly in Ω , which can be directly proved by (2.15) and the maximum principle with Robin boundary condition.

Now, for $i = 1, 2$ and $j = 1, \dots, m$, multiplying (3.2) by $P Z_{ij}$ and integrating by parts we find

$$\sum_{l=1}^2 \sum_{k=1}^m c_{lk} \int_{\Omega} e^{U_{\delta_k, \xi_k}} Z_{lk} P Z_{ij} + \int_{\Omega} h P Z_{ij} = \int_{\Omega} W_\xi \phi P Z_{ij} - \int_{\Omega} e^{U_{\delta_j, \xi_j}} Z_{ij} \phi. \quad (3.25)$$

Since $|\int_{\Omega} hPZ_{ij}| = O(\|h\|_*)$, by (3.23)–(3.24),

$$\sum_{l=1}^2 \sum_{k=1}^m c_{lk} \int_{\Omega} e^{U_{\delta_k, \xi_k}} Z_{lk} PZ_{ij} + \int_{\Omega} hPZ_{ij} = Dc_{ij} + O\left(e^{-p/4} \sum_{l=1}^2 \sum_{k=1}^m |c_{lk}|\right) + O(\|h\|_*),$$

where $D = 64 \int_{\mathbb{R}^2} \frac{|y|^2}{(1+|y|^2)^4}$. Moreover, by Lemma 3.1,

$$\begin{aligned} Dc_{ij} + O\left(e^{-p/4} \sum_{l=1}^2 \sum_{k=1}^m |c_{lk}|\right) + O(\|h\|_*) &= \int_{B(\xi_j, \varepsilon d(\xi_j)/\sqrt{\delta_j})} W_{\xi} \phi PZ_{ij} - \int_{\Omega} e^{U_{\delta_j, \xi_j}} Z_{ij} \phi + O(\lambda\sqrt{\rho}\|\phi\|_{\infty}) \\ &= \int_{B(\xi_j, \varepsilon d(\xi_j)/\sqrt{\delta_j})} (W_{\xi} - e^{U_{\delta_j, \xi_j}}) \phi PZ_{ij} + \int_{\Omega} e^{U_{\delta_j, \xi_j}} (PZ_{ij} - Z_{ij}) \phi + O(\lambda\sqrt{\rho}\|\phi\|_{\infty}) \\ &= \frac{1}{p} \int_{B(0, \varepsilon d(\xi_j)/\sqrt{\delta_j})} \frac{32y_i}{(1+|y|^2)^3} \left(\omega_1 - U_{1,0} - \frac{1}{2}U_{1,0}^2\right) \widehat{\phi}_j(y) + O\left(\frac{1}{p^2}\|\phi\|_{\infty}\right), \end{aligned}$$

in view of (3.21)–(3.22), where $\widehat{\phi}_j(y) = \phi(\delta_j y + \xi_j)$. Accordingly, (3.25) can be reduced to

$$Dc_{ij} + O\left(e^{-p/4} \sum_{l=1}^2 \sum_{k=1}^m |c_{lk}|\right) = O\left(\|h\|_* + \frac{1}{p}\|\phi\|_{\infty}\right)$$

for each $i = 1, 2$ and $j = 1, \dots, m$. Therefore, we obtain

$$\sum_{l=1}^2 \sum_{k=1}^m |c_{lk}| = O\left(\|h\|_* + \frac{1}{p}\|\phi\|_{\infty}\right) = o(1).$$

As in Step 4, we conclude that for each $j = 1, \dots, m$,

$$\widehat{\phi}_j \rightarrow C_j \frac{|y|^2 - 1}{|y|^2 + 1} \quad \text{in } C_{\text{loc}}^0(\mathbb{R}^2)$$

for some constant $C_j \in \mathbb{R}$, and thus

$$\int_{B(0, \varepsilon d(\xi_j)/\sqrt{\delta_j})} \frac{32y_i}{(1+|y|^2)^3} \left(\omega_1 - U_{1,0} - \frac{1}{2}U_{1,0}^2\right) \widehat{\phi}_j(y) \rightarrow C_j \int_{\mathbb{R}^2} \frac{32y_i(|y|^2 - 1)}{(1+|y|^2)^4} \left(\omega_1 - U_{1,0} - \frac{1}{2}U_{1,0}^2\right)(y) = 0.$$

Therefore,

$$\sum_{l=1}^2 \sum_{k=1}^m |c_{lk}| = o\left(\frac{1}{p}\right) + O(\|h\|_*),$$

which is impossible because of (3.20).

Step 6:

We prove the solvability of problem (3.2). To this purpose, we consider the spaces

$$K_{\xi} = \left\{ \sum_{i=1}^2 \sum_{j=1}^m c_{ij} PZ_{ij} : c_{ij} \in \mathbb{R} \text{ for } i = 1, 2, j = 1, \dots, m \right\}$$

and

$$K_{\xi}^{\perp} = \left\{ \phi \in L^2(\Omega) : \int_{\Omega} e^{U_{\delta_j, \xi_j}} Z_{ij} \phi = 0 \text{ for } i = 1, 2, j = 1, \dots, m \right\}.$$

Define $\Pi_\xi: L^2(\Omega) \rightarrow K_\xi$ by

$$\Pi_\xi \phi = \sum_{i=1}^2 \sum_{j=1}^m c_{ij} P Z_{ij},$$

where the coefficients c_{ij} are uniquely determined (as it follows by (3.23)–(3.24)) by the system

$$\int_{\Omega} e^{U_{\delta_k, \xi_k}} Z_{lk} \left(\phi - \sum_{i=1}^2 \sum_{j=1}^m c_{ij} P Z_{ij} \right) = 0 \quad \text{for any } l = 1, 2, k = 1, \dots, m.$$

Let $\Pi_\xi^\perp = \text{Id} - \Pi_\xi: L^2(\Omega) \rightarrow K_\xi^\perp$. Moreover, the Hilbert space $K_\xi^\perp \cap H^1(\Omega)$ is endowed with the inner product

$$\langle \phi, \psi \rangle_H = \int_{\Omega} \nabla \phi \nabla \psi + \lambda \int_{\partial \Omega} b(x) \phi \psi.$$

Problem (3.2), expressed in a weak form, is equivalent to finding $\phi \in K_\xi^\perp \cap H^1(\Omega)$ such that

$$\langle \phi, \psi \rangle_H = \int_{\Omega} (W_\xi \phi - h) \psi \quad \text{for all } \psi \in K_\xi^\perp \cap H^1(\Omega).$$

With the aid of Riesz's theorem, this equation gets rewritten in $K_\xi^\perp \cap H^1(\Omega)$ in the operator form

$$\phi = K(\phi) + \tilde{h},$$

where

$$\tilde{h} = -\Pi_\xi^\perp \left[(-\Delta)|_{\Omega} + \left(\frac{\partial}{\partial \nu} + \lambda b(x) \right) \Big|_{\partial \Omega} \right]^{-1} h \quad \text{and} \quad K(\phi) = \Pi_\xi^\perp \left[(-\Delta)|_{\Omega} + \left(\frac{\partial}{\partial \nu} + \lambda b(x) \right) \Big|_{\partial \Omega} \right]^{-1} (W_\xi \phi)$$

is a linear compact operator in $K_\xi^\perp \cap H^1(\Omega)$. By Fredholm's alternative with Robin boundary condition (see [6, 14] and references therein), we obtain the unique solvability of this problem for any $\tilde{h} \in K_\xi^\perp$ provided that the homogeneous equation $\phi = K(\phi)$ has only the trivial solution in $K_\xi^\perp \cap H^1(\Omega)$, which in turn follows from the a priori estimate (3.19) in Step 5. Moreover, by the elliptic regularity theory the solution constructed in this way belongs to $H^2(\Omega)$. For $p > p_0$ and $\lambda > \lambda_0$ fixed and under the restrained growth condition (2.4), and by the density of $C^{0,\alpha}(\overline{\Omega})$ in $(C(\overline{\Omega}), \|\cdot\|_\infty)$, we can approximate $h \in C(\overline{\Omega})$ by smooth functions and, by (3.19) and the elliptic regularity theory, we can show that (3.4) holds for any $h \in C(\overline{\Omega})$. This completes the proof. $\square$

Remark 3.4. Given $h \in C(\overline{\Omega})$, let ϕ be the solution of problem (3.2) given by Proposition 3.3. Multiplying (3.2) against ϕ and integrating by parts, we get

$$\|\phi\|_H^2 := \int_{\Omega} |\nabla \phi|^2 + \lambda \int_{\partial \Omega} b(x) \phi^2 = \int_{\Omega} W_\xi \phi^2 - \int_{\Omega} h \phi.$$

By Lemma 3.1, we can prove that $|\int_{\Omega} W_\xi \phi^2| \leq C \|\phi\|_\infty^2$, and therefore $\|\phi\|_H \leq C (\|h\|_* + \|\phi\|_\infty)$.

4 The nonlinear problem

In what follows we want to solve the nonlinear auxiliary problem: for any points $\xi = (\xi_1, \dots, \xi_m) \in \mathcal{O}_\varepsilon$, we look for a function $\phi \in H^2(\Omega)$ such that

$$\begin{cases} \Delta(U_\xi + \phi) + (U_\xi + \phi)^p = \sum_{i=1}^2 \sum_{j=1}^m c_{ij} e^{U_{\delta_j, \xi_j}} Z_{ij} & \text{in } \Omega, \\ U_\xi + \phi > 0 & \text{in } \Omega, \\ \frac{\partial \phi}{\partial \nu} + \lambda b(x) \phi = 0 & \text{on } \partial \Omega, \\ \int_{\Omega} e^{U_{\delta_j, \xi_j}} Z_{ij} \phi = 0 & \text{for } i = 1, 2, j = 1, \dots, m, \end{cases} \quad (4.1)$$

for some coefficients c_{ij} , $i = 1, 2, j = 1, \dots, m$, which depend on ξ . Recalling that

$$N(\phi) = (U_\xi + \phi)_+^p - U_\xi^p - pU_\xi^{p-1}\phi \text{ and } R_\xi = \Delta U_\xi + U_\xi^p,$$

we can rewrite the first equation in (4.1) in the form

$$L(\phi) = -[R_\xi + N(\phi)] + \sum_{i=1}^2 \sum_{j=1}^m c_{ij} e^{U_{\delta_j, \xi_j}} Z_{ij}.$$

Using the theory developed in the previous section for the linear operator L , we have the following result.

Proposition 4.1. *Let $\varepsilon > 0$ be fixed and small. There exist $C > 0$, $p_0 > 0$ and $\lambda_0 > 0$ such that for any points $\xi = (\xi_1, \dots, \xi_m) \in \mathcal{O}_\varepsilon$, and any $p > p_0$ and $\lambda > \lambda_0$ satisfying (2.4), problem (4.1) has a unique solution ϕ_ξ for some scalars $c_{ij}(\xi)$, $i = 1, 2, j = 1, \dots, m$, such that*

$$\|\phi_\xi\|_\infty \leq \frac{C}{p^3}, \quad \sum_{i=1}^2 \sum_{j=1}^m |c_{ij}(\xi)| \leq \frac{C}{p^4} \text{ and } \|\phi_\xi\|_H \leq \frac{C}{p^3}. \quad (4.2)$$

Furthermore, the map $\xi \rightarrow \phi_\xi$ is a C^1 -function in $C(\overline{\Omega})$ and $H^1(\Omega)$.

Proof. The proof of this proposition can be done along the lines of the proof of [13, Lemma 4.1]. We omit the details. $\square$

5 Variational reduction

After problem (4.1) has been solved, we find a solution of problem (2.22) and hence to the original problem (1.1) if $\xi = (\xi_1, \dots, \xi_m) \in \mathcal{O}_\varepsilon$ satisfies

$$c_{ij}(\xi) = 0 \quad \text{for all } i = 1, 2, j = 1, \dots, m. \quad (5.1)$$

Equation (1.1) is the Euler–Lagrange equation of the functional $J_p^\lambda: H^1(\Omega) \rightarrow \mathbb{R}$, defined by

$$J_p^\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{p+1} \int_\Omega u_+^{p+1} + \frac{\lambda}{2} \int_{\partial\Omega} b(x) u^2. \quad (5.2)$$

We define

$$F_p^\lambda(\xi) = J_p^\lambda(U_\xi + \phi_\xi) \quad \text{for all } \xi \in \mathcal{O}_\varepsilon, \quad (5.3)$$

where ϕ_ξ is the unique solution to problem (4.1) given by Proposition 4.1. The critical points of F_p^λ correspond to solutions of (5.1) for all sufficiently large p and λ satisfying (2.4), as the following result states.

Proposition 5.1. *The function $F_p^\lambda: \mathcal{O}_\varepsilon \rightarrow \mathbb{R}$ is of class C^1 . Moreover, for all sufficiently large p and λ satisfying (2.4), if $D_\xi F_p^\lambda(\xi) = 0$, then ξ satisfies (5.1).*

Proof. Since the map $\xi \rightarrow \phi_\xi$ is a C^1 -map into $H^1(\Omega)$, F_p^λ is a C^1 -function of ξ . Suppose now that $\xi \in \mathcal{O}_\varepsilon$ is a point such that $D_\xi F_p^\lambda(\xi) = 0$. Then, we have that

$$\begin{aligned} 0 &= \int_\Omega \nabla(U_\xi + \phi_\xi) \nabla[D_\xi(U_\xi + \phi_\xi)] - \int_\Omega (U_\xi + \phi_\xi)^p D_\xi(U_\xi + \phi_\xi) + \lambda \int_{\partial\Omega} b(x)(U_\xi + \phi_\xi) D_\xi(U_\xi + \phi_\xi) \\ &= - \int_\Omega [\Delta(U_\xi + \phi_\xi) + (U_\xi + \phi_\xi)^p] D_\xi(U_\xi + \phi_\xi) \\ &= - \sum_{i=1}^2 \sum_{j=1}^m c_{ij}(\xi) \int_\Omega e^{U_{\delta_j, \xi_j}} Z_{ij} D_\xi(U_\xi + \phi_\xi) \\ &= - \sum_{i=1}^2 \sum_{j=1}^m c_{ij}(\xi) \int_\Omega e^{U_{\delta_j, \xi_j}} Z_{ij} D_\xi U_\xi + \sum_{i=1}^2 \sum_{j=1}^m c_{ij}(\xi) \int_\Omega D_\xi(e^{U_{\delta_j, \xi_j}} Z_{ij}) \phi_\xi, \end{aligned} \quad (5.4)$$

because $\int_{\Omega} e^{U_{\delta_j, \xi_j}} Z_{ij} \phi_{\xi} = 0$. From the definition of U_{ξ} in (2.12), we obtain

$$\begin{aligned} \partial_{(\xi_k)_l} U_{\xi} &= \sum_{j=1}^m \frac{1}{y\mu_j^{2/(p-1)}} \left\{ \partial_{(\xi_k)_l} \left[U_{\delta_j, \xi_j}(x) + \frac{1}{p} \omega_1 \left(\frac{x - \xi_j}{\delta_j} \right) + \frac{1}{p^2} \omega_2 \left(\frac{x - \xi_j}{\delta_j} \right) + y\mu_j^{2/(p-1)} H_j(x) \right] \right. \\ &\quad \left. - \frac{2\partial_{(\xi_k)_l} \log \mu_j}{p-1} \left[U_{\delta_j, \xi_j}(x) + \frac{1}{p} \omega_1 \left(\frac{x - \xi_j}{\delta_j} \right) + \frac{1}{p^2} \omega_2 \left(\frac{x - \xi_j}{\delta_j} \right) + y\mu_j^{2/(p-1)} H_j(x) \right] \right\} \\ &= \sum_{j=1}^m \frac{1}{y\mu_j^{2/(p-1)}} \left\{ \partial_{(\xi_k)_l} \left[U_{\delta_j, \xi_j}(x) + \frac{1}{p} \omega_1 \left(\frac{x - \xi_j}{\delta_j} \right) + \frac{1}{p^2} \omega_2 \left(\frac{x - \xi_j}{\delta_j} \right) + y\mu_j^{2/(p-1)} H_j(x) \right] \right. \\ &\quad \left. - \frac{2\partial_{(\xi_k)_l} \log \mu_j}{p-1} [p + O(1)] \right\}. \end{aligned}$$

Notice that by (2.1) and (3.3),

$$\partial_{(\xi_k)_l} U_{\delta_j, \xi_j}(x) = \frac{1}{\delta_j} \delta_{kj} Z_{lj} + 2Z_{0j} \partial_{(\xi_k)_l} \log \mu_j = \frac{1}{\delta_j} \delta_{kj} Z_{lj} + 2 \left(1 - \frac{2\delta_j^2}{|x - \xi_j|^2 + \delta_j^2} \right) \partial_{(\xi_k)_l} \log \mu_j,$$

while for $i = 1, 2$, by (2.9) and (3.3),

$$\begin{aligned} \partial_{(\xi_k)_l} \omega_i \left(\frac{x - \xi_j}{\delta_j} \right) &= -\omega'_i \left(\frac{|x - \xi_j|}{\delta_j} \right) \left\{ \frac{(x - \xi_j)_l}{|x - \xi_j|} \frac{1}{\delta_j} \delta_{kj} + \frac{|x - \xi_j|}{\delta_j} \partial_{(\xi_k)_l} \log \mu_j \right\} \\ &= -\frac{1}{\delta_j} \delta_{kj} \left[\frac{C_i}{4} Z_{lj} + O \left(\frac{\delta_j^2}{|x - \xi_j|^2 + \delta_j^2} \right) \right] - C_i \left[1 + O \left(\frac{\delta_j^2}{|x - \xi_j|^2 + \delta_j^2} \right) \right] \partial_{(\xi_k)_l} \log \mu_j, \end{aligned}$$

where δ_{kj} denotes Kronecker's symbol. Moreover, as in the proof of Lemma 2.1, we can prove that

$$\begin{aligned} \partial_{(\xi_k)_l} [y\mu_j^{2/(p-1)} H_j(x)] &= \delta_{kj} \left(1 - \frac{C_1}{4p} - \frac{C_2}{4p^2} \right) 8\pi \partial_{(\xi_k)_l} H_{\lambda}(x, \xi_j) + \left(-2 + \frac{C_1}{p} + \frac{C_2}{p^2} \right) \partial_{(\xi_k)_l} \log \mu_j \\ &\quad + \frac{1}{p} O \left(\frac{\rho}{d(\xi_j)^3} + \frac{\rho}{d(\xi_j)^2} |\partial_{(\xi_k)_l} \log \mu_j| \right). \end{aligned}$$

Furthermore, by (2.3)–(2.5), (2.19) and (3.21), we get

$$\begin{aligned} \partial_{(\xi_k)_l} U_{\xi} &= \frac{1}{\delta_k y \mu_k^{2/(p-1)}} \left\{ Z_{lk} - \frac{1}{p} \omega'_1 \left(\frac{|x - \xi_k|}{\delta_k} \right) \frac{(x - \xi_k)_l}{|x - \xi_k|} - \frac{1}{p^2} \omega'_2 \left(\frac{|x - \xi_k|}{\delta_k} \right) \frac{(x - \xi_k)_l}{|x - \xi_k|} \right. \\ &\quad \left. + \left(1 - \frac{C_1}{4p} - \frac{C_2}{4p^2} \right) 8\pi \delta_k \partial_{(\xi_k)_l} H_{\lambda}(x, \xi_k) \right\} \\ &\quad + \sum_{j=1}^m \frac{1}{y\mu_j^{2/(p-1)}} \left\{ \left[-2 + O \left(\frac{1}{p} + \frac{\delta_j^2}{|x - \xi_j|^2 + \delta_j^2} \right) \right] \partial_{(\xi_k)_l} \log \mu_j + O \left(\frac{\lambda^3 \rho}{p} \right) \right\} \\ &= \frac{1}{\delta_k y \mu_k^{2/(p-1)}} \left\{ \left(1 - \frac{C_1}{4p} - \frac{C_2}{4p^2} \right) P Z_{lk} + O \left(\lambda^3 \rho^3 + \frac{1}{p} \frac{\delta_k^2}{|x - \xi_k|^2 + \delta_k^2} \right) \right\} + O \left(\frac{\sqrt{\lambda}}{y} \right). \quad (5.5) \end{aligned}$$

On the other hand, by (2.1) and (3.3), for $i = 1, 2$, we have that

$$\partial_{(\xi_k)_l} (e^{U_{\delta_j, \xi_j}} Z_{ij}) = -4\delta_j e^{U_{\delta_j, \xi_j}} \left[\frac{\delta_{il}}{|x - \xi_j|^2 + \delta_j^2} - 6 \frac{(x - \xi_j)_i (x - \xi_k)_l}{(|x - \xi_j|^2 + \delta_j^2)^2} \right] \delta_{jk} + 3e^{U_{\delta_j, \xi_j}} Z_{0j} \partial_{(\xi_k)_l} \log \mu_j. \quad (5.6)$$

Consequently, for each $l = 1, 2$ and $k = 1, \dots, m$, (5.4) can be written as

$$-\sum_{i,j} \frac{c_{ij}(\xi) [1 + O(\frac{1}{p})]}{\delta_k y \mu_k^{2/(p-1)}} \int_{\Omega} e^{U_{\delta_j, \xi_j}} Z_{ij} P Z_{lk} + \sum_{i,j} |c_{ij}(\xi)| O \left(\frac{\sqrt{\lambda}}{y} + \|\phi_{\xi}\|_{\infty} \int_{\Omega} |\partial_{(\xi_k)_l} (e^{U_{\delta_j, \xi_j}} Z_{ij})| \right) = 0,$$

so that, using (2.19), (3.23), (4.2) and (5.6),

$$-\frac{64c_{lk}(\xi)}{\delta_k y \mu_k^{2/(p-1)}} \int_{\mathbb{R}^2} \frac{|y|^2}{(1+|y|^2)^4} dy + O\left(\frac{1}{\delta_k p y} + \frac{\lambda}{y} + \frac{1}{p^3 \delta_k} + \frac{\sqrt{\lambda}}{p^3}\right) \sum_{i=1}^2 \sum_{j=1}^m |c_{ij}(\xi)| = 0,$$

which, together with (2.5), implies $c_{lk}(\xi) = 0$ for each $l = 1, 2$ and $k = 1, \dots, m$. $\square$

Next, we need to write the expansion of $F_p^\lambda(\xi)$ in terms of $\varphi_m^\lambda(\xi)$ as p and λ go to $+\infty$ and satisfy (2.4).

Proposition 5.2. *Let $\varepsilon > 0$ be fixed and small. With the choice (2.17) for the parameters μ_j , there exist $p_0 > 0$ and $\lambda_0 > 0$ such that for any points $\xi = (\xi_1, \dots, \xi_m) \in \mathcal{O}_\varepsilon$, and any $p > p_0$ and $\lambda > \lambda_0$ satisfying (2.4), the following expansion holds uniformly:*

$$F_p^\lambda(\xi) = \frac{4\pi m p}{y^2} - \frac{16\pi m \log \lambda}{y^2} - \frac{32\pi^2}{y^2} \varphi_m^\lambda(\xi) + \frac{4\pi m}{y^2} + \frac{m}{2y^2} \int_{\mathbb{R}^2} \left(\frac{8}{(1+|y|^2)^2} U_{1,0} - \Delta \omega_1 \right) + O\left(\frac{\log \lambda}{p^3}\right). \quad (5.7)$$

Proof. Multiplying the first equation in (4.1) by $U_\xi + \phi_\xi$ and integrating by parts, we obtain

$$\int_{\Omega} (U_\xi + \phi_\xi)^{p+1} = \int_{\Omega} |\nabla(U_\xi + \phi_\xi)|^2 + \lambda \int_{\partial\Omega} b(x)(U_\xi + \phi_\xi)^2 + \sum_{i=1}^2 \sum_{j=1}^m c_{ij}(\xi) \int_{\Omega} e^{U_{\delta_j, \xi_j}} Z_{ij} U_\xi.$$

Since U_ξ is a bounded function, by (4.2) we get

$$\int_{\Omega} (U_\xi + \phi_\xi)^{p+1} = \int_{\Omega} |\nabla(U_\xi + \phi_\xi)|^2 + \lambda \int_{\partial\Omega} b(x)(U_\xi + \phi_\xi)^2 + O\left(\frac{1}{p^4}\right)$$

uniformly for any points $\xi = (\xi_1, \dots, \xi_m) \in \mathcal{O}_\varepsilon$. Hence, by (5.2)–(5.3) we have

$$\begin{aligned} F_p^\lambda(\xi) &= \left(\frac{1}{2} - \frac{1}{p+1}\right) \left[\int_{\Omega} |\nabla(U_\xi + \phi_\xi)|^2 + \lambda \int_{\partial\Omega} b(x)(U_\xi + \phi_\xi)^2 \right] + O\left(\frac{1}{p^4}\right) \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right) \left[\left(\int_{\Omega} |\nabla U_\xi|^2 + \lambda \int_{\partial\Omega} b(x) U_\xi^2 \right) + 2 \left(\int_{\Omega} \nabla U_\xi \nabla \phi_\xi + \lambda \int_{\partial\Omega} b(x) U_\xi \phi_\xi \right) \right. \\ &\quad \left. + \left(\int_{\Omega} |\nabla \phi_\xi|^2 + \lambda \int_{\partial\Omega} b(x) \phi_\xi^2 \right) \right] + O\left(\frac{1}{p^4}\right). \end{aligned} \quad (5.8)$$

In view of (2.16), (2.20) and (2.25) we have

$$\begin{aligned} \int_{\Omega} |\nabla U_\xi|^2 + \lambda \int_{\partial\Omega} b(x) U_\xi^2 &= \int_{\Omega} (-\Delta U_\xi) U_\xi \\ &= \sum_{j=1}^m \frac{1}{y \mu_j^{2/(p-1)}} \int_{B(\xi_j, \varepsilon d(\xi_j))} \left[e^{U_{\delta_j, \xi_j}} - \frac{1}{p \delta_j^2} \Delta \omega_1 \left(\frac{x - \xi_j}{\delta_j} \right) - \frac{1}{p^2 \delta_j^2} \Delta \omega_2 \left(\frac{x - \xi_j}{\delta_j} \right) \right] U_\xi \\ &\quad + O(\rho^2 \lambda^2 \log \lambda) \\ &= \sum_{j=1}^m \frac{1}{y^2 \mu_j^{4/(p-1)}} \int_{B(0, \varepsilon d(\xi_j)/\delta_j)} \left[\frac{8}{(1+|y|^2)^2} - \frac{1}{p} \Delta \omega_1(y) - \frac{1}{p^2} \Delta \omega_2(y) \right] \\ &\quad \times \left[p + U_{1,0}(y) + \frac{1}{p} \omega_1(y) + \frac{1}{p^2} \omega_2(y) + O(\lambda \rho |y|) + \frac{1}{p} O(\lambda \rho) \right] dy + O(\rho^2 \lambda^2 \log \lambda) \\ &= \sum_{j=1}^m \frac{1}{y^2 \mu_j^{4/(p-1)}} \left[8\pi p + \int_{\mathbb{R}^2} \left(\frac{8}{(1+|y|^2)^2} U_{1,0} - \Delta \omega_1 \right) + O\left(\frac{1}{p}\right) \right] \\ &= \frac{8\pi m p}{y^2} - \frac{32\pi}{y^2} \sum_{j=1}^m \log \mu_j + \frac{m}{y^2} \int_{\mathbb{R}^2} \left(\frac{8}{(1+|y|^2)^2} U_{1,0} - \Delta \omega_1 \right) + O\left(\frac{1}{p^3}\right), \end{aligned}$$

since $\mu_j^{-4/(p-1)} = 1 - \frac{4}{p} \log \mu_j + O(\frac{1}{p^2})$. Recalling the expansion (2.17) of μ_j and the definition (1.9) of $\varphi_m^\lambda(\xi)$, we get

$$\begin{aligned} \int_{\Omega} |\nabla U_\xi|^2 + \lambda \int_{\partial\Omega} b(x) U_\xi^2 &= \frac{8\pi m p}{y^2} - \frac{32\pi m \log \lambda}{y^2} - \frac{64\pi^2}{y^2} \varphi_m^\lambda(\xi) \\ &\quad + \frac{24\pi m}{y^2} + \frac{m}{y^2} \int_{\mathbb{R}^2} \left(\frac{8}{(1+|y|^2)^2} U_{1,0} - \Delta \omega_1 \right) + O\left(\frac{\log \lambda}{p^3}\right). \end{aligned} \quad (5.9)$$

On the other hand, by virtue of (4.2), we have

$$2 \left(\int_{\Omega} \nabla U_\xi \nabla \phi_\xi + \lambda \int_{\partial\Omega} b(x) U_\xi \phi_\xi \right) + \left(\int_{\Omega} |\nabla \phi_\xi|^2 + \lambda \int_{\partial\Omega} b(x) \phi_\xi^2 \right) = O\left(\frac{1}{p^{7/2}}\right). \quad (5.10)$$

Consequently, inserting (5.9) and (5.10) in (5.8), we obtain (5.7). $\square$

Finally, we want to show that the expansion of $F_p^\lambda(\xi)$ in terms of $\varphi_m^\lambda(\xi)$ holds in a C^1 -sense.

Proposition 5.3. *Let $\varepsilon > 0$ be fixed and small. Then, there exist $p_0 > 0$ and $\lambda_0 > 0$ such that for any points $\xi = (\xi_1, \dots, \xi_m) \in \mathcal{O}_\varepsilon$, and any $p > p_0$ and $\lambda > \lambda_0$ satisfying (2.4), the following expansion holds uniformly:*

$$\nabla_{(\xi_k)_l} F_p^\lambda(\xi) = -\frac{32\pi^2}{y^2} \nabla_{(\xi_k)_l} \varphi_m^\lambda(\xi_1, \dots, \xi_m) + O\left(\frac{\lambda}{p^5}\right) + O\left(\frac{1}{p^3}\right),$$

where $l = 1, 2$ and $k = 1, \dots, m$.

Proof. Observe that for any $l = 1, 2$ and $k = 1, \dots, m$,

$$\partial_{(\xi_k)_l} F_p^\lambda(\xi) = - \int_{\Omega} (\Delta u_\xi + u_\xi^p) \partial_{(\xi_k)_l} U_\xi - \int_{\Omega} (\Delta u_\xi + u_\xi^p) \partial_{(\xi_k)_l} \phi_\xi, \quad (5.11)$$

where $u_\xi = U_\xi + \phi_\xi$. For each $k = 1, \dots, m$, let η_k be a radial smooth cut-off function such that $0 \leq \eta_k \leq 1$, $\eta_k = 1$ if $|x| \leq \varepsilon d(\xi_k)$, and $\eta_k = 0$ if $|x| \geq 2\varepsilon d(\xi_k)$. In view of (4.1), we deduce

$$\begin{aligned} \int_{\Omega} (\Delta u_\xi + u_\xi^p) \partial_{(\xi_k)_l} \phi_\xi &= \sum_{i=1}^2 \sum_{j=1}^m c_{ij}(\xi) \int_{\Omega} e^{U_{\delta_j, \xi_j}} Z_{ij} \partial_{(\xi_k)_l} \phi_\xi \\ &= - \sum_{i=1}^2 \sum_{j=1}^m c_{ij}(\xi) \int_{\Omega} \partial_{(\xi_k)_l} (e^{U_{\delta_j, \xi_j}} Z_{ij}) \phi_\xi \\ &= \sum_{i=1}^2 \sum_{j=1}^m c_{ij}(\xi) \int_{\Omega} \partial_{x_l} (e^{U_{\delta_j, \xi_j}} Z_{ij}) \eta_k(x - \xi_k) \phi_\xi \\ &\quad - \sum_{i=1}^2 \sum_{j=1}^m c_{ij}(\xi) \int_{\Omega} [\partial_{(\xi_k)_l} (e^{U_{\delta_j, \xi_j}} Z_{ij}) + \eta_k(x - \xi_k) \partial_{x_l} (e^{U_{\delta_j, \xi_j}} Z_{ij})] \phi_\xi. \end{aligned}$$

Observe that $p^2 \phi_\xi(x) \rightarrow 0$ in $C(\overline{\Omega})$ and thus, by the elliptic regularity theory,

$$p^2 \phi_\xi(x) \rightarrow 0 \quad \text{uniformly in } C_{\text{loc}}^1 \left(\Omega \setminus \bigcup_{h=1}^m B(\xi_h, \varepsilon d(\xi_h)) \right). \quad (5.12)$$

Since $\eta_k(x - \xi_k) \equiv 0$ on $\partial\Omega$, using an integration by parts of the derivative in x_l and (4.2), we get

$$\begin{aligned} \int_{\Omega} \partial_{x_l} (e^{U_{\delta_j, \xi_j}} Z_{ij}) \eta_k(x - \xi_k) \phi_\xi &= \int_{\Omega} \partial_{x_l} (e^{U_{\delta_j, \xi_j}} Z_{ij} \eta_k(x - \xi_k) \phi_\xi) - \int_{\Omega} e^{U_{\delta_j, \xi_j}} Z_{ij} \partial_{x_l} (\eta_k(x - \xi_k) \phi_\xi) \\ &= - \int_{\Omega} e^{U_{\delta_j, \xi_j}} Z_{ij} \partial_{x_l} (\eta_k(x - \xi_k) \phi_\xi) \\ &= - \int_{B(\xi_k, \varepsilon d(\xi_k))} e^{U_{\delta_j, \xi_j}} Z_{ij} \partial_{x_l} \phi_\xi + O\left(\frac{\lambda^4 p^3}{p^3}\right), \end{aligned}$$

in view of $|\partial_{x_l} \eta_k(x - \xi_k)| = O(\lambda)$. On the other hand, by (5.6) we have

$$\begin{aligned} & \partial_{(\xi_k)_l} (e^{U_{\delta_j, \xi_j}} Z_{ij}) + \eta_k(x - \xi_k) \partial_{x_l} (e^{U_{\delta_j, \xi_j}} Z_{ij}) \\ &= 4\delta_j e^{U_{\delta_j, \xi_j}} \frac{\eta_k(x - \xi_k) - \delta_{jk}}{|x - \xi_j|^2 + \delta_j^2} \delta_{il} + 24\delta_j e^{U_{\delta_j, \xi_j}} \frac{(x - \xi_j)_i (x - \xi_k)_l}{(|x - \xi_j|^2 + \delta_j^2)^2} (\delta_{jk} - \eta_k(x - \xi_k)) + 3e^{U_{\delta_j, \xi_j}} Z_{0j} Z_{ij} \partial_{(\xi_k)_l} \log \mu_j \\ &= 3e^{U_{\delta_j, \xi_j}} Z_{0j} Z_{ij} \partial_{(\xi_k)_l} \log \mu_j + O(\lambda^6 \rho^3), \end{aligned}$$

which, together with (2.19), (4.2), and the fact that $|Z_{0j} Z_{ij}| \leq 2$, implies

$$\left| \sum_{i=1}^2 \sum_{j=1}^m c_{ij}(\xi) \int_{\Omega} [\partial_{(\xi_k)_l} (e^{U_{\delta_j, \xi_j}} Z_{ij}) + \eta_k(x - \xi_k) \partial_{x_l} (e^{U_{\delta_j, \xi_j}} Z_{ij})] \phi_{\xi} \right| = O\left(\frac{\sqrt{\lambda}}{p^7}\right).$$

Therefore,

$$\begin{aligned} \int_{\Omega} (\Delta u_{\xi} + u_{\xi}^p) \partial_{(\xi_k)_l} \phi_{\xi} &= - \sum_{i=1}^2 \sum_{j=1}^m c_{ij}(\xi) \int_{B(\xi_k, \varepsilon d(\xi_k))} e^{U_{\delta_j, \xi_j}} Z_{ij} \partial_{x_l} \phi_{\xi} + O\left(\frac{\sqrt{\lambda}}{p^7}\right) \\ &= - \int_{B(\xi_k, \varepsilon d(\xi_k))} (\Delta u_{\xi} + u_{\xi}^p) \partial_{x_l} \phi_{\xi} + O\left(\frac{\sqrt{\lambda}}{p^7}\right). \end{aligned} \quad (5.13)$$

Now, by the expression of U_{ξ} in (2.12) we obtain

$$\partial_{x_l} U_{\xi} = \sum_{j=1}^m \frac{1}{\gamma \mu_j^{2/(p-1)}} \partial_{x_l} \left[U_{\delta_j, \xi_j}(x) + \frac{1}{p} \omega_1 \left(\frac{x - \xi_j}{\delta_j} \right) + \frac{1}{p^2} \omega_2 \left(\frac{x - \xi_j}{\delta_j} \right) + \gamma \mu_j^{2/(p-1)} H_j(x) \right].$$

Obviously, by (2.1) and (3.3),

$$\partial_{x_l} U_{\delta_j, \xi_j}(x) = -\frac{1}{\delta_j} Z_{lj},$$

while for $i = 1, 2$, by (2.9) and (3.3),

$$\partial_{x_l} \omega_i \left(\frac{x - \xi_j}{\delta_j} \right) = \omega'_i \left(\frac{|x - \xi_j|}{\delta_j} \right) \frac{(x - \xi_j)_l}{|x - \xi_j|} \frac{1}{\delta_j} = \frac{1}{\delta_j} \left[\frac{C_i}{4} Z_{lj} + O\left(\frac{\delta_j^2}{|x - \xi_j|^2 + \delta_j^2} \right) \right].$$

Moreover, as in the proof of Lemma 2.1, for $|x - \xi_k| \leq 2\varepsilon d(\xi_k)$, we have that

$$\partial_{x_l} [\gamma \mu_j^{2/(p-1)} H_j(x)] = \left(1 - \frac{C_1}{4p} - \frac{C_2}{4p^2} \right) 8\pi \partial_{x_l} H_{\lambda}(x, \xi_j) + \frac{1}{p} O\left(\frac{\rho}{d(\xi_j) d(\xi_k)} \right).$$

Then, for $|x - \xi_k| \leq 2\varepsilon d(\xi_k)$,

$$\begin{aligned} \partial_{x_l} U_{\xi} &= \sum_{j=1}^m \frac{1}{\delta_j \gamma \mu_j^{2/(p-1)}} \left\{ -Z_{lj} + \frac{1}{p} \omega'_1 \left(\frac{|x - \xi_j|}{\delta_j} \right) \frac{(x - \xi_j)_l}{|x - \xi_j|} + \frac{1}{p^2} \omega'_2 \left(\frac{|x - \xi_j|}{\delta_j} \right) \frac{(x - \xi_j)_l}{|x - \xi_j|} \right. \\ &\quad \left. + \left(1 - \frac{C_1}{4p} - \frac{C_2}{4p^2} \right) 8\pi \delta_j \partial_{x_l} H_{\lambda}(x, \xi_j) + \frac{1}{p} O(\lambda^2 \rho^2) \right\} \\ &= \frac{1}{\delta_k \gamma \mu_k^{2/(p-1)}} \left\{ -Z_{lk} + \frac{1}{p} \omega'_1 \left(\frac{|x - \xi_k|}{\delta_k} \right) \frac{(x - \xi_k)_l}{|x - \xi_k|} + \frac{1}{p^2} \omega'_2 \left(\frac{|x - \xi_k|}{\delta_k} \right) \frac{(x - \xi_k)_l}{|x - \xi_k|} \right. \\ &\quad \left. + \left(1 - \frac{C_1}{4p} - \frac{C_2}{4p^2} \right) 8\pi \delta_k \partial_{x_l} H_{\lambda}(x, \xi_k) \right\} + O\left(\frac{1}{\gamma} \right). \end{aligned} \quad (5.14)$$

Combing (5.5) and (5.14), we find

$$\partial_{(\xi_k)_l} U_{\xi} + \eta_k(x - \xi_k) \partial_{x_l} U_{\xi} = \frac{8\pi [1 + O(\frac{1}{p})]}{\gamma \mu_k^{2/(p-1)}} [\partial_{(\xi_k)_l} H_{\lambda}(x, \xi_k) + \eta_k(x - \xi_k) \partial_{x_l} H_{\lambda}(x, \xi_k)] + O\left(\frac{\sqrt{\lambda}}{\gamma} \right) = O\left(\frac{\lambda}{\gamma} \right).$$

Furthermore,

$$\begin{aligned}
 \int_{\Omega} (\Delta u_{\xi} + u_{\xi}^p) \partial_{(\xi_k)_l} U_{\xi} &= \sum_{i=1}^2 \sum_{j=1}^m c_{ij}(\xi) \int_{\Omega} e^{U_{\delta_j, \xi_j}} Z_{ij} [\partial_{(\xi_k)_l} U_{\xi} + \eta_k(x - \xi_k) \partial_{x_l} U_{\xi}] \\
 &\quad - \sum_{i=1}^2 \sum_{j=1}^m c_{ij}(\xi) \int_{\Omega} e^{U_{\delta_j, \xi_j}} Z_{ij} \eta_k(x - \xi_k) \partial_{x_l} U_{\xi} \\
 &= - \sum_{i=1}^2 \sum_{j=1}^m c_{ij}(\xi) \int_{B(\xi_k, \varepsilon d(\xi_k))} e^{U_{\delta_j, \xi_j}} Z_{ij} \partial_{x_l} U_{\xi} + O\left(\frac{\lambda}{p^5}\right) \\
 &= - \int_{B(\xi_k, \varepsilon d(\xi_k))} (\Delta u_{\xi} + u_{\xi}^p) \partial_{x_l} U_{\xi} + O\left(\frac{\lambda}{p^5}\right). \tag{5.15}
 \end{aligned}$$

Inserting (5.13) and (5.15) in (5.11), we get

$$\partial_{(\xi_k)_l} F_p^{\lambda}(\xi) = \int_{B(\xi_k, \varepsilon d(\xi_k))} (\Delta u_{\xi} + u_{\xi}^p) \partial_{x_l} u_{\xi} + O\left(\frac{\lambda}{p^5}\right). \tag{5.16}$$

Finally, by integrating by parts we have the following Pohozaev-type identities. For any ball $B \subset \Omega$ and for any function u ,

$$\int_B \Delta u \partial_{x_l} u = \int_{\partial B} \left(\partial_{\nu} u \partial_{x_l} u - \frac{1}{2} |\nabla u|^2 \nu_l \right) \text{ and } \int_B u^p \partial_{x_l} u = \frac{1}{p+1} \int_{\partial B} u^{p+1} \nu_l, \tag{5.17}$$

where $\nu_l(x)$ denotes the l -component of the exterior unit normal vector $\nu(x)$ of ∂B at $x \in \partial B$. Let us define

$$\varphi_k(x) = H_{\lambda}(x, \xi_k) + \sum_{j=k} G_{\lambda}(x, \xi_j).$$

Observe that for sufficiently large p and λ satisfying (2.4), by (2.16) and (3.15), we have

$$\gamma U_{\xi}(x) = 8\pi \sum_{j=1}^m G_{\lambda}(x, \xi_j) + O\left(\frac{1}{p}\right) = 8\pi \left(\frac{1}{2\pi} \log \frac{1}{|x - \xi_k|} + \varphi_k(x) \right) + O\left(\frac{1}{p}\right) \tag{5.18}$$

uniformly in $C_{\text{loc}}^1(\Omega \setminus \bigcup_{h=1}^m B(\xi_h, \varepsilon d(\xi_h)))$. Using (5.12), (5.17), (5.18) and the fact that $u_{\xi} = U_{\xi} + \phi_{\xi}$, we obtain

$$\begin{aligned}
 \int_{B(\xi_k, \varepsilon d(\xi_k))} (\Delta u_{\xi} + u_{\xi}^p) \partial_{x_l} u_{\xi} &= \int_{\partial B(\xi_k, \varepsilon d(\xi_k))} \left(\partial_{\nu} u_{\xi} \partial_{x_l} u_{\xi} - \frac{1}{2} |\nabla u_{\xi}|^2 \nu_l + \frac{1}{p+1} u_{\xi}^{p+1} \nu_l \right) \\
 &= \frac{64\pi^2}{\gamma^2} \left[\int_{\partial B(\xi_k, \varepsilon d(\xi_k))} \left(-\frac{1}{2\pi \varepsilon d(\xi_k)} + \partial_{\nu} \varphi_k \right) \left(-\frac{1}{2\pi \varepsilon d(\xi_k)} \nu_l + \partial_{x_l} \varphi_k \right) \right. \\
 &\quad \left. - \frac{1}{2} \int_{\partial B(\xi_k, \varepsilon d(\xi_k))} \left| -\frac{1}{2\pi \varepsilon d(\xi_k)} \nu + \nabla \varphi_k \right|^2 \nu_l \right] + O\left(\frac{1}{p^3}\right) \\
 &= \frac{64\pi^2}{\gamma^2} \int_{\partial B(\xi_k, \varepsilon d(\xi_k))} \left[-\frac{1}{2\pi \varepsilon d(\xi_k)} \partial_{x_l} \varphi_k + \left(\partial_{\nu} \varphi_k \partial_{x_l} \varphi_k - \frac{1}{2} |\nabla \varphi_k|^2 \nu_l \right) \right] + O\left(\frac{1}{p^3}\right) \\
 &= -\frac{64\pi^2}{\gamma^2} \partial_{x_l} \varphi_k(\xi_k) + O\left(\frac{1}{p^3}\right),
 \end{aligned}$$

where the last equality is due to the harmonicity of φ_k near ξ_k , more precisely,

$$\frac{1}{2\pi \varepsilon d(\xi_k)} \int_{\partial B(\xi_k, \varepsilon d(\xi_k))} \partial_{x_l} \varphi_k = \partial_{x_l} \varphi_k(\xi_k),$$

and by (5.17),

$$\int_{\partial B(\xi_k, \varepsilon d(\xi_k))} \left(\partial_{x_l} \varphi_k \partial_{x_l} \varphi_k - \frac{1}{2} |\nabla \varphi_k|^2 v_l \right) = \int_{B(\xi_k, \varepsilon d(\xi_k))} \Delta \varphi_k \partial_{x_l} \varphi_k = 0.$$

Accordingly, by (5.16) we get

$$\partial_{(\xi_k)_l} F_p^\lambda(\xi) = -\frac{64\pi^2}{y^2} \partial_{x_l} \varphi_k(\xi_k) + O\left(\frac{\lambda}{p^5} + \frac{1}{p^3}\right) = -\frac{32\pi^2}{y^2} \partial_{(\xi_k)_l} \varphi_m^\lambda(\xi) + O\left(\frac{\lambda}{p^5}\right) + O\left(\frac{1}{p^3}\right),$$

since

$$\partial_{x_l} \varphi_k(\xi_k) = \frac{1}{2} \partial_{(\xi_k)_l} \varphi_m^\lambda(\xi).$$

This completes the proof. $\square$

6 Proof of theorems

Proof of Theorem 1.1. According to Proposition 5.1, the function $u_{p,\lambda} = U_\xi + \phi_\xi$, where U_ξ is defined in (2.12) and ϕ_ξ is the unique solution to problem (4.1) given by Proposition 4.1, is a solution to problem (1.1) if we adjust ξ so that it is a critical point of $F_p^\lambda(\xi)$, defined by (5.3). This is equivalent to finding a critical point of

$$\tilde{F}_p^\lambda(\xi) = -\frac{y^2}{32\pi^2} \left[F_p^\lambda(\xi) - \frac{4\pi m p}{y^2} - \frac{4\pi m}{y^2} - \frac{m}{2y^2} \int_{\mathbb{R}^2} \left(\frac{8}{(1+|y|^2)^2} U_{1,0} - \Delta \omega_1 \right) \right].$$

On the other hand, from Proposition 5.2 and 5.3, for $\xi = (\xi_1, \dots, \xi_m) \in \mathcal{O}_\varepsilon$, we have that

$$\tilde{F}_p^\lambda(\xi) = \varphi_m^\lambda(\xi) + \frac{m}{2\pi} \log \lambda + \frac{\log \lambda}{p} \Theta_{p,\lambda}(\xi), \quad (6.1)$$

where φ_m^λ is given by (1.9), and $\Theta_{p,\lambda}$ and $\nabla_\xi \Theta_{p,\lambda}$ are uniformly bounded in the considered region as p and λ go to $+\infty$ such that $\frac{\log \lambda}{p} = o(1)$ and $\frac{\lambda}{p^2 \log \lambda} = O(1)$.

Let us first claim that for any small $\varepsilon > 0$, there exists a constant $C > 0$ such that

$$\inf_{\xi \in \partial \mathcal{O}_\varepsilon} \tilde{F}_p^\lambda(\xi) \geq \frac{m}{2\pi} \inf_{x \in \partial \Omega} h_\lambda(\theta_{b(x)}, b(x)) + \frac{m}{2\pi} \log \lambda + \frac{C}{\varepsilon \lambda} - C$$

uniformly for any large p and λ satisfying $\frac{\log \lambda}{p} = o(1)$ and $\lambda p^2 \log \lambda = O(1)$. To prove this, we recall that

$$\mathcal{O}_\varepsilon = \left\{ \xi = (\xi_1, \dots, \xi_m) \in \Omega^m : |\xi_j - \xi_k| \geq \varepsilon, |\lambda d(\xi_j) - \theta_{b(\xi_j)}| \leq \frac{1}{\varepsilon \lambda^{1/2}}, j, k = 1, \dots, m, j \neq k \right\},$$

where $\varepsilon > 0$ is a sufficiently small but fixed number. Obviously, if $\xi = (\xi_1, \dots, \xi_m) \in \partial \mathcal{O}_\varepsilon$, then either we have $|\lambda d(\xi_{j_0}) - \theta_{b(\xi_{j_0})}| = \frac{1}{\varepsilon \lambda^{1/2}}$ for some j_0 , or $|\xi_{j_0} - \xi_{k_0}| = \varepsilon$ for some $j_0 \neq k_0$. Using (1.5)–(1.8) and the fact that $h_\lambda(\theta, \tilde{b})$ has a unique non-degenerate minimum $\theta_{\tilde{b}} \in (0, +\infty)$ independent of λ , we have, for $|\lambda d(\xi_{j_0}) - \theta_{b(\xi_{j_0})}| = \frac{1}{\varepsilon \lambda^{1/2}}$, that

$$\begin{aligned} \tilde{F}_p^\lambda(\xi) &= \sum_{j=1}^m H_\lambda(\xi_j, \xi_j) + \sum_{j=k}^m G_\lambda(\xi_j, \xi_k) + \frac{m}{2\pi} \log \lambda + \frac{\log \lambda}{p} \Theta_{p,\lambda}(\xi) \\ &\geq \frac{1}{2\pi} h_\lambda\left(\theta_{b(\xi_{j_0})} \pm \frac{1}{\varepsilon \lambda^{1/2}}, b(\xi_{j_0})\right) + \frac{1}{2\pi} \sum_{j=j_0} h_\lambda(\theta_{b(\xi_j)}, b(\xi_j)) + \frac{m}{2\pi} \log \lambda + O\left(\frac{1}{\lambda}\right) + O\left(\frac{1}{p} \log \lambda\right) \\ &\geq \frac{m}{2\pi} \inf_{x \in \partial \Omega} h_\lambda(\theta_{b(x)}, b(x)) + \frac{m}{2\pi} \log \lambda + \frac{C}{\varepsilon^2 \lambda} - C, \end{aligned}$$

while for $|\xi_{j_0} - \xi_{k_0}| = \varepsilon$, by (3.15),

$$\begin{aligned}\tilde{F}_p^\lambda(\xi) &= \sum_{j=1}^m H_\lambda(\xi_j, \xi_j) + \sum_{j=k} G_\lambda(\xi_j, \xi_k) + \frac{m}{2\pi} \log \lambda + \frac{\log \lambda}{p} \Theta_{p,\lambda}(\xi) \\ &\geq \frac{m}{2\pi} \inf_{x \in \partial\Omega} h_\lambda(\theta_{b(x)}, b(x)) + G_\lambda(\xi_{j_0}, \xi_{k_0}) + \frac{m}{2\pi} \log \lambda + O\left(\frac{1}{\lambda}\right) + O\left(\frac{1}{p} \log \lambda\right) \\ &\geq \frac{m}{2\pi} \inf_{x \in \partial\Omega} h_\lambda(\theta_{b(x)}, b(x)) + \frac{m}{2\pi} \log \lambda + \frac{C}{\varepsilon \lambda} - C.\end{aligned}$$

We shall show that $\tilde{F}_p^\lambda$ has at least two distinct critical points in the region $\mathcal{O}_\varepsilon$, a fact that will prove our result. Observe that $\tilde{F}_p^\lambda$ is C^1 and uniformly bounded from below in $\mathcal{O}_\varepsilon$, and such that

$$\inf_{\xi \in \partial\mathcal{O}_\varepsilon} \tilde{F}_p^\lambda(\xi) \rightarrow +\infty \quad \text{as } \varepsilon \rightarrow 0.$$

Hence, since ε is sufficiently small, $\tilde{F}_p^\lambda$ has an absolute minimum $M_1^{p,\lambda}$ in $\mathcal{O}_\varepsilon$ whenever p and λ are sufficiently large such that $\frac{\log \lambda}{p} = o(1)$ and $\frac{\lambda}{p^2 \log \lambda} = O(1)$. On the other hand, the Lusternik–Schnirelmann theory is applicable in our setting, so that the number of critical points of $\tilde{F}_p^\lambda$ in $\mathcal{O}_\varepsilon$ can be estimated from below by $\text{cat}_{\mathcal{O}_\varepsilon}(\mathcal{O}_\varepsilon)$, the Lusternik–Schnirelmann category of $\mathcal{O}_\varepsilon$ relative to $\mathcal{O}_\varepsilon$. Let us recall that $\text{cat}_{\mathcal{O}_\varepsilon}(A)$, the Lusternik–Schnirelmann category of a closed subset A of $\mathcal{O}_\varepsilon$ relative to $\mathcal{O}_\varepsilon$, is the minimal integer l such that A can be covered by l closed and contractible subsets of $\mathcal{O}_\varepsilon$.

We claim that $\text{cat}_{\mathcal{O}_\varepsilon}(\mathcal{O}_\varepsilon) > 1$. Indeed, by contradiction, assume that $\text{cat}_{\mathcal{O}_\varepsilon}(\mathcal{O}_\varepsilon) = 1$. This means that $\mathcal{O}_\varepsilon$ is contractible in itself, namely there exist a point $\xi^0 \in \mathcal{O}_\varepsilon$ and a continuous function $\Gamma: [0, 1] \times \mathcal{O}_\varepsilon \rightarrow \mathcal{O}_\varepsilon$ such that for all $\xi \in \mathcal{O}_\varepsilon$,

$$\Gamma(0, \xi) = \xi \text{ and } \Gamma(1, \xi) = \xi^0.$$

Given one connected component $\mathcal{C}_0$ of $\partial\Omega$, let $\Lambda: S^1 \rightarrow \mathcal{C}_0$ be a continuous bijective function that parametrizes $\mathcal{C}_0$. Observe that for any $x \in \Omega$ close to $\partial\Omega$, there exists a unique point $\hat{x}$ on $\partial\Omega$ closest to x . Then, we can define a continuous map $\hat{P}: \mathcal{O}_\varepsilon \rightarrow \partial\Omega$ such that $\hat{P}\xi = \hat{\xi}_1$ for any $\xi = (\xi_1, \dots, \xi_m) \in \mathcal{O}_\varepsilon$. Set

$$g_\lambda(x) = x - \frac{\theta_{b(x)}}{\lambda} \nu(x), \quad x \in \mathcal{C}_0,$$

where ν is the exterior unit normal vector of $\partial\Omega$. Obviously, for λ sufficiently large, g_λ is a continuous invertible function. We represent $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Let $f: S^1 \rightarrow \mathcal{O}_\varepsilon$ be the continuous function defined by

$$f_\lambda(z) = \left(g_\lambda(\Lambda(z)), g_\lambda(\Lambda(ze^{i2\pi \frac{1}{m}})), \dots, g_\lambda(\Lambda(ze^{i2\pi \frac{m-1}{m}})) \right), \quad z \in S^1.$$

Let $\eta: [0, 1] \times S^1 \rightarrow S^1$ be the well-defined continuous map given by

$$\eta_\lambda(t, z) = \Lambda^{-1} \circ \hat{P} \circ \Gamma(t, f_\lambda(z)).$$

The function η_λ is a contraction of S^1 to a point and this gives a contradiction. Thus, we conclude that $\text{cat}_{\mathcal{O}_\varepsilon}(\mathcal{O}_\varepsilon) \geq 2$.

Let us now define

$$M_2^{p,\lambda} = \inf_{A \in \Xi} \sup_{\xi \in A} \tilde{F}_p^\lambda(\xi),$$

where

$$\Xi = \{A \subset \mathcal{O}_\varepsilon : A \text{ closed and } \text{cat}_{\mathcal{O}_\varepsilon}(A) \geq 2\}.$$

By the above construction of f_λ , we have that

$$\Lambda^{-1} \circ \hat{P} \circ f_\lambda(z) = z \quad \text{for any } z \in S^1,$$

so $\Lambda^{-1} \circ \hat{P} \circ f_\lambda$ has a nonzero winding number, which implies that $\text{cat}_{\mathcal{O}_\varepsilon}(A_0) \geq 2$ with $A_0 = \{f_\lambda(z) : z \in S^1\}$. Note that for any $\xi = (\xi_1, \dots, \xi_m) \in A_0$, the m coordinates of ξ are uniformly separated and independent of ε and λ . Then, for any $\xi \in A_0$,

$$\begin{aligned} \tilde{F}_p^\lambda(\xi) &= \sum_{j=1}^m H_\lambda(\xi_j, \xi_j) + \sum_{j=k} G_\lambda(\xi_j, \xi_k) + \frac{m}{2\pi} \log \lambda + \frac{\log \lambda}{p} \Theta_{p,\lambda}(\xi) \\ &= \frac{1}{2\pi} \sum_{j=1}^m h_\lambda(\theta_{b(\hat{\xi}_j)}, b(\hat{\xi}_j)) + \frac{m}{2\pi} \log \lambda + O(1) + O\left(\frac{1}{\lambda}\right) + O\left(\frac{1}{p} \log \lambda\right), \end{aligned}$$

so that, using (1.7) and (2.3),

$$M_2^{p,\lambda} = \inf_{A \in \Xi} \sup_{\xi \in A} \tilde{F}_p^\lambda(\xi) \leq \sup_{\xi \in A_0} \tilde{F}_p^\lambda(\xi) \leq C < +\infty$$

uniformly for any large p and λ such that $\frac{\log \lambda}{p} = o(1)$ and $\frac{\lambda}{p^2 \log \lambda} = O(1)$. Thus, Lusternik–Schnirelmann theory gives that $M_2^{p,\lambda}$ is a critical level for $\tilde{F}_p^\lambda$.

Obviously, $M_1^{p,\lambda} \leq M_2^{p,\lambda}$. If $M_1^{p,\lambda} < M_2^{p,\lambda}$, then we conclude that there are at least two distinct critical points for $\tilde{F}_p^\lambda$ (distinct up to permutations) in $\mathcal{O}_\varepsilon$ corresponding to two distinct solutions of problem (1.1). If $M_1^{p,\lambda} = M_2^{p,\lambda}$, then we get that there is at least one set A , with $\text{cat}_{\mathcal{O}_\varepsilon}(A) \geq 2$, where the function $\tilde{F}_p^\lambda$ reaches its absolute minimum. In this case, we conclude that there are infinitely many critical points for $\tilde{F}_p^\lambda$ in $\mathcal{O}_\varepsilon$. Since permutations are only finite in number, the result is thus proven. $\square$

Proof of Theorem 1.2. For the case $m = 1$, it suffices to prove the existence of critical points of $\tilde{F}_p^\lambda$ given by (6.1), whose proof is based on the asymptotic formulas (1.5)–(1.6) for $H_\lambda(\xi, \xi)$ combined with a linking argument.

In fact, for $m = 1$, by (6.1) for any large p and λ satisfying $\frac{1}{p} \log \lambda = o(1)$, we get that

$$\tilde{F}_p^\lambda(\xi) = H_\lambda(\xi, \xi) + \frac{1}{2\pi} \log \lambda + \frac{\log \lambda}{p} \Theta_{p,\lambda}(\xi), \quad \xi \in \mathcal{O}_\varepsilon, \quad (6.2)$$

where

$$\frac{\log \lambda}{p} |\Theta_{p,\lambda}(\xi)| = O\left(\frac{\log \lambda}{p}\right) = o(1), \quad \frac{\log \lambda}{p} |\nabla_\xi \Theta_{p,\lambda}(\xi)| = O\left(\frac{\lambda}{p^3}\right) + O\left(\frac{1}{p}\right) \quad (6.3)$$

and

$$\mathcal{O}_\varepsilon = \left\{ \xi \in \Omega : |\lambda d(\xi) - \theta_{b(\hat{\xi})}| \leq \frac{1}{\varepsilon \lambda^{1/2}} \right\}. \quad (6.4)$$

Using the asymptotic formulas of $H_\lambda(\xi, \xi)$ in (1.5)–(1.6), for $\xi \in \mathcal{O}_\varepsilon$, we have that

$$\tilde{F}_p^\lambda(\xi) = \frac{1}{2\pi} h_\lambda(\lambda d(\xi), b(\hat{\xi})) + \frac{1}{2\pi} \log \lambda + O\left(\frac{1}{\lambda}\right) + O\left(\frac{1}{p} \log \lambda\right). \quad (6.5)$$

Note that for any $\tilde{b} > 0$ the function $h_\lambda(\cdot, \tilde{b})$ has a unique non-degenerate minimum $\theta_{\tilde{b}} \in (0, +\infty)$, which is a continuous function with respect to the variable $\tilde{b}$. We define

$$S^* := \{ \xi \in \Omega : \lambda d(\xi) = \theta_{b(\hat{\xi})} \} \subset \mathcal{O}_\varepsilon.$$

Then, for $\varepsilon > 0$ sufficiently small, it follows that

$$\inf_{\xi \in \partial \mathcal{O}_\varepsilon} h_\lambda(\lambda d(\xi), b(\hat{\xi})) > \sup_{\xi \in S^*} h_\lambda(\lambda d(\xi), b(\hat{\xi})),$$

and therefore, by (6.5),

$$\inf_{\xi \in \partial \mathcal{O}_\varepsilon} \tilde{F}_p^\lambda(\xi) > \sup_{\xi \in S^*} \tilde{F}_p^\lambda(\xi) \quad (6.6)$$

uniformly for any large p and λ satisfying $\frac{1}{p} \log \lambda = o(1)$. Accordingly, there exists $\xi_{1,p,\lambda} \in \mathcal{O}_\varepsilon$ such that

$$\inf_{\xi \in \mathcal{O}_\varepsilon} \tilde{F}_p^\lambda(\xi) = \tilde{F}_p^\lambda(\xi_{1,p,\lambda}).$$

We now seek another critical point of $\tilde{F}_p^\lambda$ in $\mathcal{O}_\varepsilon$. For each $x \in \partial\Omega$, we consider the set

$$Q_x := \{\xi \in \mathcal{O}_\varepsilon : \hat{\xi} = x\}.$$

Let us assume that there exists $x_0 \in \partial\Omega$ such that

$$\inf_{\xi \in Q_{x_0}} \tilde{F}_p^\lambda(\xi) > \tilde{F}_p^\lambda(\xi_{1,p,\lambda}). \quad (6.7)$$

Otherwise, if such a point does not exist then the theorem is proved. Let ∂Q_{x_0} denote the boundary of Q_{x_0} . Obviously, the sets ∂Q_{x_0} and S^* link in $\mathcal{O}_\varepsilon$. Moreover, by (6.6) we have

$$\inf_{\xi \in \partial Q_{x_0}} \tilde{F}_p^\lambda(\xi) > \sup_{\xi \in S^*} \tilde{F}_p^\lambda(\xi).$$

Let

$$\mathcal{P} = \{f \in C^0(\overline{Q_{x_0}}, \overline{\mathcal{O}_\varepsilon}) : f|_{\partial Q_{x_0}} = \text{Id}|_{\partial Q_{x_0}}\}.$$

Then,

$$\beta = \sup_{f \in \mathcal{P}} \inf_{\xi \in Q_{x_0}} \tilde{F}_p^\lambda(\xi)$$

is a critical value of $\tilde{F}_p^\lambda$ which is different from $\tilde{F}_p^\lambda(\xi_{1,p,\lambda})$ by (6.7). Hence, we obtain the existence of the second critical point $\xi_{2,p,\lambda}$ for $\tilde{F}_p^\lambda$ in $\mathcal{O}_\varepsilon$, which is different from $\xi_{1,p,\lambda}$. $\square$

Proof of Theorem 1.3. Let $x_0 \in \partial\Omega$ be a non-degenerate critical point of the mean curvature κ and $b \equiv 1$. As in the proof of Theorem 1.2, it suffices to prove the existence of critical points of $\tilde{F}_p^\lambda$ near x_0 , where $\tilde{F}_p^\lambda$ is given by (6.2)–(6.4). For this purpose, we will take advantage of the asymptotic formula (1.6) of $H_\lambda(\xi, \xi)$ to relate the topological degree of the $\nabla \tilde{F}_p^\lambda$ in a suitable small set close to x_0 with that of the $\nabla \kappa$.

Let $R_\lambda(\xi) = H_\lambda(\xi, \xi)$ and ∇_T denote the tangential gradient which is defined in a neighborhood of $\partial\Omega$. For any $\alpha \in (0, 1)$, there exist the following expansions for the gradient of $R_\lambda(\xi)$ (see [7]):

$$\nabla_T R_\lambda(\xi) = \frac{1}{2\pi\lambda} \nabla \kappa(\hat{\xi}) v(\lambda d(\xi)) + O\left(\frac{1}{\lambda^{1+\alpha}}\right), \quad (6.8)$$

$$\langle \nabla R_\lambda(\xi), v(\hat{\xi}) \rangle = -\frac{\lambda}{2\pi} \frac{\partial h_\lambda}{\partial \theta}(\lambda d(\xi), 1) - \frac{1}{2\pi} \kappa(\hat{\xi}) v'(\lambda d(\xi)) + O\left(\frac{1}{\lambda^\alpha}\right) \quad (6.9)$$

uniformly for $|\lambda d(\xi) - \theta_b| \leq \frac{1}{\varepsilon \lambda^{1/2}}$ with $b \equiv 1$.

Since $x_0 \in \partial\Omega$ is a non-degenerate critical point of κ , there exist $C > 0$ and $\sigma > 0$ such that

$$|\nabla \kappa(\hat{\xi})| \geq C |\hat{\xi} - x_0| \quad \text{for all } \hat{\xi} \in \partial\Omega \text{ such that } |\hat{\xi} - x_0| \leq \sigma. \quad (6.10)$$

On the other hand, we know that the function $h_\lambda(\cdot, 1)$ has a unique minimum $\theta_b > 0$, which is non-degenerate, and hence by taking $C > 0$ and $\sigma > 0$ smaller if necessary, we have

$$\left| \frac{\partial}{\partial \theta} h_\lambda(\theta, 1) \right| \geq C |\theta - \theta_b| \quad \text{for all } |\theta - \theta_b| \leq \sigma. \quad (6.11)$$

Using the fact that the function v is continuous and negative in $\mathbb{R}$, we can choose $\sigma > 0$ smaller such that

$$\inf_{\theta \in [\theta_b - \sigma, \theta_b + \sigma]} |v(\theta)| > 0. \quad (6.12)$$

Let $0 < \beta < \alpha$. For λ large and ε small but fixed, we can also assume $\max\{\frac{1}{\varepsilon \lambda^{1/2}}, \frac{1}{\lambda^\beta}\} < \sigma < \theta_b$, since $\theta_b > 0$. Consider the compact set

$$\mathcal{K}_\lambda := \left\{ \xi \in \Omega : |\lambda d(\xi) - \theta_b| \leq \frac{1}{\varepsilon \lambda^{1/2}}, |\hat{\xi} - x_0| \leq \frac{1}{\lambda^\beta} \right\}.$$

Then, $\mathcal{K}_\lambda \subset \mathcal{O}_\varepsilon$ with $\mathcal{O}_\varepsilon$ defined in (6.4). Let

$$\tilde{F}_\lambda^0(\xi) = \frac{1}{2\pi} h_\lambda(\lambda d(\xi), 1) + \frac{1}{2\pi\lambda} \kappa(\hat{\xi}) v(\lambda d(\xi)).$$

Obviously, $\tilde{F}_\lambda^0$ has a unique non-degenerate critical point in the interior of $\mathcal{K}_\lambda$. Now we define the homotopy between $\tilde{F}_\lambda^p$ and $\tilde{F}_\lambda^0$ as follows:

$$\tilde{F}_\lambda^{p,t}(\xi) = t\tilde{F}_\lambda^p(\xi) + (1-t)\tilde{F}_\lambda^0(\xi), \quad t \in [0, 1].$$

We observe that for any large p and λ satisfying $\frac{1}{p} \log \lambda = o(1)$, there exists $C' > 0$ such that if $\xi \in \partial\mathcal{K}_\lambda$ then:

1. if $|\lambda d(\xi) - \theta_b| = \frac{1}{\varepsilon\lambda^{1/2}}$, then by (6.2), (6.3), (6.9) and (6.11), we have

$$|\nabla \tilde{F}_\lambda^{p,t}(\xi)| \geq |\langle \nabla \tilde{F}_\lambda^{p,t}(\xi), \nu(\hat{\xi}) \rangle| \geq C' \sqrt{\lambda} + O\left(\frac{\lambda}{p^3}\right) + O\left(\frac{1}{p}\right);$$

2. if $|\hat{\xi} - x_0| = \frac{1}{\lambda^\beta}$, then by (6.2), (6.3), (6.8), (6.10) and (6.12), we have

$$|\nabla \tilde{F}_\lambda^{p,t}(\xi)| \geq |\nabla_T \tilde{F}_\lambda^{p,t}(\xi)| \geq \frac{C'}{\lambda^{1+\beta}} + O\left(\frac{\lambda}{p^3}\right) + O\left(\frac{1}{p}\right).$$

Then, it is easily checked that by choosing sufficiently large p and λ such that $\frac{1}{p}\lambda^{1+\beta} = o(1)$, $|\nabla \tilde{F}_\lambda^{p,t}(\xi)|$ remains uniformly positive on $\partial\mathcal{K}_\lambda$. By the degree theory $\tilde{F}_\lambda^p(\xi) = \tilde{F}_\lambda^{p,1}(\xi)$ has a critical point in the interior of $\mathcal{K}_\lambda$, and hence it is located at distance $O(\lambda^{-\beta})$ from x_0 . $\square$

Proof of Theorem 1.4. Let $x_0 \in \partial\Omega$ be a non-degenerate critical point of the function b . The proof is similar to that of Theorem 1.3, that is, to prove the existence of critical points of $\tilde{F}_\lambda^p$ defined in (6.2)–(6.4), we will take advantage of the asymptotic formula (1.5) of $H_\lambda(\xi, \xi)$ to relate the topological degree of the $\nabla \tilde{F}_\lambda^p$ in a suitable small set close to x_0 with that of the ∇b .

Note that from [7], the gradient of $R_\lambda(\xi)$ has the following expansions:

$$\nabla_T R_\lambda(\xi) = \frac{1}{2\pi} \frac{\partial h_\lambda}{\partial \tilde{b}}(\lambda d(\xi), b(\hat{\xi})) \nabla b(\hat{\xi}) + O\left(\frac{1}{\lambda}\right), \quad (6.13)$$

$$\langle \nabla R_\lambda(\xi), \nu(\hat{\xi}) \rangle = -\frac{\lambda}{2\pi} \frac{\partial h_\lambda}{\partial \theta}(\lambda d(\xi), b(\hat{\xi})) + O(1) \quad (6.14)$$

uniformly for $|\lambda d(\xi) - \theta_{b(\hat{\xi})}| \leq \frac{1}{\varepsilon\lambda^{1/2}}$.

Since $x_0 \in \partial\Omega$ is a non-degenerate critical point of the function b , there exist $C > 0$ and $\sigma > 0$ such that

$$|\nabla b(\hat{\xi})| \geq C|\hat{\xi} - x_0| \quad \text{for all } \hat{\xi} \in \partial\Omega \text{ such that } |\hat{\xi} - x_0| \leq \sigma. \quad (6.15)$$

On the other hand, we know that for any $\tilde{b} > 0$ the function $h_\lambda(\cdot, \tilde{b})$ has a unique non-degenerate minimum $\theta_{\tilde{b}} \in (0, +\infty)$, which is a continuous function with respect to the variable $\tilde{b}$, and hence taking $C > 0$ and $\sigma > 0$ smaller if necessary, we have

$$\left| \frac{\partial}{\partial \theta} h_\lambda(\theta, b(\hat{\xi})) \right| \geq C|\theta - \theta_{b(\hat{\xi})}| \quad \text{for all } |\theta - \theta_{b(\hat{\xi})}| \leq \sigma \text{ and } \hat{\xi} \in \partial\Omega. \quad (6.16)$$

Using that the function $\frac{\partial h_\lambda}{\partial \tilde{b}}(\theta, \tilde{b})$ is continuous and negative in $\mathbb{R}_+ \times \mathbb{R}_+$, we can choose $\sigma > 0$ smaller such that

$$\inf_{\hat{\xi} \in \partial\Omega, \theta \in [\theta_{b(\hat{\xi})-\sigma}, \theta_{b(\hat{\xi})+\sigma}]} \left| \frac{\partial}{\partial \tilde{b}} h_\lambda(\theta, b(\hat{\xi})) \right| > 0. \quad (6.17)$$

Let $0 < \beta < 1$. For large λ and small but fixed ε , we can also assume $\max\{\frac{1}{\varepsilon\lambda^{1/2}}, \frac{1}{\lambda^\beta}\} < \sigma < \min_{\hat{\xi} \in \partial\Omega} \theta_{b(\hat{\xi})}$, since the function $\theta_{b(\cdot)}$ is continuous and positive on $\partial\Omega$. Consider the compact set

$$\mathcal{K}_\lambda := \left\{ \xi \in \Omega : |\lambda d(\xi) - \theta_{b(\hat{\xi})}| \leq \frac{1}{\varepsilon\lambda^{1/2}}, |\hat{\xi} - x_0| \leq \frac{1}{\lambda^\beta} \right\}.$$

Then, $\mathcal{K}_\lambda \subset \mathcal{O}_\varepsilon$ with $\mathcal{O}_\varepsilon$ defined in (6.4). Let

$$\tilde{F}_\lambda^0(\xi) = \frac{1}{2\pi} h_\lambda(\lambda d(\xi), b(\hat{\xi})).$$

Obviously, $\tilde{F}_\lambda^0$ has a unique non-degenerate critical point ξ_λ located in the interior of $\mathcal{K}_\lambda$ such that $\hat{\xi}_\lambda = x_0$ and $\lambda d(\xi_\lambda) = \theta_{b(x_0)}$. Now we define the homotopy between $\tilde{F}_\lambda^p$ and $\tilde{F}_\lambda^0$ as follows:

$$\tilde{F}_\lambda^{p,t}(\xi) = t\tilde{F}_\lambda^p(\xi) + (1-t)\tilde{F}_\lambda^0(\xi), \quad t \in [0, 1].$$

We observe that for any large p and λ satisfying $\frac{1}{p} \log \lambda = o(1)$, there exists $C' > 0$ such that if $\xi \in \partial\mathcal{K}_\lambda$ then:

1. if $|\lambda d(\xi) - \theta_{b(\hat{\xi})}| = \frac{1}{\varepsilon\lambda^{1/2}}$, then by (6.2), (6.3), (6.14) and (6.16), we have

$$|\nabla \tilde{F}_\lambda^{p,t}(\xi)| \geq |\langle \nabla \tilde{F}_\lambda^{p,t}(\xi), \nu(\hat{\xi}) \rangle| \geq C' \sqrt{\lambda} + O\left(\frac{\lambda}{p^3}\right) + O\left(\frac{1}{p}\right);$$

2. if $|\hat{\xi} - x_0| = \frac{1}{\lambda^\beta}$, then by (6.2), (6.3), (6.13), (6.15) and (6.17), we have

$$|\nabla \tilde{F}_\lambda^{p,t}(\xi)| \geq |\nabla_T \tilde{F}_\lambda^{p,t}(\xi)| \geq \frac{C'}{\lambda^\beta} + O\left(\frac{1}{\lambda}\right) + O\left(\frac{\lambda}{p^3}\right) + O\left(\frac{1}{p}\right).$$

Then, it is easily checked that, choosing sufficiently large p and λ such that $\frac{1}{p}(\lambda^{1/3(1+\beta)} + \lambda^\beta) = o(1)$, $|\nabla \tilde{F}_\lambda^{p,t}(\xi)|$ remains uniformly positive on $\partial\mathcal{K}_\lambda$. By the degree theory, $\tilde{F}_\lambda^p(\xi) = \tilde{F}_\lambda^{p,1}(\xi)$ has a critical point in the interior of $\mathcal{K}_\lambda$, and hence it is located at distance $O(\lambda^{-\beta})$ from x_0 . $\square$

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