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On the convergence analysis of a time dependent elliptic equation with discontinuous coefficients

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Abstract: In this paper, we consider a heat equation with diffusion coefficient that varies depending on the heterogeneity of the domain. We propose a spectral elements discretization of this problem with the mortar domain decomposition method on the space variable and Euler's implicit scheme with respect to the time. The convergence analysis and an optimal error estimates are proved.

Keywords: Elliptic equation, Discontinuous coefficient, Mortar spectral element method, Error analysis

MSC: 35J57, 65M70

1 Introduction

This paper is devoted to the numerical analysis of the mortar spectral element discretization of the heat equation in an heterogeneous medium with a variable diffusion coefficient λ formulated by the problem (1),

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(\lambda \operatorname{grad} u) = f & \text{in } \Omega \times]0, T[\\ u = 0 & \text{in } \partial\Omega \times]0, T[\\ u(\cdot, 0) = u_0 & \text{on } \Omega. \end{cases} \quad (1)$$

The connected two-dimensional domain Ω is open and bounded with a Lipschitz-continuous boundary $\partial\Omega$. Let T be a fixed positive real. We suppose that the function λ is positive and does not depend on time.

This problem was handled in some previous works on different cases [1]. The case where the function λ is not globally continuous is presented in [2–4]. The a priori and a posteriori analysis were proposed based on the finite element method and the spectral discretization. For the case where λ is piecewise constant and such that the ratio of its maximal value to its minimal value is large enough, the discretization of the stationary problem is studied in [5] by conforming finite elements and in [6] by the mortar spectral discretization. In the present work, we consider a non stationary problem with λ is piecewise constant. We proceed to the domain decomposition in two steps. Firstly, We associate a decomposition based on the value of λ (i.e. λ is constant on each sub-domain). Secondly, each obtained sub-domain is itself decomposed on rectangles using the mortar spectral method. The later is considered as the most suitable method for handling nonconforming decomposition (i.e the intersection of two sub-domains is not restricted to be a corner or a whole edge of both of them) [7]. The number of sub-domains can be highly reduced thanks to the non-conformity property. We refer to [8] for a first application of this method to discontinuous coefficient in the finite element method.

The discretization in time of our problem is based on the implicit Euler method. We prove that the semi-discrete problem on time is well posed and we give a time error estimate of order one. On each sub-domain,

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we consider a spectral discretization which approaches the solution by high degree polynomials. Since the basis of polynomials are tensorized, the sub-domains are rectangles. Different degrees of polynomials are chosen on each sub-domain according to the different values of λ . We prove that the discrete problem is well posed and we show an optimal error estimate for a good choice of domain decomposition. An outline of the paper is as follows :

- In section 2 we present the continuous problem and some regularity results.
- The section 3 is about the analysis and the error estimate of the semi-discrete problem on time.
- The mortar spectral element discretization is developed in section 4.
- In section 5, we perform the estimation of the error.
- Section 6 is an annex of the proof of the error estimation since it is quite technical.

2 The continuous problem

We denote by $\mathbf{x}=(x,y)$ the generic point in $\mathbb{R}^2$, and we suppose that there exists a finite number of sub-domain Ω_i° , $1 \leq i \leq I^\circ$ such that:

1)

$$\overline{\Omega} = \bigcup_{i=1}^{I^\circ} \overline{\Omega}_i^\circ, \quad \Omega_i^\circ \cap \Omega_j^\circ = \emptyset, \quad 1 \leq i < j \leq I^\circ,$$

2) the restriction of λ to each $\overline{\Omega}_i^\circ$ is continuous on Ω_i° , $1 \leq i \leq I^\circ$,

3) λ is bounded on each $\overline{\Omega}_i^\circ$, let

$$\lambda_i^{max} = \sup_{\mathbf{x} \in \Omega_i^\circ} \lambda(\mathbf{x}) \quad \text{and} \quad \lambda_i^{min} = \min_{\mathbf{x} \in \Omega_i^\circ} \lambda(\mathbf{x}).$$

We define

$$\lambda^{max} = \max_{1 \leq i \leq I^\circ} \lambda_i^{max}, \quad \text{and} \quad \lambda^{min} = \max_{1 \leq i \leq I^\circ} \lambda_i^{min}. \quad (2)$$

Let $H^s(\Omega)$, $s > 0$, the Sobolev spaces associated with the norm $\|\cdot\|_{s,\Omega}$ and the semi-norm $|\cdot|_{s,\Omega}$. The space $H_0^1(\Omega)$ stands for the closure in $H^1(\Omega)$ of the space of infinitely differentiable functions with compact support in Ω and $H^{-1}(\Omega)$ is its dual space. The scalar product and its associate norm on the space $L^2(\Omega)$ are denoted by $(\cdot, \cdot)$ and $\|\cdot\|_{0,\Omega}$. $H^{\frac{1}{2}}(\partial\Omega)$ is the space of trace of functions in $H^1(\Omega)$. Let $\gamma \subset \partial\Omega$, $H_{00}^{\frac{1}{2}}(\gamma)$ is the space of functions in $H^{\frac{1}{2}}(\gamma)$ such that their extension by zero to $\partial\Omega/\gamma$ belongs to $H^{\frac{1}{2}}(\partial\Omega)$.

We introduce some notions to clarify the spaces of functions that depend on time. The function $u(\mathbf{x}, t)$, defined on the domain $\Omega \times]0, T[$, can be written as:

$$\begin{aligned} u :]0, T[&\longrightarrow X \\ t &\longmapsto u(t) = u(\cdot, t) \end{aligned}$$

where X is a separable Banach space. We define $\mathcal{C}^j(0, T; X)$ the set of time $\mathcal{C}^j$ classes functions with a value on X . $\mathcal{C}^j(0, T; X)$ is a Banach space for the norm

$$\|u\|_{\mathcal{C}^j(0, T; X)} = \sup_{0 \leq t \leq T} \sum_{l=0}^j \|\partial_t^l u\|_X$$

where $\partial_t^l u$ is the partial derivative of order l in time of the function u . We define also the spaces

$$L^p(0, T; X) = \{v \text{ measurable on }]0, T[\text{ such that } \int_0^T \|v(t)\|_X^p dt < \infty\}$$

and

$$H^s(0, T; X) = \{v \in L^2(0, T; X); \partial^k v \in L^2(0, T; X); k \leq s\}.$$

$L^p(0, T; X)$ is a Banach space for the norm

$$\|v\|_{L^p(0, T; X)} = \begin{cases} \left(\int_0^T \|v(t)\|_X^p dt \right)^{\frac{1}{p}}, & \text{for } 1 \leq p < +\infty \\ \sup_{0 \leq t \leq T} \|v(t)\|_X, & \text{for } p = +\infty, \end{cases}$$

and $H^s(0, T; X)$ is an Hilbert space for the following scalar product:

$$(u, v) = ((u, v)_{L^2(0, T; X)} + \sum_{k=0}^s (\partial^k u, \partial^k v)_{L^2(0, T; X)})^{\frac{1}{2}}.$$

Problem (1) admits the equivalent variational formulation:

For $t \in]0, T[$ and $f \in L^2(0, T; H^{-1}(\Omega))$, find $u \in C^0(0, T; H_0^1(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ such that: for all $v \in H_0^1(\Omega)$

$$\int_{\Omega} \frac{\partial u}{\partial t}(\mathbf{x}, t) v(\mathbf{x}) d\mathbf{x} + \sum_{i=1}^{I^{\circ}} \int_{\Omega_i^{\circ}} \lambda(\mathbf{x}) \nabla u(\mathbf{x}, t) \nabla v(\mathbf{x}) d\mathbf{x} = \langle f(\cdot, t), v \rangle \quad (3)$$

where $\langle \cdot, \cdot \rangle$ is the duality product between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$.

We introduce the energy norm

$$\|u\|_{\lambda}(T) = (\|u\|_{0, \Omega}^2 + \sum_{i=1}^{I^{\circ}} \int_0^T \|\lambda(\mathbf{x})^{\frac{1}{2}} \nabla u(\mathbf{x}, t)\|_{0, \Omega_i^{\circ}}^2 dt)^{\frac{1}{2}}. \quad (4)$$

We recall the following proposition (see [9], chap 3 for its proof).

Proposition 1. For $f \in L^2(0, T; H^{-1}(\Omega))$ and $u_0 \in L^2(\Omega)$, the problem (3) has a unique solution $u \in L^2(0, T; H_0^1(\Omega))$ verifying the following estimation:

$$\|u\|_{\lambda}(T) \leq (\|u_0\|_{0, \Omega}^2 + (\frac{1}{\lambda_{\min}}) \|f\|_{L^2(0, T; H^{-1}(\Omega))}^2)^{\frac{1}{2}}. \quad (5)$$

We have the regularity result proved in ([5], Prop 2.2) and [10].

Proposition 2. We suppose that the restriction of the function λ on each sub-domain Ω_i° , $1 \leq i \leq I^{\circ}$ is constant. There exists a real $0 < s_0 < \frac{1}{2}$, depending on the geometry of the domain and the quotient $\frac{\lambda_{\max}}{\lambda_{\min}}$, such that for $f \in L^2(0, T; H^{s-1}(\Omega))$, the solution u of problem (3) belongs to $L^2(0, T; H^{s+1}(\Omega) \cap H_0^1(\Omega))$ for any $0 \leq s \leq s_0$.

Remark 1. The maximum value of the real s_0 is bounded in the following way (see [10])

$$0 \leq s_0 \leq \min \left(\frac{1}{2}, C \left| \log \left(1 - \frac{\lambda_{\min}}{\lambda_{\max}} \right) \right| \right),$$

where C is a constant that depends only on the domain Ω .

3 The time semi discrete problem

To find the discrete problem in time, we introduce a partition of the interval $[0, T]$. Let $[t_{n-1}, t_n]$ the sub-interval of this partition, such that $0 = t_0 < t_1 < \dots < t_{n-1} < \dots < t_M = T$ where M is a positive integer. We notice $\tau = t_n - t_{n-1}$, $1 \leq n \leq M$ the step of the partition that we suppose constant. We denote by $v(\cdot, t_n) = v^n$, $0 \leq n \leq M$. We define the function v_{τ} which is affine on each interval $[t_{n-1}, t_n]$ by

$$v_{\tau}(\cdot, t) = v^n - \frac{t_n - t}{\tau} (v^n - v^{n-1}). \quad (6)$$

Using Euler implicit method, the semi discrete problem is written as follows :

$$\begin{cases} \frac{u^n - u^{n-1}}{\tau} - \operatorname{div}(\lambda \nabla u^n) = f^n, & \text{in } \Omega, \quad 1 \leq n \leq M \\ u^n = 0, & \text{on } \partial\Omega, \quad 1 \leq n \leq M \\ u^0 = u_0, & \text{in } \Omega. \end{cases} \quad (7)$$

The problem (7) has the equivalent variational formulation: Find $(u^n)_{0 \leq n \leq M} \in L^2(\Omega) \times H_0^1(\Omega)^M$ such that for all $v \in H_0^1(\Omega)$,

$$\int_{\Omega} u^n(\mathbf{x})v(\mathbf{x})d\mathbf{x} + \tau \sum_{i=1}^{I^\circ} \int_{\Omega_i^\circ} \lambda(\mathbf{x})\nabla u^n(\mathbf{x})\nabla v(\mathbf{x})d\mathbf{x} = \int_{\Omega} u^{n-1}(\mathbf{x})v(\mathbf{x})d\mathbf{x} + \tau \int_{\Omega} f^n(\mathbf{x})v(\mathbf{x})d\mathbf{x}. \quad (8)$$

Let the bilinear form $a^n(., .)$ and the linear form $L^n(.)$ defined respectively by

$$a^n(u^n, v) = \int_{\Omega} u^n(\mathbf{x})v(\mathbf{x})d\mathbf{x} + \tau \sum_{i=1}^{I^\circ} \int_{\Omega_i^\circ} \lambda(\mathbf{x})\nabla u^n(\mathbf{x})\nabla v(\mathbf{x})d\mathbf{x}$$

and

$$L^n(v) = \int_{\Omega} u^{n-1}(\mathbf{x})v(\mathbf{x})d\mathbf{x} + \tau \int_{\Omega} f^n(\mathbf{x})v(\mathbf{x})d\mathbf{x}.$$

It is easy to prove that the bilinear form $a^n(., .)$ is continuous on the space $H_0^1(\Omega) \times H_0^1(\Omega)$, coercive on the space $H_0^1(\Omega)$ and that the linear form L^n is continuous on the space $H_0^1(\Omega)$. So according to the Lax Milgram theorem, we deduce the following proposition.

Proposition 3. For any function f in $\mathcal{C}^0(0, T; H^{-1}(\Omega))$ and $u_0 \in L^2(\Omega)$, problem (8) has a unique solution $(u^n)_{0 \leq n \leq M} \in L^2(\Omega) \times (H_0^1(\Omega))^M$.

If we take $v = u^n$ in problem (8), we deduce the following inequality :

$$\|u^n\|_{0,\Omega}^2 + \tau \sum_{i=1}^{I^\circ} \|\lambda^{\frac{1}{2}} \nabla u^n\|_{0,\Omega_i^\circ}^2 \leq \|u^{n-1}\|_{0,\Omega}^2 + \frac{\tau}{\lambda_{\min}} \|f^n\|_{-1,\Omega}^2,$$

and by making the sum on n we conclude :

$$\|u^n\|_{0,\Omega}^2 + \tau \sum_{j=1}^n \sum_{i=1}^{I^\circ} \|\lambda^{\frac{1}{2}} \nabla u^j\|_{0,\Omega_i^\circ}^2 \leq \|u_0\|_{0,\Omega}^2 + \frac{\tau}{\lambda_{\min}} \sum_{j=1}^n \|f^j\|_{-1,\Omega}^2. \quad (9)$$

Proposition 4. For f in $\mathcal{C}^0(0, T; H^{-1}(\Omega))$ and $u_0 \in H^1(\Omega)$, the solution $(u^n)_{0 \leq n \leq M}$ of problem (8) satisfies the following estimation

$$\begin{aligned} \frac{1}{4} (\|u_0\|_{0,\Omega}^2 + \frac{\tau}{\lambda_{\min}} \sum_{j=1}^n \|f^j\|_{-1,\Omega}^2) &\leq \|u_\tau\|_\lambda^2 \\ &\leq \|u_0\|_{0,\Omega}^2 + \frac{\tau}{\lambda_{\min}} \sum_{j=1}^n \|f^j\|_{-1,\Omega}^2 + \frac{1}{2} \tau \|\lambda^{\frac{1}{2}} \nabla u_0\|_{0,\Omega}^2. \end{aligned} \quad (10)$$

Proof. To prove the estimation (10), we have to compare the two terms

$$A_k = \int_{t_{k-1}}^{t_k} \|\lambda^{\frac{1}{2}} \nabla u_\tau(., t)\|_{0,\Omega}^2 dt \text{ and } B_k = \tau \|\lambda^{\frac{1}{2}} \nabla u^k\|_{0,\Omega}^2.$$

According to the definition of the function u_τ defined in (6), we have $\forall \mathbf{x} \in \Omega$

$$\int_{t_{k-1}}^{t_k} |\nabla u_\tau(\mathbf{x}, t)|^2 dt = \frac{\tau}{3} \left(|\nabla u^k(\mathbf{x})|^2 + |\nabla u^{k-1}(\mathbf{x})|^2 + \nabla u^k(\mathbf{x}) \cdot \nabla u^{k-1}(\mathbf{x}) \right),$$

where $\cdot$ is the scalar product in $\mathbb{R}^2$ and $|\cdot|$ its associate norm.

Then

$$A_k = \frac{\tau}{3} \left(\|\lambda^{\frac{1}{2}} \nabla u^k(\mathbf{x})\|_{0,\Omega}^2 + \|\lambda^{\frac{1}{2}} \nabla u^{k-1}(\mathbf{x})\|_{0,\Omega}^2 + (\lambda^{\frac{1}{2}} \nabla u^k(\mathbf{x}), \lambda^{\frac{1}{2}} \nabla u^{k-1}(\mathbf{x})) \right).$$

Given that $xy \geq -\frac{1}{4}x^2 - y^2$, thus

$$A_k \geq \frac{\tau}{4} \|\lambda^{\frac{1}{2}} \nabla u^k\|_{0,\Omega}^2 = \frac{1}{4} B_k.$$

We deduce the first inequality of (10) by doing the sum on k .

Now using the fact that $xy \leq \frac{1}{2}x^2 + \frac{1}{2}y^2$, we conclude that

$$A_k \leq \frac{\tau}{2} \left(\|\lambda^{\frac{1}{2}} \nabla u^k\|_{0,\Omega}^2 + \|\lambda^{\frac{1}{2}} \nabla u^{k-1}\|_{0,\Omega}^2 \right).$$

By doing the sum on k and using the estimation (9) we prove the second inequality of (10). $\square$

We define the norm $\|\cdot\|_n$ by :

$$\|u^n\|_n = \left(\|u^n\|_{0,\Omega}^2 + \tau \sum_{j=1}^n \sum_{i=1}^{I^\circ} \|\lambda^{\frac{1}{2}} \nabla u^j\|_{0,\Omega_i^\circ}^2 \right)^{\frac{1}{2}}. \quad (11)$$

The a priori error estimate is the object of the following theorem.

Theorem 3.1. *If the solution u of problem (3) verifies that $\partial_t^2 u(\cdot, t) \in L^2(0, T, H^{-1}(\Omega))$, then*

$$\|u - u_\tau\|_n \leq c\tau \|u\|_{H^2(0,T,H^{-1}(\Omega))} \quad (12)$$

where c is a positive constant.

Proof. Let $e^j = u(t_j) - u^j$, $1 \leq j \leq M$ and $e^0 = 0$. Taking $t = t_j$ in problem (3), we obtain

$$\int_{\Omega} \frac{\partial u}{\partial t}(\mathbf{x}, t_j) v(\mathbf{x}) d\mathbf{x} + \sum_{i=1}^{I^\circ} \int_{\Omega_i^\circ} \lambda(\mathbf{x}) \nabla u(\mathbf{x}, t_j) \nabla v(\mathbf{x}) d\mathbf{x} = \int_{\Omega} f(\mathbf{x}, t_j) v(\mathbf{x}) d\mathbf{x}.$$

Since

$$\int_{\Omega} \left(\int_{t_{j-1}}^{t_j} \frac{\partial u}{\partial t} dt \right) v(\mathbf{x}) d\mathbf{x} = \int_{\Omega} (u(\mathbf{x}, t_j) - u(\mathbf{x}, t_{j-1})) v(\mathbf{x}) d\mathbf{x},$$

then using the variational formulation (8), we deduce that for any $v \in H_0^1(\Omega)$, the sequence (e^j) , $1 \leq j \leq M$ is a solution of problem

$$\begin{aligned} \int_{\Omega} e^j(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} + \tau \sum_{i=1}^{I^\circ} \int_{\Omega_i^\circ} \lambda(\mathbf{x}) \nabla e^j(\mathbf{x}) \nabla v(\mathbf{x}) d\mathbf{x} = \int_{\Omega} e^{j-1}(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} + \\ \tau \left(\int_{\Omega} \left(\frac{1}{\tau} \int_{t_{j-1}}^{t_j} \frac{\partial u}{\partial t}(\mathbf{x}, t) dt - \frac{\partial u}{\partial t}(\mathbf{x}, t_j) \right) v(\mathbf{x}) d\mathbf{x} \right). \end{aligned}$$

We remark that the error e^j is the solution of problem (8) for a data function

$$f^j = \frac{1}{\tau} \int_{t_{j-1}}^{t_j} \frac{\partial u}{\partial t}(\mathbf{x}, t) dt - \frac{\partial u}{\partial t}(\mathbf{x}, t_j).$$

By applying the mean value theorem and (9), we obtain, for $\partial_t^2 u(\cdot, t) \in L^2(0, T, H^{-1}(\Omega))$, the following estimation

$$\|e^n\|_{0,\Omega}^2 + \tau \sum_{j=1}^n \sum_{i=1}^{I^\circ} \|\lambda^{\frac{1}{2}} \nabla e^j\|_{0,\Omega_i^\circ}^2 \leq c\tau \|\partial_t^2 u\|_{L^2(0,T,H^{-1}(\Omega))}^2,$$

where c is a positive constant.

We conclude by using the proposition 4. $\square$

4 The mortar spectral element discretization

In this section we consider the function λ piecewise constant. The spectral discretization requires that the elements be rectangles, which leads us to make another partition without overlapping of the domain Ω

$$\overline{\Omega} = \bigcup_{i=1}^{i=I} \overline{\Omega}_i, \quad \Omega_i \cap \Omega_j = \emptyset, \quad i \neq j. \quad (13)$$

We suppose the function λ is constant on each Ω_i , $1 \leq i \leq I$. We remark that for any $1 \leq i \leq I$, there exists $1 \leq j \leq I^\circ$, such that $\Omega_i \subset \Omega_j^\circ$ and $I > I^\circ$.

To explain this problem, we take the case where $I^\circ = 2$. This means that Ω is composed of two heterogeneous regions (see figure 1). To handle this domain by spectral discretization, we need 5 rectangles ($I = 5$).

Fig. 1: The domain Ω

However, 9 rectangles are required for a conforming decomposition. We mean by conforming that if the intersection of two rectangles $\overline{\Omega}_i$ and $\overline{\Omega}_j$, $i \neq j$ is not empty, it is necessarily equal to a corner or to a hole edge of Ω_i and Ω_j .

We suppose that the intersection of each boundary $\partial\Omega_i$ of the sub-domain Ω_i with the boundary $\partial\Omega$ of the domain Ω is a corner or a hole edge of Ω_i . The skeleton of the decomposition

$$\mathbf{S} = \bigcup_{i=1}^I \partial\Omega_i \setminus \partial\Omega$$

is equal to

$$\overline{\mathbf{S}} = \bigcup_{m=1}^{\mathcal{M}} \overline{y_m}, \quad y_m \cap y_{m'} = \emptyset, \quad 1 \leq m \neq m' \leq \mathcal{M}, \quad (14)$$

where y_m is called mortar, which is equal to a hole edge of one sub-domain Ω_i that we note $\Omega_{i(m)}$. Let $\mathbb{P}_{N_i}(\Omega_i)$, $N_i \geq 2$, $1 \leq i \leq I$, the space of the polynomial functions defined on Ω_i , with degree $\leq N_i$, for the two variables x and y .

We define the mortar discrete space $\mathbb{X}_\delta$, ($\delta = (N_1, \dots, N_I)$ is the discretization parameter) as the space of functions u_δ such that (see [7]):

- u_{δ/Ω_i} , $1 \leq i \leq I$, belongs to the polynomial space $\mathbb{P}_{N_i}(\Omega_i)$,
- u_δ vanishes on the boundary $\partial\Omega$,
- let ϕ the mortar function such that $\phi|_{y_m} = u_{\delta/\Omega_{i(m)}/y_m}$, for each Ω_i , $1 \leq i \leq I$ and an edge Γ of Ω_i , which is not included on the boundary $\partial\Omega$, we have the following matching condition :

$$\forall \chi_\delta \in \mathbb{P}_{N_i-2}(\Gamma), \quad \int_{\Gamma} (u_{\delta/\Omega_i} - \phi)(\xi) \chi_\delta(\xi) d\xi = 0, \quad (15)$$

where $\mathbb{P}_{N_i-2}(\Gamma)$ is the space of polynomials with degree $\leq (N_i - 2)$, defined on Γ . Since Γ does not always coincide with a mortar γ_m , $1 \leq m \leq \mathcal{M}$, this allows us to say that the discretization is not conforming ($\mathbb{X}_\delta$ is not a subspace of $H^1(\Omega)$).

We remind the Gauss-Lobatto quadrature formula on the interval $\Lambda =]-1, 1[$:

If $N \geq 2$ is an integer, let $\epsilon_0 = -1$ and $\epsilon_N = 1$, there exists a unique set of nodes ϵ_k , $1 \leq k \leq (N-1)$ and weights ϱ_k , $0 \leq k \leq N$, such that :

$$\forall \psi \in \mathbb{P}_{2N-1}(\Lambda), \quad \int_{-1}^1 \psi(\xi) d\xi = \sum_{k=1}^N \psi(\epsilon_k) \varrho_k. \quad (16)$$

Hereinafter, We recall the following property (see [11]):

$$\forall \kappa_N \in \mathbb{P}_N(\Lambda), \quad \|\kappa_N\|_{0,\Lambda}^2 \leq \sum_{k=0}^N \kappa_N^2(\epsilon_k) \varrho_k \leq 3 \|\kappa_N\|_{0,\Lambda}^2. \quad (17)$$

We find the value of the nodes and weights ϵ_{ik}^x and ϱ_{ik}^x (respectively ϵ_{ik}^y and ϱ_{ik}^y) in the direction x (respectively in the direction y) by homothety and translation of the domain Ω_i to the reference domain Λ^2 . So, we have the discrete scalar product defined as :

For φ and ψ continuous on each Ω_i , $1 \leq i \leq I$

$$(\varphi, \psi)_\delta = \sum_{i=1}^I (\varphi, \psi)_{N_i}, \quad (18)$$

where

$$(\varphi, \psi)_{N_i} = \sum_{k=0}^{N_i} \sum_{l=0}^{N_i} \varphi(\epsilon_{ik}^x, \epsilon_{il}^y) \psi(\epsilon_{ik}^x, \epsilon_{il}^y) \varrho_{ik}^x \varrho_{il}^y.$$

Let $|\varphi|_{d,\Omega} = (\varphi, \varphi)_\delta^{\frac{1}{2}}$ the associate discrete norm.

We introduce the auxiliary space

$$\mathbb{Y}_\delta = \{\kappa_\delta \in L^2(\Omega); \kappa_\delta|_{\Omega_i} \in \mathbb{P}_{N_i}(\Omega_i); 1 \leq i \leq I\}$$

and $\mathfrak{I}_\delta$ the Lagrange interpolation operator defined as :

For all $\varphi \in \mathbb{X}_\delta$ such as $\varphi|_{\Omega_i}$, $1 \leq i \leq I$ is continuous on $\overline{\Omega_i}$, $\mathfrak{I}_\delta(\varphi) \in \mathbb{Y}_\delta$, with $\mathfrak{I}_\delta(\varphi)(\epsilon_{ik}^x, \epsilon_{il}^y) = \varphi(\epsilon_{ik}^x, \epsilon_{il}^y)$.

We suppose for any $0 \leq n \leq M$, f^n is continuous on each sub-domain Ω_i , $1 \leq i \leq I$. Then we define the discrete problem :

Find $u_\delta^n \in \mathbb{X}_\delta$ for each $1 \leq n \leq M$, such that

$$u_\delta^0 = \mathfrak{I}_\delta(u_0),$$

and

$$\forall v_\delta \in \mathbb{X}_\delta, \quad a_\delta^n(u_\delta^n, v_\delta) = L_\delta^n(v_\delta). \quad (19)$$

The bilinear form $a_\delta^n(., .)$, and the linear form $L_\delta^n(., .)$, for $1 \leq n \leq M$ are defined as :

$$a_\delta^n(u_\delta^n, v_\delta) = (u_\delta^n, v_\delta)_\delta + \tau \sum_{i=1}^I \lambda_i (\nabla u_\delta^n, \nabla v_\delta)_{N_i} \quad (20)$$

and

$$L_\delta^n(v_\delta) = (u_\delta^{n-1}, v_\delta)_\delta + \tau (f^n, v_\delta)_\delta. \quad (21)$$

Since the discretization is not conforming, we define the broken energy norm on $\mathbb{X}_\delta$

$$\|v_\delta\|_{\mathbb{X}_\delta} = \left(\|v_\delta\|_{0,\Omega}^2 + \tau \sum_{i=1}^I \lambda_i |v_\delta|_{1,\Omega_i}^2 \right)^{\frac{1}{2}}. \quad (22)$$

Lemma 1. *There exist two constants c_1 and c_2 independent of δ such that for all v_δ in $\mathbb{X}_\delta$, we have the following equivalence :*

$$c_1 \min(1, \lambda^{\min}) \sum_{i=1}^I \|v_\delta\|_{1,\Omega_i}^2 \leq \|v_\delta\|_{\mathbb{X}_\delta} \leq c_2 \max(1, \lambda^{\max}) \sum_{i=1}^I \|v_\delta\|_{1,\Omega_i}^2. \quad (23)$$

Proof. From (22) we deduce that

$$\left(\|v_\delta\|_{0,\Omega}^2 + \tau \lambda^{\min} \sum_{i=1}^I \|v_\delta\|_{1,\Omega_i}^2 \right)^{\frac{1}{2}} \leq \|v_\delta\|_{\mathbb{X}_\delta} \leq \left(\|v_\delta\|_{0,\Omega}^2 + \tau \lambda^{\max} \sum_{i=1}^I \|v_\delta\|_{1,\Omega_i}^2 \right)^{\frac{1}{2}}.$$

We conclude (23) since

$$\left(\sum_{i=1}^I \|v_\delta\|_{1,\Omega_i}^2 \right)^{\frac{1}{2}} \quad \text{and} \quad \left(\sum_{i=1}^I \|v_\delta\|_{1,\Omega_i}^2 \right)^{\frac{1}{2}}$$

are equivalent with constants c_1 and c_2 independent of δ (see [12]). $\square$

We prove using (17), Cauchy-Schwarz inequality and lemma 1 that the bilinear form $a_\delta^n(\cdot, \cdot)$ is continuous on $\mathbb{X}_\delta \times \mathbb{X}_\delta$, coercive on $\mathbb{X}_\delta$ and that the linear form $L_\delta^n(\cdot)$ is continuous on $\mathbb{X}_\delta$. Using the Lax Milgram theorem, we obtain the following result.

Theorem 1. *For f continuous on $\overline{\Omega} \times [0, T]$ and u_0 continuous on $\overline{\Omega}$, problem (19) has a unique solution $(u_\delta^n)_{0 \leq n \leq M}$ in $\mathbb{Y}_\delta \times (\mathbb{X}_\delta)^M$ such that*

$$\|u_\delta^n\|_{0,\Omega}^2 + \tau \sum_{j=1}^n \sum_{i=1}^I \lambda_i \|\nabla u_\delta^j\|_{0,\Omega_i}^2 \leq 9 \|\mathcal{I}_N u_0\|_{0,\Omega}^2 + \frac{81 C^2 \lambda^{\max}}{\lambda^{\min}} \sum_{j=1}^n \|\mathcal{I}_N f^j\|_{0,\Omega}^2$$

where C is the Poincaré-Friedrichs constant which only depends on the domain Ω .

Proof 1. Let $v_\delta = u_\delta^n$ in problem (19). Using the Cauchy-Schwarz inequality, we have

$$|u_\delta^n|_{d,\Omega}^2 + \tau \sum_{i=1}^I \lambda_i |\nabla u_\delta^n|_{d,\Omega_i}^2 \leq |u_\delta^{n-1}|_{d,\Omega} |u_\delta^n|_{d,\Omega} + |\mathcal{I}_N f^n|_{d,\Omega} |u_\delta^n|_{d,\Omega}.$$

Using (17), Poincaré-Friedrichs inequality and the fact that $ab \leq \frac{a^2}{2\mu} + \frac{\mu b^2}{2}$, $\forall \mu > 0$

$$\frac{1}{2} |u_\delta^n|_{d,\Omega}^2 + \tau \sum_{i=1}^I \lambda_i \|\nabla u_\delta^n\|_{0,\Omega_i}^2 \leq \frac{|u_\delta^{n-1}|_{d,\Omega}^2}{2} + \frac{9 \|\mathcal{I}_N f^n\|_{0,\Omega}^2}{2\mu} + \lambda^{\max} \frac{9 C^2 \mu}{2 \lambda^{\min}} \sum_{i=1}^I \lambda_i \|\nabla u_\delta^n\|_{0,\Omega_i}^2,$$

where C is the Poincaré-Friedrichs constant.

Doing the sum on n and using (17)

$$\frac{1}{2} \|u_\delta^n\|_{0,\Omega}^2 + \tau \sum_{j=1}^n \sum_{i=1}^I \lambda_i \|\nabla u_\delta^j\|_{0,\Omega_i}^2 \leq \frac{9 \|\mathcal{I}_N u_0\|_{0,\Omega}^2}{2} + \frac{9}{2\mu} \sum_{j=1}^n \|\mathcal{I}_N f^j\|_{0,\Omega}^2 + \lambda^{\max} \frac{9 C^2 \mu}{2 \lambda^{\min}} \sum_{j=1}^n \sum_{i=1}^I \lambda_i \|\nabla u_\delta^j\|_{0,\Omega_i}^2.$$

Then we conclude by choosing $\mu = \frac{\lambda^{\min}}{9 C^2 \lambda^{\max}}$.

5 Error estimate

For $1 \leq n \leq M$, we recall that u^n is the solution of problem (7). In the case where λ is piecewise constant, problem (7) is written :

$$\begin{cases} u^n + \tau \lambda \Delta u^n = u^{n-1} + \tau f^n & \text{in } \Omega \\ u^n = 0 & \text{on } \partial\Omega \\ u^0 = u_0 & \text{in } \Omega. \end{cases} \quad (24)$$

Multiplying the first equation in (24) by $v_\delta \in \mathbb{X}_\delta$ and integrating by parts gives

$$a^n(u^n, v_\delta) = L^n(v_\delta) + \sum_{i=1}^I \int_{\partial\Omega_i} \lambda_i(\partial_{\mathbf{n}_i} u^n) v_\delta d\xi,$$

where $\mathbf{n}_i$ is the unit normal vector to $\partial\Omega_i$.

If we define $[v_\delta]$ the jump of v_δ through the skeleton $\mathbf{S}$, we obtain

$$a^n(u^n, v_\delta) = L^n(v_\delta) + \frac{1}{2} \int_{\mathbf{S}} \lambda(\partial_{\mathbf{n}} u^n) [v_\delta] d\xi. \quad (25)$$

Proposition 5. *If f and u^0 are respectively continuous on $\overline{\Omega} \times [0, T]$ and $\overline{\Omega}$, then the error estimate between $(u^n)_{0 \leq n \leq M} \in L^2(\Omega) \times (H_0^1(\Omega))^M$ solution of (8) and $(u_\delta^n)_{0 \leq n \leq M} \in \mathbb{Y}_\delta \times (\mathbb{X}_\delta)^M$ solution of (19) is*

$$\|u^n - u_\delta^n\|_n \leq c \left(\inf_{v_\delta^n \in \mathbb{X}_\delta} \|u^n - v_\delta^n\|_n + \left[\inf_{v_\delta^0 \in \mathbb{Y}_\delta} \|u^0 - v_\delta^0\|_{0,\Omega} + \frac{\lambda^{\max}}{\lambda^{\min}} \sum_{j=1}^n (E_a^{1,j} + E_a^{2,j} + E_f^j + E_c^j) \right] \right) \quad (26)$$

where

$$\begin{aligned} E_a^{1,j} &= \frac{1}{\tau} \sup_{w_\delta \in \mathbb{X}_\delta} \frac{\int_{\Omega} (u^j - u^{j-1})(\mathbf{x}) w_\delta(\mathbf{x}) d\mathbf{x} - (v_\delta^j - v_\delta^{j-1}, w_\delta)_\delta}{\|w_\delta\|_{\mathbb{X}_\delta}}, \\ E_a^{2,j} &= \sup_{w_\delta \in \mathbb{X}_\delta} \frac{\sum_{i=1}^I \lambda_i \left(\int_{\Omega_i} \nabla u^j \nabla w_\delta d\mathbf{x} - (\nabla v_\delta^j, \nabla w_\delta)_{N_i} \right)}{\|w_\delta\|_{\mathbb{X}_\delta}}, \\ E_f^j &= \sup_{w_\delta \in \mathbb{X}_\delta} \frac{\int_{\Omega} f^j(\mathbf{x}) w_\delta(\mathbf{x}) d\mathbf{x} - (f, w_\delta)_\delta}{\|w_\delta\|_{\mathbb{X}_\delta}}, \\ E_c^j &= \sup_{w_\delta \in \mathbb{X}_\delta} \frac{\int_{\mathbf{S}} \lambda \partial_{\mathbf{n}} u^j [w_\delta] d\xi}{\|w_\delta\|_{\mathbb{X}_\delta}} \end{aligned}$$

and c is a positive constant independent of δ .

Proof 2. Let $(v_\delta^n)_{0 \leq n \leq M} \in \mathbb{X}_\delta$. By triangular inequality

$$\|u^n - u_\delta^n\|_n \leq \|u^n - v_\delta^n\|_n + \|v_\delta^n - u_\delta^n\|_n.$$

To estimate the term $\|u_\delta^n - v_\delta^n\|_n$, we consider the two problems (25) and (19)

for $w_\delta \in \mathbb{X}_\delta$:

$$\begin{aligned} \int_{\Omega} u^n(\mathbf{x}) w_\delta(\mathbf{x}) d\mathbf{x} + \tau \sum_{i=1}^I \lambda_i \int_{\Omega_i} \nabla u^n(\mathbf{x}) \nabla w_\delta(\mathbf{x}) d\mathbf{x} &= \int_{\Omega} u^{n-1}(\mathbf{x}) w_\delta(\mathbf{x}) d\mathbf{x} \\ &+ \tau \int_{\Omega} f^n(\mathbf{x}) w_\delta(\mathbf{x}) d\mathbf{x} + \frac{1}{2} \int_{\mathbf{S}} \lambda \partial_{\mathbf{n}} u^n [w_\delta] d\xi \end{aligned}$$

and

$$(u_\delta^n, w_\delta)_\delta + \tau \sum_{i=1}^I \lambda_i (\nabla u_\delta^n, \nabla w_\delta)_{N_i} = (u_\delta^{n-1}, w_\delta)_\delta + \tau (f^n, w_\delta)_\delta.$$

By doing the difference term by term, we obtain

$$(u_\delta^n - v_\delta^n, w_\delta)_\delta + \tau \sum_{i=1}^I \lambda_i (\nabla (u_\delta^n - v_\delta^n), \nabla w_\delta)_{N_i} = (u_\delta^{n-1} - v_\delta^{n-1}, w_\delta)_\delta + \tau \mathcal{L}^n(w_\delta)$$

where

$$\begin{aligned}\mathcal{L}^n(w_\delta) = & \frac{1}{\tau} \left[\int_{\Omega} (u^n - u^{n-1}) w_\delta \, d\mathbf{x} - (v_\delta^n - v_\delta^{n-1}, w_\delta)_\delta \right] \\ & + \sum_{i=1}^I \lambda_i \left(\int_{\Omega_i} \nabla u^n \nabla w_\delta \, d\mathbf{x} - (\nabla v_\delta^n, \nabla w_\delta)_{N_i} \right) \\ & + \int_{\Omega} f^n w_\delta \, d\mathbf{x} - (f^n, w_\delta)_\delta + \frac{1}{2} \int_{\mathbf{S}} \lambda \partial_{\mathbf{n}} u^n [w_\delta] \, d\xi.\end{aligned}$$

We remark that $\mathcal{L}^n$ is linear and continuous on $\mathbb{X}_\delta$ which is an Hilbert space for the scalar product $(\cdot, \cdot)_\delta$. Then, by Riesz theorem, there exists a unique element $g_\delta^n \in \mathbb{X}_\delta$ such that

$$\mathcal{L}^n(w_\delta) = (g_\delta^n, w_\delta)_\delta.$$

Therefore, we conclude that $z_\delta^n = u_\delta^n - v_\delta^n$ is solution to problem (8) with a data function equal to g_δ^n and $z_\delta^0 = \mathcal{I}_\delta u_0 - v_\delta^0$. Consequently using theorem 1, we have

$$\|u_\delta^n - v_\delta^n\|_n \leq c \left(\|\mathcal{I}_\delta u_0 - v_\delta^0\|_{0,\Omega}^2 + \frac{\lambda_{\min}}{\lambda_{\max}} \sum_{j=1}^n \|g_\delta^j\|_{0,\Omega}^2 \right)^{1/2}.$$

We remark that

$$\|g_\delta^j\|_{0,\Omega} \leq c \sup_{w_\delta \in \mathbb{X}_\delta} \frac{(g_\delta^j, w_\delta)_\delta}{\|w_\delta\|_{\mathbb{X}_\delta}},$$

which permits to conclude (26).

Let for each mortar $y_m \subset \mathbf{S}$, $1 \leq m \leq \mathcal{M}$, $\zeta(m)$ is the set of subscripts i , $1 \leq i \leq I$, such that $\partial\Omega_i \cap y_m$ has a positive measure. By estimating each term in (26), we obtain the following result proved in section 6.

Theorem 2. For λ constant on each Ω_i , $1 \leq i \leq I$. Let f such that $f|_{\Omega_i} \in \mathcal{C}^0(0, T; H^{\sigma_i}(\Omega_i))$; $\sigma_i > 1$, u_0 is such that $u_0|_{\Omega_i} \in H^{\mu_i}(\Omega_i)$; $\mu_i > 1$ and the solution $(u^n)_{0 \leq n \leq M}$ of problem (8) is such that $u_{/ \Omega_i}^n \in H^{s_i+1}(\Omega_i)$; $s_i \geq 0$. Then the error between $(u^n)_{0 \leq n \leq M}$ and $(u_\delta^n)_{0 \leq n \leq M}$ solution of problem (19) is

$$\begin{aligned}\|u^n - u_\delta^n\|_n \leq c \tau \left[(1 + \beta + \beta_\delta) \left(\left(\frac{\lambda^{\max}}{\lambda^{\min}} \right) \sum_{i=1}^I \lambda_i N_i^{-2s_i} \log(N_i) \|u^n\|_{s_i+1, \Omega_i}^2 \right)^{1/2} \right. \\ \left. + \left(\frac{1}{\min(1, \lambda^{\min})} \right)^{1/2} \left(\sum_{i=1}^I N_i^{-2\sigma_i} \|f\|_{\mathcal{C}^0(0, T; H^{\sigma_i}(\Omega_i))}^2 \right)^{1/2} \right. \\ \left. + \left(\sum_{i=1}^I N_i^{-2\mu_i} \|\mathcal{I}_\delta u_0\|_{\mu_i, \Omega_i}^2 \right)^{1/2} \right],\end{aligned}\tag{27}$$

where c is a positive constant independent of δ ,

$$\beta = \max_{1 \leq m \leq \mathcal{M}} \max_{k \in \zeta(m)} \left(\frac{\lambda_k}{\lambda_{i(m)}} \right)^{1/2}$$

and

$$\beta_\delta = \max_{1 \leq m \leq \mathcal{M}} \max_{k \in \zeta(m)} \left(\frac{\lambda_k N_{i(m)}}{\lambda_{i(m)} N_k} \right)^{1/2}.$$

Remark 2. For a conforming decomposition, the term β_δ vanishes. Since there is no restriction on choosing of mortars, we opt, if it is possible, for those leading to

$$\forall k \in \zeta(m), \quad \lambda_k \leq \lambda_{i(m)}.\tag{28}$$

Thus $\beta \leq 1$. If it is not possible, we choose the mortars and the degrees of approximation polynomials such that

$$\forall k \in \zeta(m), \quad \lambda_k N_k^{-1} \leq \lambda_{i(m)} N_{i(m)}^{-1}, \quad (29)$$

which may force us to make a small change in the decomposition. So we can optimize the estimation (27) without forcing the conformity of the decomposition.

6 Annex

This section is devoted to the proof of theorem 2. We need to estimate each term in (26).

6.1 Estimation of $E_a^{1,j}$

We pose $\kappa^j = u^j - u^{j-1}$. By remarking that $\Pi_{N_{i-1}}^1 \kappa^j = v_\delta^j - v_\delta^{j-1}$, we deduce that

$$E_a^{1,j} \leq \|\kappa^j - \Pi_{N_{i-1}}^1 \kappa^j\|_{0,\Omega}$$

where $\Pi_{N_{i-1}}^1$ is the orthogonal projection from $H^1(\Omega)$ to $P_{N_{i-1}}(\Omega)$.

The approximation properties of the operator $\Pi_{N_{i-1}}^1$ are well known (see [11], Theorem 7.3 or [13], Proposition 2.6). This permit to obtain

$$E_a^{1,j} \leq c N_i^{-s_i} \|\kappa^j\|_{s_i, \Omega_i},$$

where, $\kappa^j \in H^{s_i}(\Omega_i)$; for $s_i \geq 1$.

6.2 Estimation of $E_a^{2,j}$

Using the exactness of the quadrature formula for a polynomial of degree $\leq 2N - 1$, we write :

$$\sum_{i=1}^I \lambda_i \left[\int_{\Omega_i} \nabla u^j \nabla w_\delta d\mathbf{x} - (\nabla v_\delta^j, \nabla w_\delta)_{N_i} \right] = \sum_{i=1}^I \lambda_i \left[\int_{\Omega_i} \nabla (u^j - \Pi_{N_{i-1}}^1 u^j) \nabla w_\delta d\mathbf{x} - (\nabla (v_\delta^j - \Pi_{N_{i-1}}^1 v_\delta^j), \nabla w_\delta)_{N_i} \right].$$

By the triangular and Cauchy-Schwarz inequalities, we obtain :

$$\sup_{w_\delta \in \mathbb{X}_\delta} \frac{\sum_{i=1}^I \lambda_i \left[\int_{\Omega_i} \nabla u^j \nabla w_\delta d\mathbf{x} - (\nabla v_\delta^j, \nabla w_\delta)_{N_i} \right]}{\|w_\delta\|_{\mathbb{X}_\delta}} \leq c \lambda^{\max} \left[\sum_{i=1}^I |u^j - \Pi_{N_{i-1}}^1 u^j|_{1, \Omega_i} + |v_\delta^j - \Pi_{N_{i-1}}^1 v_\delta^j|_{1, \Omega_i} \right],$$

then we conclude by the properties of operator $\Pi_{N_{i-1}}^1$.

6.3 Estimation of E_f^j

Let $\Pi_{N_{i-1}}$ the orthogonal projection from $L^2(\Omega_i)$ to $\mathbb{P}_{N_{i-1}}(\Omega_i)$. We have by the exactness of the quadrature formula, for a polynomial of degree $\leq 2N_i - 1$,

$$\int_{\Omega} f^j(\mathbf{x}) w_\delta(\mathbf{x}) d\mathbf{x} - (f^j, w_\delta)_\delta = \sum_{i=1}^I \left[\int_{\Omega_i} (f^j - \Pi_{N_{i-1}} f^j)(\mathbf{x}) w_\delta(\mathbf{x}) d\mathbf{x} - (\mathcal{I}_\delta f^j - \Pi_{N_{i-1}} f^j, w_\delta)_{N_i} \right]$$

for all $w_\delta \in \mathbb{X}_\delta$.

Using (17) in each direction, we obtain

$$\int_{\Omega} f^j(\mathbf{x}) w_\delta(\mathbf{x}) d\mathbf{x} - (f^j, w_\delta)_\delta \leq \left[10 \left(\sum_{i=1}^I \|f^j - \Pi_{N_i-1} f^j\|_{0,\Omega}^2 \right)^{1/2} + 9 \|f^j - \mathcal{I}_\delta f^j\|_{0,\Omega}^2 \right] \|w_\delta\|_{0,\Omega}.$$

Using lemma 1 to bound $\|w_\delta\|_{0,\Omega}$ by $\|w_\delta\|_{\mathbb{X}_\delta}$ and the approximation properties of operator Π_{N_i-1} (see [11], Theorem 7.1) and $\mathcal{I}_\delta$ (see [11], Theorem 14.2) for $f^j \in H^{\sigma_i}(\Omega_i)$; $\sigma_i > 1$, we obtain

$$\sup_{w_\delta \in \mathbb{X}_\delta} \frac{\int_{\Omega} f^j(\mathbf{x}) w_\delta(\mathbf{x}) d\mathbf{x} - (f^j, w_\delta)_\delta}{\|w_\delta\|_{\mathbb{X}_\delta}} \leq c \left(\frac{1}{\min(1, \lambda^{\min})} \right)^{1/2} \left(\sum_{i=1}^I N_i^{-2\sigma_i} \|f^j\|_{\sigma_i, \Omega_i}^2 \right)^{1/2}.$$

6.4 Estimation of E_c^j

Let the operator

$$(\partial_{\lambda \mathbf{n}} u)_{/\partial \Omega_i} = \lambda_i \partial_{\mathbf{n}_i} u, \quad 1 \leq i \leq I.$$

This operator is discontinuous through the skeleton $\mathbf{S}$. It means that if $\Gamma = \partial \Omega_i \cap \partial \Omega_{i'}$, $1 \leq i < i' \leq I$, we have

$$(\partial_{\lambda \mathbf{n}} u)_{/\partial \Omega_i} + (\partial_{\lambda \mathbf{n}} u)_{/\partial \Omega_{i'}} = 0 \quad \text{on } \Gamma.$$

Lemma 2. For u^j in $\mathbb{X}_\delta$, $1 \leq j \leq n$, such that $u^j_{/\Omega_i} \in H^{s_i+1}(\Omega_i)$; $s_i > 0$, we have:

$$\sup_{w_\delta \in \mathbb{X}_\delta} \frac{\int_{\Omega} \partial_{\lambda \mathbf{n}} u^j [w_\delta] d\xi}{\|w_\delta\|_{\mathbb{X}_\delta}} \leq c(1 + \beta) \left(\sum_{i=1}^I \lambda_i N_i^{-2s_i} (\log(N_i)) \|u^j\|_{s_i+1, \Omega_i}^2 \right)^{1/2}$$

where $\beta = \max_{1 \leq m \leq \mathcal{M}} \max_{k \in \zeta(m)} \left(\frac{\lambda_k}{\lambda_{i(m)}} \right)^{1/2}$ and c is a positive constant independent of δ .

Proof 3. Let ψ the mortar function associated to $w_\delta \in \mathbb{X}_\delta$. By the matching condition (15), for each edge Γ of $\partial \Omega_i$, $1 \leq i \leq I$ which is not mortar, we have :

$$\int_{\Gamma} \partial_{\lambda \mathbf{n}} u^j [w_\delta] d\xi = \lambda_i \int_{\Gamma} (\partial_{\mathbf{n}_i} u^j - \pi_{N_i-2}^\Gamma \partial_{\mathbf{n}_i} u^j) (w_{\delta/\Omega_i} - \psi) d\xi$$

where $\pi_{N_i-2}^\Gamma$ is the projection operator on $\mathbb{P}_{N_i-2}(\Gamma)$.

We suppose now that the decomposition is conforming, then $\Gamma = \overline{\Omega_i} \cap \overline{\Omega_{i'}}$, $1 \leq i \neq i' \leq I$. We know that $u^j_{/\Omega_i} \in H^{s_i+1}(\Omega_i)$, we study the two following cases.

i) If $0 < s_i < \frac{1}{2}$, we define the trace of $\lambda_i \partial_{\mathbf{n}_i} u^j$ by duality:

For $v \in H^{1/2-s_i}(\Gamma)$

$$\int_{\Gamma} \lambda_i \partial_{\mathbf{n}_i} u^j v d\xi = \lambda_i \int_{\Omega_i} \Delta u^j \tilde{v}(\mathbf{x}) d\mathbf{x} + \lambda_i \int_{\Omega_i} \nabla u^j \nabla \tilde{v} d\mathbf{x}$$

where $\tilde{v}$ is the lifting function in $H^{1-s_i}(\Omega_i)$ of v .

Therefore, since $\pi_{N_i-2}^\Gamma$ is the orthogonal operator from $H^{s_i-1/2}(\Omega_i)$ into $\mathbb{P}_{N_i-2}(\Gamma)$, we conclude then for any w_{N_i-2} and ψ_{N_i-2} in $\mathbb{P}_{N_i-2}(\Gamma)$,

$$\int_{\Gamma} \partial_{\lambda \mathbf{n}} u^j [w_\delta] d\xi \leq \|\lambda_i^{1/2} \partial_{\mathbf{n}_i} u^j\|_{s_i-1/2, \Gamma} \lambda_i^{1/2} \left[\|w_\delta - w_{N_i-2}\|_{1/2-s_i, \Gamma} + \|\psi - \psi_{N_i-2}\|_{1/2-s_i, \Gamma} \right].$$

Thus, the approximation properties (see [11]) allows us to deduce

$$\int_{\Gamma} \partial_{\lambda \mathbf{n}} u^j [w_{\delta}] d\xi \leq c N_i^{-s_i} \|\lambda_i^{1/2} \partial_{\mathbf{n}_i} u^j\|_{s_i-1/2, \Gamma} \lambda_i^{1/2} \left[\|w_{\delta}\|_{1/2, \Gamma} + \|\psi\|_{1/2, \Gamma} \right].$$

We remark that $\|w_{\delta}\|_{1/2, \Gamma}$ is bounded by $\|w_{\delta}\|_{1, \Omega_i}$ and since the decomposition is conforming, $\|\psi\|_{1/2, \Gamma}$ is bounded by $\|w_{\delta}\|_{1, \Omega_i}$, which permit to conclude.

ii) In the case where $s_i > \frac{1}{2}$, we define $\pi_{N_i-2}^{\Gamma}$ the orthogonal projection operator from $L^2(\Gamma)$ to $\mathbb{P}_{N_i-2}(\Gamma)$, then

$$\int_{\Gamma} \partial_{\lambda \mathbf{n}} u^j [w_{\delta}] d\tau \leq \|\lambda_i^{1/2} (\partial_{\mathbf{n}_i} u^j - \pi_{N_i-2}^{\Gamma}(\partial_{\mathbf{n}_i} u^j))\|_{0, \Gamma} \lambda_i^{1/2} \left[\|w_{\delta/\Omega_i} - w_{N_i-2}\|_{0, \Gamma} + \|\psi - \psi_{N_i-2}\|_{0, \Gamma} \right].$$

We conclude thanks to the approximation properties of $\pi_{N_i-2}^{\Gamma}$ (see [11], theorem 6.1).

The estimation in the case $s_i = 1/2$ is given by interpolation argument. In the case where $\Gamma = \partial\Omega_i \cap \partial\Omega$, the mortar function $\psi \notin H^{1/2}(\Gamma)$ and a modification make appears $(\log(N_i))^{1/2}$ ([13], proposition 21).

Remark 3. The term $(\log(N_i))^{1/2}$ is negligible compared to $N_i^{-s_i}$ when N_i is big enough. This term disappears when the edges of any sub-domain Ω_i , $1 \leq i \leq I$, which are not a mortar, are in the boundary $\partial\Omega$.

6.5 Estimation of $\inf_{v_{\delta}^n \in \mathbb{X}_{\delta}} \|u^n - v_{\delta}^n\|_n$

Lemma 3. Let $u^n \in \mathbb{X}_{\delta}$; $1 \leq n \leq M$ such that $u_{/\Omega_i}^n \in H^{s_i+1}(\Omega_i)$, $s_i \geq 2$, then we have

$$\inf_{v_{\delta}^n \in \mathbb{X}_{\delta}} \|u^n - v_{\delta}^n\|_n \leq c(1 + \beta + \beta_{\delta}) \left(\sum_{i=1}^I \lambda_i N^{-2s_i} \|u^n\|_{s_i+1, \Omega_i}^2 \right)^{1/2}$$

where c is a positive constant independent on δ and

$$\beta_{\delta} = \max_{1 \leq m \leq \mathcal{M}} \max_{k \in \xi(m)} \left(\frac{\lambda_k N_{i(m)}}{\lambda_{i(m)} N_k} \right)^{1/2}$$

depends on δ .

Proof 4. The proof will be done in three steps to construct $v_{\delta}^n \in \mathbb{X}_{\delta}$ which represents the best approximation of u^n in the space $\mathbb{X}_{\delta}$.

1) Let π_N^0 the operator from $H^{3/2}(\Lambda)$ in $\mathbb{P}_N(\Lambda)$ which preserves ± 1 (see [11], theorem 6.4), then $\forall \varphi \in H^s(\Lambda)$, $s \geq 3/2$ and $0 \leq t \leq 3/2$,

$$\|\varphi - \pi_N^0 \varphi\|_{t, \Lambda} \leq c N^{t-s} \|\varphi\|_{s, \Lambda}.$$

Let $\pi_{N_i}^0$ the operator defined from π_N^0 by translation and homothety. We choose v_{δ}^{n1} such that

$$v_{\delta/\Omega_i}^{n1} = (\pi_{N_i}^{0,x} \circ \pi_{N_i}^{0,y})(u_{/\Omega_i}^n),$$

where $\pi_{N_i}^{0,x}$ and $\pi_{N_i}^{0,y}$ are the operators in the x - and y -directions on Ω_i . We conclude then, if $s_i \geq 2$ (see [11]):

$$\lambda_i^{1/2} \|u^n - v_{\delta}^{n1}\|_{t, \Omega_i} \leq c \lambda_i^{1/2} N_i^{t-s_i-1} \|u^n\|_{s_i+1, \Omega_i}, \quad (30)$$

where the function v_{δ}^{n1} vanishes on $\partial\Omega$ and coincides with u^n at the corners of the sub-domain Ω_i , $1 \leq i \leq I$.

- 2) Let ϖ the set of vertices of Ω_i which are not a boundary of an edge of an other sub-domain $\Omega_{i'}$. For $c \in \varpi$, $\zeta(c)$ is the set of subscripts i , $1 \leq i \leq I$ such that $c \in \partial\Omega_i$. We denote by $i(c)$ the index of $\zeta(c)$ such that

$$\forall i \in \zeta(c); \quad \lambda_i \leq \lambda_{i(c)}$$

and for $i \in \zeta(c)/\{i(c)\}$, we define ψ_i^c the polynomial of minimal degree such that $\psi_i^c(c) = 1$ and vanishes on the edges of Ω_i which do not contain c and on the other points of $\varpi \cap \partial\Omega_i$, then :

$$v_\delta^{n2} = \sum_{c \in \varpi} \sum_{i \in \zeta(c)/\{i(c)\}} \left(v_\delta^{1n}/_{\Omega_{i(c)}} - v_\delta^{1n}/_{\Omega_i} \right) (c) \psi_i^c.$$

So, we have for $0 \leq t \leq 3/2$

$$\begin{aligned} \lambda_i^{1/2} \|v_\delta^{n2}\|_{t, \Omega_i} &\leq c \lambda_i^{1/2} \sum_{c \in \varpi \cap \partial\Omega_i} \left| \left(v_\delta^{1n}/_{\Omega_{i(c)}} - v_\delta^{1n}/_{\Omega_i} \right) (c) \right| \\ &\leq c \lambda_i^{1/2} \sum_{c \in \varpi \cap \partial\Omega_i} \left(\|u^n - v_\delta^{n1}\|_{L^\infty(\Omega_i)} + \|u^n - v_\delta^{n1}\|_{L^\infty(\Omega_{i(c)})} \right). \end{aligned}$$

Thanks to the Gagliardo-Nirenberg inequality, for $0 < \alpha_i < 1/2$

$$\begin{aligned} \lambda_i^{1/2} \|v_\delta^{n2}\|_{t, \Omega_i} &\leq c \lambda_i^{1/2} \left(\sum_{c \in \varpi \cap \partial\Omega_i} \left(\|u^n - v_\delta^{n1}\|_{1-\alpha_i, \Omega_i}^{1/2} \|u^n - v_\delta^{n1}\|_{1+\alpha_i, \Omega_i}^{1/2} \right. \right. \\ &\quad \left. \left. + \|u^n - v_\delta^{n1}\|_{1-\alpha_i, \Omega_{i(c)}}^{1/2} \|u^n - v_\delta^{n1}\|_{1+\alpha_i, \Omega_{i(c)}}^{1/2} \right) \right). \end{aligned}$$

Since $\lambda_i \leq \lambda_{i(c)}$ and using (30) (for $t = 1 - \alpha_i$ and $t = 1 + \alpha_i$), we conclude that :

$$\lambda_i^{1/2} \|v_\delta^{n2}\|_{t, \Omega_i} \leq c \lambda_i^{1/2} N_i^{-s_i} \|u^n\|_{s_i+1, \Omega_i} + \sum_{c \in \varpi \cap \partial\Omega_i} \lambda_{i(c)}^{1/2} N_i^{-s_{i(c)}} \|u^n\|_{s_{i(c)}+1, \Omega_{i(c)}}. \quad (31)$$

Therefore, we end by doing the sum on i , $1 \leq i \leq I$.

- 3) Let $\mathcal{H}_i$, $1 \leq i \leq I$ the set of edges of Ω_i which are not include in $\partial\Omega$ and are not mortars. If φ is the mortar function associated to $v_\delta^{n1} + v_\delta^{n2}$, then

$$v_\delta^{n3} = \sum_{i=1}^I \sum_{\Gamma \in \mathcal{H}_i} R_i^L \left(\pi_{N_i}^* \varphi - (v_\delta^{n1} + v_\delta^{n2})/_\Omega_i \right)$$

where

- R_i^L is the lifting operator from $\mathbb{P}_{N_i}(\Gamma) \cap H_{00}^{1/2}(\Gamma)$ into $\mathbb{P}_{N_i}(\Omega_i)$, such that $\forall \varphi_\delta \in \mathbb{P}_{N_i}(\Gamma) \cap H_{00}^{1/2}(\Gamma)$ (see [14])

$$\|R_i^L \varphi_\delta\|_{1, \Omega_i} \leq c \|\varphi_\delta\|_{H_{00}^{1/2}(\Gamma)}, \quad (32)$$

- $\pi_{N_i}^*$ is the operator of projection from $H_{00}^{1/2}(\Gamma)$ to $\mathbb{P}_{N_i}(\Gamma) \cap H_{00}^{1/2}(\Gamma)$, such that $\forall \varphi \in H_{00}^{1/2}(\Gamma)$

$$\int_{\Gamma} (\varphi - \pi_{N_i}^* \varphi) \psi_\delta(\tau) d\xi = 0 \quad \forall \psi_\delta \in \mathbb{P}_{N_i-2}(\Gamma). \quad (33)$$

Then following (31)

$$\lambda_i^{1/2} \|v_\delta^{n3}\|_{1, \Omega_i} \leq c \lambda_i^{1/2} \sum_{\Gamma \in \mathcal{H}_i} \|\pi_{N_i}^* \varphi - (v_\delta^{n1} + v_\delta^{n2})/_\Omega_i\|_{H_{00}^{1/2}(\Gamma)}.$$

Since $\pi_{N_i}^* \varphi - (v_\delta^{n1} + v_\delta^{n2})/_\Omega_i$ coincides with $(v_\delta^{n1} + v_\delta^{n2})/_\Omega_i$ on Γ , then :

$$\begin{aligned} \lambda_i^{1/2} \|v_\delta^{n3}\|_{1, \Omega_i} &\leq c \lambda_i^{1/2} \sum_{\Gamma \in \mathcal{H}_i} \left(\| [u^n - (v_\delta^{n1} + v_\delta^{n2})]_{H_{00}^{1/2}(\Gamma)} \right. \\ &\quad \left. + \|(id - \pi_{N_i}^*) [u^n - (v_\delta^{n1} + v_\delta^{n2})] \|_{H_{00}^{1/2}(\Gamma)} \right) \end{aligned} \quad (34)$$

where $[\cdot]$ is the jump through Γ .

Using the trace theorem, (30) and (31), we conclude for $t = 1$ that

$$\lambda_i^{1/2} \| [u^n - (v_\delta^{n1} + v_\delta^{n2})] \|_{H_{00}^{1/2}(\Gamma)} \leq \lambda_i^{1/2} \left(N_i^{-s_i} \|u^n\|_{s_i+1, \Omega_i} + \sum_{\text{mes}(\partial\Omega_i \cap \gamma_m) > 0} N_{i(m)}^{-s_{i(m)}} \|u^n\|_{s_{i(m)}+1, \Omega_{i(m)}} \right). \quad (35)$$

To evaluate the second term in (34), we remark that the operator $\pi_{N_i}^*$ is equal to the operator of projection from $H_0^1(\Gamma)$ into $\mathbb{P}_{N_i}(\Gamma) \cap H_0^1(\Gamma)$. Then, $\forall \psi \in H_0^1(\Gamma)$

$$\|\psi - \pi_{N_i}^* \psi\|_{H_{00}^{1/2}(\Gamma)} \leq c N_i^{-1/2} \| [u^n - (v_\delta^{n1} + v_\delta^{n2})] \|_{1, \Gamma}.$$

Therefore, we proceed as before with $t = 3/2$, we obtain

$$\lambda_i^{1/2} \|(id - \pi_{N_i}^*) [u^n - (v_\delta^{n1} + v_\delta^{n2})]\|_{H_{00}^{1/2}(\Gamma)} \leq c \lambda_i^{1/2} N_i^{-1/2} \left(N_i^{1/2-s_i} \|u^n\|_{s_i+1, \Omega_i} + \sum_{\text{mes}(\partial\Omega_i \cap \gamma_m) > 0} N_{i(m)}^{1/2-s_{i(m)}} \|u^n\|_{s_{i(m)}+1, \Omega_{i(m)}} \right). \quad (36)$$

This is where we introduce β_δ .

To conclude the proof of Theorem 2, we choose $v_\delta^n = v_\delta^{n1} + v_\delta^{n2} + v_\delta^{n3}$ in $\mathbb{X}_\delta$ and combine (30), (31), (35) and (36).

Conclusion

This work concerns the numerical analysis of the mortar spectral elements method discretization of the heat equation with a diffusion coefficient λ , depending on the heterogeneity of the domain. To solve the problem of the solution singularity due to the discontinuity of λ , we use a non conform geometric decomposition of the domain. We prove an optimal error estimate that depends only on the local regularity of the solution. The numerical validation of this result will be the subject of a forthcoming work.

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References

- [1] N.S. Papageorgiou, V.D. Radulescu and D.D. Repovs, Nonlinear analysis – theory and methods, Springer Monographs in Mathematics. Springer, Cham, 2019.
- [2] A. Bergam, C. Bernardi and Z. Mghazli, A posteriori analysis of the finite element discretization of some parabolic equations, Math. Comp. 74 (2005), no. 251, 1117-1138.
- [3] N. Chorfi, M. Abdelwahed and I. Ben Omrane, A posteriori analysis of the spectral element discretization of heat equation, An. St. Univ. Ovidius Constanta. 22 (3) (2014), 13-35.
- [4] V. Thomée, Galerkin Finite Element Methods for Parabolic Problems, Springer Series in Computational Mathematics 25, Springer, Paris, 1997.
- [5] C. Bernardi and R. Verfürth, Adaptive finite element methods for elliptic equations with non-smooth coefficients, Numer. Math. 85 (2000) 579-608.
- [6] C. Bernardi and N. Chorfi, Mortar spectral element methods for elliptic equations with discontinuous coefficients, Math. Models Methods Appl. Sci. 12 (2002), no. 4, 497-524.
- [7] C. Bernardi, Y. Maday and A. T. Patera, A new nonconforming approach to domain decomposition: the mortar element method, in Nonlinear Partial Differential Equations and their Applications, Collège de France Seminar, H. Brézis, J.-L. Lions, Eds, (1991).

- [8] B. I. Wohlmuth, Discretisation Methods and Iterative Solvers Based on Domain Decomposition, Lecture Notes in Computational Science and Engineering, Vol. 17 (Springer 2001).
- [9] J. L. Lions and E. Magenes, Problèmes aux Limites non Homogène et Applications, 1, Dunod, (1968).
- [10] N. G. Meyers, An L^p -estimate for the gradient of solutions of second order elliptic divergence equations, Ann. Sci. Norm. Sup. Pisa 17 (1963) 189-206.
- [11] C. Bernardi and Y. Maday, Spectral Methods, in Handbook of Numerical Analysis V, P.G. Ciarlet and J.-L. Lions, eds., North-Holland, Amsterdam, 1997, pp. 209-485.
- [12] M. Azaiez, C. Bernardi and Y. Maday, Some tools for adaptivity in the spectral element method, in Proc. of Third Int. Conf. On Spectral And High Order Methods, Houston J. Math. (1996) 243-253.
- [13] C. Bernardi and Y. Maday, Spectral, spectral element and mortar element methods, in Theory and Numerics of Differential Equations, Durham 2000, eds. J. F. Blowey, J. P. coleman and A. W. Craig (Springer, 2001), pp. 1-57.
- [14] C. Bernardi, M. Dauge, and Y. Maday, Polynomials in the Sobolev World, Internal Report, Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, Paris, France, 2003.