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The superposition operator in the space of functions continuous and converging at infinity on the real half-axis

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Abstract: We will consider the so-called superposition operator in the space $CC(\mathbb{R}_+)$ of real functions defined, continuous on the real half-axis $\mathbb{R}_+$ and converging to finite limits at infinity.

We will assume that the function $f = f(t, x)$ generating the mentioned superposition operator is locally uniformly continuous with respect to the variable x uniformly for $t \in \mathbb{R}_+$. Moreover, we require that the function $t \rightarrow f(t, x)$ satisfies the Cauchy condition at infinity uniformly with respect to the variable x . Under the above indicated assumptions a few properties of the superposition operator in question are derived. Examples illustrating our considerations will be included.

Keywords: Banach space; space of functions defined, continuous on the half-axis and converging at infinity; superposition operator; Cauchy condition at infinity; equicontinuous functions; relatively compact set

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1. Introduction

The superposition operator plays an important role in numerous mathematical investigations connected with operator theory, theory of integral equations and with considerations conducted in nonlinear functional analysis (cf. [1–4]). Especially, a lot of properties related to the superposition operator are utilized in the theory of Hammerstein integral equations [4].

Our aim in this paper is to establish some properties of the superposition operator acting in the space $CC(\mathbb{R}_+)$ consisting of real functions defined, continuous on the real half-axis $\mathbb{R}_+$ and converging to finite limits at infinity.

Firstly, we are going to establish conditions guaranteeing that the superposition operator transforms the space $CC(\mathbb{R}_+)$ into itself and is continuous. In order to find those conditions we will impose appropriate assumptions on the function $f(t, x)$ generating the superposition operator in question.

It is worthwhile mentioning that in several research works devoted to the superposition operator one can find a lot of papers stating properties in the mentioned spirit which are dedicated to investigations of properties of the superposition operator in several function spaces [2]. However, the research in the case of the space $CC(\mathbb{R}_+)$ seems to be not thoroughly conducted up to now.

Secondly, we will look for conditions (imposed on the generating function $f(t, x)$ which ensure that the superposition operator generated by $f(t, x)$ transforms some kind of relatively compact sets in the space $CC(\mathbb{R}_+)$ into itself. Such investigations are, in general, not easy what is mainly caused by the fact that we do not know necessary and sufficient conditions for relative compactness in the space $CC(\mathbb{R}_+)$ which are related to

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the structure of this space (such, for example, as Arzelà-Ascoli criterion in the classical space $C([a, b])$ or Kolmogorov or Riesz criteria in the space $L^p(a, b)$).

Results which will be obtained in the realization of the above formulated goals will be applied in the formulation of additional properties of the superposition operator connected with the property of transforming continuously some class of subsets of the space $CC(\mathbb{R}_+)$ being relatively compact into that class.

The results of the paper create an extension of those ones obtained up to now in the theory of superposition operators (cf. [1, 2, 5, 6], for example).

2. Notation, definitions and auxiliary facts

This section is devoted to establish some notation and to present definitions of basic concepts utilized in the paper. Moreover, we are going to recall a few results connected with investigations conducted in the paper.

At the beginning let us present some notation.

Denote by $\mathbb{R}$ the set of real numbers and put $\mathbb{R}_+ = [0, \infty)$. The symbol $\mathbb{N}$ will stand for the set of natural numbers (positive integers).

In this paper we will mainly deal with the so-called superposition operator. To formulate the definition of that operator let us take a function $f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$. Next, consider the set X consisting of functions $x : \mathbb{R}_+ \rightarrow \mathbb{R}$. Then the operator $F : X \rightarrow X$ defined on the set X in the following way

$$(Fx)(t) = f(t, x(t)), \quad t \in \mathbb{R}_+,$$

is called the superposition operator generated by the function $f = f(t, x)$.

Let us pay attention to the fact that we can define the superposition operator in a more general setting [1, 2], but for our further purposes the above given definition will be entirely sufficient.

Considerations of this paper will be located in the space $CC(\mathbb{R}_+)$ consisting of all functions $x(t) = x : \mathbb{R}_+ \rightarrow \mathbb{R}$ which are continuous on $\mathbb{R}_+$ and converging at infinity to finite limits. The space $CC(\mathbb{R}_+)$ forms a Banach space with the standard supremum norm

$$\|x\|_\infty = \sup \{ |x(t)| : t \in \mathbb{R}_+ \}.$$

Observe that the space $CC(\mathbb{R}_+)$ can be defined equivalently as the space consisting of real functions $x = x(t)$ defined, continuous and bounded on $\mathbb{R}_+$ and converging at infinity.

We can also consider the space $BC(\mathbb{R}_+)$ of all functions which are defined, continuous and bounded on $\mathbb{R}_+$. Obviously, $BC(\mathbb{R}_+)$ forms a Banach space with the norm $\|\cdot\|_\infty$ defined above. It is also worthwhile mentioning that $CC(\mathbb{R}_+)$ forms a closed subspace of the space $BC(\mathbb{R}_+)$. We omit the standard proof of this assertion.

Keeping in mind our further considerations we now recall a convenient necessary and sufficient condition for the function $x : \mathbb{R}_+ \rightarrow \mathbb{R}$ to be convergent at infinity to a finite limit. It is easy to recognize this condition as the classical Cauchy condition.

Lemma 2.1. *The function $x : \mathbb{R}_+ \rightarrow \mathbb{R}$ is convergent at infinity to a finite limit if and only if it satisfies the following condition*

$$\forall \varepsilon > 0 \quad \exists T > 0 \quad \forall t, s \geq T \quad |x(t) - x(s)| \leq \varepsilon.$$

Finally, we recall a sufficient condition for a bounded subset of the space $CC(\mathbb{R}_+)$ to be relatively compact in $CC(\mathbb{R}_+)$. That condition can be easily deduced from a suitable condition in the space $BC(\mathbb{R}_+)$ given in [7]. In order to formulate a sufficient condition mentioned above we quote definitions of two concepts which will be needed in the presentation of the announced condition.

Definition 2.2. Let X be a subset of the space $BC(\mathbb{R}_+)$. We say that functions of the set X are locally equicontinuous on the interval $\mathbb{R}_+$ if the following condition is satisfied

$$\forall \varepsilon > 0 \quad \forall T > 0 \quad \exists \delta > 0 \quad \forall x \in X \quad \forall t, s \in [0, T] \quad [|t - s| \leq \delta \Rightarrow |x(t) - x(s)| \leq \varepsilon].$$

Definition 2.3. Let X be a subset of the space $BC(\mathbb{R}_+)$. We will say that functions of the set X satisfy uniformly the Cauchy condition at infinity if

$$\forall \varepsilon > 0 \quad \exists T > 0 \quad \forall x \in X \quad \forall t, s \geq T \quad |x(t) - x(s)| \leq \varepsilon.$$

Observe that the above given definitions remain the same if we formulate them in the space $CC(\mathbb{R}_+)$. Moreover, it is worthwhile mentioning that Definition 2.3 is closely related to Lemma 2.1.

Now, we formulate a theorem providing the above announced sufficient condition for a bounded subset of the space $CC(\mathbb{R}_+)$ to be relatively compact in $CC(\mathbb{R}_+)$.

Theorem 2.4. Let X be a bounded set in the space $CC(\mathbb{R}_+)$. If functions of the set X are locally equicontinuous on the interval $\mathbb{R}_+$ and satisfy uniformly the Cauchy condition at infinity then the set X is relatively compact in the space $CC(\mathbb{R}_+)$.

In what follows we will call the condition contained in the above theorem as *the condition (A) for relative compactness*.

3. Results concerning the superposition operator

In this section we are going to present the main results of the paper which are concerning some properties of the superposition operator in the Banach function space $CC(\mathbb{R}_+)$ described in the previous section.

Throughout this section we will assume that $f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ is a given function. The superposition operator generated by the function $f(t, x)$ will be denoted by F .

In what follows we will assume that the function $f = f(t, x)$ satisfies the below presented assumptions.

- (i) The function f is continuous on the set $\mathbb{R}_+ \times \mathbb{R}$.
- (ii) For any $r > 0$ the function $x \rightarrow f(t, x)$ is uniformly continuous on the interval $[-r, r]$ uniformly with respect to the variable $t \in \mathbb{R}_+$ i.e., the following condition is satisfied

$$\forall r > 0 \quad \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x, y \in [-r, r] \quad \forall t \in \mathbb{R}_+ \quad [|x - y| \leq \delta \Rightarrow |f(t, x) - f(t, y)| \leq \varepsilon].$$

- (iii) The function $t \rightarrow f(t, x)$ satisfies the Cauchy condition at infinity uniformly with respect to the variable $x \in \mathbb{R}$ what means that the following condition is satisfied

$$\forall \varepsilon > 0 \quad \exists T > 0 \quad \forall t, s \geq T \quad \forall x \in \mathbb{R} \quad |f(t, x) - f(s, x)| \leq \varepsilon.$$

Now, let us scrutinize assumptions imposed on the function $f(t, x)$ which are formulated above.

First of all let us observe that from assumption (iii) and Lemma 2.1 we infer that for each fixed $x \in \mathbb{R}$ there exists a finite limit $\lim_{t \rightarrow \infty} f(t, x)$. Let us denote this limit by f_x i.e., let us put

$$f_x = \lim_{t \rightarrow \infty} f(t, x). \quad (3.1)$$

Then we have the following theorem.

Theorem 3.1. Let the function $f(t, x) = f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy assumptions (i)–(iii). Then the function $x \rightarrow f_x$ is nearly uniformly continuous on $\mathbb{R}$ what means that the function $\tilde{F} : \mathbb{R} \rightarrow \mathbb{R}$ defined by the equality

$$\tilde{F}(x) = f_x$$

is nearly uniformly continuous on $\mathbb{R}$ i.e.,

$$\forall \varepsilon > 0 \quad \forall r > 0 \quad \exists \delta > 0 \quad \forall x, y \in [-r, r] \quad [|x - y| \leq \delta \Rightarrow |f_x - f_y| \leq \varepsilon].$$

Proof. Fix arbitrarily numbers $\varepsilon > 0$ and $r > 0$. Then, in view of the definition of the limit of a function at infinity and (3.1), we can choose to the number $\frac{\varepsilon}{3}$ a number $T_1 > 0$ such that

$$|f(t, x) - f_x| \leq \frac{\varepsilon}{3} \quad (3.2)$$

for $t \geq T_1$. Similarly, we can find a number $T_2 > 0$ such that

$$|f(t, y) - f_y| \leq \frac{\varepsilon}{3} \quad (3.3)$$

for $t \geq T_2$.

Next, let us take $T = \max \{T_1, T_2\}$. Then, in virtue of (3.2) and (3.3), for $t \geq T$ we infer that the following inequalities are satisfied

$$|f(t, x) - f_x| \leq \frac{\varepsilon}{3}, \quad |f(t, y) - f_y| \leq \frac{\varepsilon}{3}. \quad (3.4)$$

Further, using assumption (ii), we can choose $\delta > 0$ to the numbers $\frac{\varepsilon}{3}$ and r ($r > 0$) such that for $x, y \in [-r, r]$, $|x - y| \leq \delta$ and for an arbitrary number $t \in \mathbb{R}_+$ the following inequality holds

$$|f(t, x) - f(t, y)| \leq \frac{\varepsilon}{3}. \quad (3.5)$$

Now, keeping in mind (3.4), (3.5) and taking arbitrary numbers $x, y \in [-r, r]$ such that $|x - y| \leq \delta$, for an arbitrary number $t \geq T$ we get

$$\begin{aligned} |f_x - f_y| &\leq |f_x - f(t, x)| + |f(t, x) - f(t, y)| + |f(t, y) - f_y| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Thus the proof of our theorem is complete. $\square$

In the sequel of this section we will consider an assumption being stronger than assumption (ii). Namely, let us formulate the announced assumption.

(ii') The function $x \rightarrow f(t, x)$ is uniformly continuous on $\mathbb{R}$ uniformly with respect to the variable $t \in \mathbb{R}_+$ i.e., the following condition is satisfied

$$\forall_{\varepsilon > 0} \exists_{\delta > 0} \forall_{x, y \in \mathbb{R}} \forall_{t \in \mathbb{R}_+} [|x - y| \leq \delta \Rightarrow |f(t, x) - f(t, y)| \leq \varepsilon].$$

Then, remaining assumptions (i), (iii) and replacing assumption (ii) by the above formulated assumption (ii'), we obtain the following counterpart of Theorem 3.1.

Theorem 3.2. *Let the function $f = f(t, x)$ satisfy assumptions (i), (ii') and (iii). Then the function $x \rightarrow f_x$ is uniformly continuous on $\mathbb{R}$ i.e., the function $\tilde{F} : \mathbb{R} \rightarrow \mathbb{R}$ defined by the equality $\tilde{F}(x) = f_x$, where $f_x = \lim_{t \rightarrow \infty} f(t, x)$, is uniformly continuous on $\mathbb{R}$.*

Proof. The proof runs in the same way as the proof of Theorem 3.1 to the moment when we obtain inequalities (3.4). Next, taking into account assumption (ii'), for a number $\frac{\varepsilon}{3}$ we can choose $\delta > 0$ such that if $x, y \in \mathbb{R}$ and $|x - y| \leq \delta$, then for an arbitrary number $t \in \mathbb{R}_+$ the following inequality is satisfied

$$|f(t, x) - f(t, y)| \leq \frac{\varepsilon}{3}. \quad (3.6)$$

Further, based on (3.4) and (3.6), for arbitrary numbers $x, y \in \mathbb{R}$ such that $|x - y| \leq \delta$ and for $t \geq T$ we obtain

$$\begin{aligned} |f_x - f_y| &\leq |f_x - f(t, x)| + |f(t, x) - f(t, y)| + |f(t, y) - f_y| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since the choice of the number T has not an effect on the validity of the inequality $|f_x - f_y| \leq \varepsilon$ which is true for all $x, y \in \mathbb{R}$ such that $|x - y| \leq \delta$.

The proof is complete. $\square$

Now, we illustrate our considerations by two simple examples.

Example 3.3. Let us consider the function $f(t, x) = f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ of the form

$$f(t, x) = a(t)b(x),$$

where the functions $a : \mathbb{R}_+ \rightarrow \mathbb{R}$, $b : \mathbb{R} \rightarrow \mathbb{R}$ satisfy the following assumptions:

(α) $a \in CC(\mathbb{R}_+)$.

(β) The function b is continuous and bounded on $\mathbb{R}$.

We show that the function f satisfies assumptions (i), (ii) and (iii). Obviously, the function f satisfies assumption (i). In order to show that the function f satisfies assumption (ii), let us fix arbitrarily a number $r > 0$. Next, choose arbitrary numbers $x, y \in [-r, r]$ and $t \in \mathbb{R}_+$. Then we obtain

$$|f(t, x) - f(t, y)| = |a(t)||b(x) - b(y)|. \quad (3.7)$$

Since $a \in CC(\mathbb{R}_+)$ then the function a is bounded on $\mathbb{R}_+$ i.e., there exists a constant $A > 0$ such that $|a(t)| \leq A$ for $t \in \mathbb{R}_+$.

Next, fix arbitrarily $\varepsilon > 0$ and choose a number $\delta > 0$ in such a way that there is satisfied assumption (β). Further, taking into account that the function b is uniformly continuous on the interval $[-r, r]$, in view of (3.7) we obtain

$$|f(t, x) - f(t, y)| \leq A|b(x) - b(y)| \leq A\varepsilon.$$

This shows that assumption (ii) is met.

To show that f satisfies assumption (iii) observe that based on assumption (β) we can indicate a constant $B > 0$ such that $|b(x)| \leq B$ for $x \in \mathbb{R}$. Further, fix arbitrarily $\varepsilon > 0$. Then, in virtue of assumption (α) we can find $T > 0$ such that for $t, s \geq T$ we have $|a(t) - a(s)| \leq \frac{\varepsilon}{B}$. Then, for arbitrary numbers $x \in \mathbb{R}$ and $t, s \geq T$ we obtain

$$|f(t, x) - f(s, x)| \leq |b(x)||a(t) - a(s)| \leq \varepsilon.$$

This proves that assumption (iii) is met.

Now, let us observe that a slight modification of assumption (β) causes that the function $f(t, x)$ satisfies assumptions (i), (ii') and (iii).

It is shown by the next example.

Example 3.4. Let $f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ be defined in the same way as previously i.e., $f(t, x) = a(t)b(x)$, where the functions $a : \mathbb{R}_+ \rightarrow \mathbb{R}$, $b : \mathbb{R} \rightarrow \mathbb{R}$ satisfy the following conditions:

(α) $a \in CC(\mathbb{R}_+)$.

(β') The function $b = b(x)$ is uniformly continuous and bounded on the set $\mathbb{R}$.

Then, similarly as in Example 3.3 we can show that the function $f(t, x)$ satisfies assumptions (i), (ii') and (iii).

In what follows we are going to investigate further properties of the superposition operator F generated by the function $f = f(t, x)$ satisfying assumptions (i)–(iii).

We have the following theorem.

Theorem 3.5. Let the function f satisfy assumptions (i)–(iii). Then, the superposition operator F generated by the function f maps the space $CC(\mathbb{R}_+)$ into itself.

Proof. Fix arbitrarily a function $x \in CC(\mathbb{R}_+)$.

In view of assumption (i) we deduce that the function $(Fx)(t) = f(t, x(t))$ is continuous on $\mathbb{R}_+$ as the composition of continuous functions. Thus, in order to prove our theorem it is sufficient to show that the function $f(t, x(t))$ is convergent to a finite limit at infinity. Applying Lemma 2.1 we see that to realize this goal it is sufficient to show that

$$\forall \varepsilon > 0 \quad \exists T > 0 \quad \forall t_1, t_2 \geq T \quad |f(t_2, x(t_2)) - f(t_1, x(t_1))| \leq \varepsilon.$$

To this end let us fix arbitrarily $\varepsilon > 0$. Since the function $t \rightarrow f(t, x)$ satisfies condition (iii) we infer that there exists $T_1 > 0$ such that for arbitrary numbers $t_1, t_2 \geq T_1$ and $x \in \mathbb{R}$ we have

$$|f(t_2, x) - f(t_1, x)| \leq \frac{\varepsilon}{2}. \quad (3.8)$$

Obviously the function x is bounded on $\mathbb{R}_+$ which implies that there exists a number $M > 0$ such that

$$\bigvee_{t \in \mathbb{R}_+} x(t) \in [-M, M].$$

In view of assumption (ii) we conclude that for any $r > 0$ the function $x \rightarrow f(t, x)$ is uniformly continuous on the interval $[-r, r]$ uniformly with respect to the variable $t \in \mathbb{R}_+$. Hence it follows that taking $r = M$, on the basis of (ii) we have that there exists a number $\delta > 0$ such that for arbitrary numbers $x, y \in [-M, M]$ and $t \in \mathbb{R}_+$ we obtain

$$|x - y| \leq \delta \Rightarrow |f(t, x) - f(t, y)| \leq \frac{\varepsilon}{2}. \quad (3.9)$$

Since the function $x = x(t)$ is convergent to a proper limit at infinity we infer that for the number δ from condition (3.9) there exists a number $T_2 > 0$ such that for arbitrary $t_1, t_2 \geq T_2$ we have

$$|x(t_2) - x(t_1)| \leq \delta.$$

Now, taking $T = \max \{T_1, T_2\}$ and applying (3.8) and (3.9), for arbitrary numbers $t_1, t_2 \geq T$ we obtain

$$\begin{aligned} |f(t_2, x(t_2)) - f(t_1, x(t_1))| &\leq |f(t_2, x(t_2)) - f(t_2, x(t_1))| \\ &\quad + |f(t_2, x(t_1)) - f(t_1, x(t_1))| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus the proof is complete. $\square$

Our next result is devoted to the continuity of the superposition operator F .

Theorem 3.6. *Let the function $f(t, x) = f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy assumptions (i)–(iii). Then the superposition operator F generated by the function f acts continuously from the space $CC(\mathbb{R}_+)$ into itself.*

Proof. Fix a function $x_0 \in CC(\mathbb{R}_+)$ and take a sequence $(x_n) \subset CC(\mathbb{R}_+)$ converging to x_0 in the space $CC(\mathbb{R}_+)$. This implies that for an arbitrary $t \in \mathbb{R}_+$ the real sequence $(x_n(t))$ is convergent to the limit $x_0(t)$. Thus, for an arbitrarily fixed number $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|x_n(t) - x_0(t)| \leq \varepsilon$ for $n \geq n_0, n \in \mathbb{N}$. Taking, for example, $\varepsilon = 1$ we can find $n_0 \in \mathbb{N}$ such that

$$|x_n(t) - x_0(t)| \leq 1 \quad (3.10)$$

for $n \geq n_0$. Since $x_0 \in CC(\mathbb{R}_+)$ we infer that there exists a finite limit $a = \lim_{t \rightarrow \infty} x_0(t)$. Hence it follows that we can find a number $T > 0$ such that for $t \geq T$ we have

$$|x_0(t) - a| \leq 1.$$

Utilizing the above inequality and (3.10), for $n \geq n_0$ and $t \geq T$ we obtain

$$|x_n(t) - a| \leq |x_n(t) - x_0(t)| + |x_0(t) - a| \leq 2.$$

This implies that $x_n(t) \in [a - 2, a + 2]$ for $n \geq n_0$ and $t \geq T$.

Further, let us notice that in view of the fact that the function x_0 is an element of the space $CC(\mathbb{R}_+)$ we can find a constant $M > 0$ such that $|x_0(t)| \leq M$ for $t \in \mathbb{R}_+$. Hence, in virtue of (3.10), we get

$$|x_n(t)| \leq 1 + |x_0(t)| \leq 1 + M$$

for $n \geq n_0$ and for $t \in \mathbb{R}_+$.

Next, let us observe that since the functions $x_1, x_2, \dots, x_{n_0-1}$ are bounded, thus we can find a number $P > 0$ which creates the common bound of the functions x_i for $i = 1, 2, \dots, n_0 - 1$. More precisely, let us put

$$P = \max \left\{ p > 0 : \bigvee_{t \in \mathbb{R}_+} |x_i(t)| \leq p \text{ for } i = 1, 2, \dots, n_0 - 1 \right\}.$$

Next, let us denote $r = \max \{|a + 2|, |a - 2|, 1 + M, P\}$. Then we have that $x_n(t) \in [-r, r]$ for $n \in \mathbb{N}, t \in \mathbb{R}_+$ and $x_0(t) \in [-r, r]$ for $t \in \mathbb{R}_+$.

In what follows fix an arbitrary number $\varepsilon > 0$ and choose $\delta > 0$ according to assumption (ii) with the above defined number $r > 0$. Since the sequence (x_n) is convergent to x_0 in the space $CC(\mathbb{R}_+)$, there exists a number $n_1 \in \mathbb{N}$ such that for $n \geq n_1, n \in \mathbb{N}$, we have

$$\|x_n - x_0\|_\infty \leq \delta.$$

Obviously, this yields that

$$|x_n(t) - x_0(t)| \leq \delta$$

for $t \in \mathbb{R}_+$ and $n \geq n_1$. Thus, applying assumption (ii) we obtain

$$|f(t, x_n(t)) - f(t, x_0(t))| \leq \varepsilon \quad (3.11)$$

for $n \geq n_1$ and for $t \in \mathbb{R}_+$. On the other hand we have

$$\|Fx_n - Fx_0\|_\infty = \sup \{ |(Fx_n)(t) - (Fx_0)(t)| : t \in \mathbb{R}_+ \}. \quad (3.12)$$

Finally, joining (3.11) and (3.12) we conclude that $\|Fx_n - Fx_0\|_\infty \leq \varepsilon$ for $n \geq n_1$ and the proof is complete. $\square$

Further on, let us notice that both Theorem 3.5 and Theorem 3.6 remain true if we replace assumption (ii) by the stronger assumption (ii'). Obviously, assumptions (i) and (iii) remain the same.

In the sequel of this section we prove that the image of a subset of the space $CC(\mathbb{R}_+)$, which is locally equicontinuous on the interval $\mathbb{R}_+$ (cf. Definition 2.2), by the superposition operator F is also locally equicontinuous on $\mathbb{R}_+$.

Theorem 3.7. Assume that the function $f(t, x) = f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies assumptions (i)–(iii). Let X be a subset of the space $CC(\mathbb{R}_+)$ which is bounded and locally equicontinuous on $\mathbb{R}_+$. Then the image $F(X)$ of the set X by the superposition operator F generated by the function $f(t, x)$ is locally equicontinuous on $\mathbb{R}_+$.

Proof. Since the set X is bounded in the space $CC(\mathbb{R}_+)$, there exists a constant $M > 0$ such that $|x(t)| \leq M$ for each $x \in X$ and for any $t \in \mathbb{R}_+$. Keeping in mind the fact that X is locally equicontinuous on $\mathbb{R}_+$, on the basis of Definition 2.2 we infer that the following condition holds

$$\bigvee_{\varepsilon > 0} \bigvee_{T > 0} \exists \delta > 0 \bigvee_{t, s \in [0, T]} \bigvee_{x \in X} [|t - s| \leq \delta \Rightarrow |x(t) - x(s)| \leq \varepsilon]. \quad (3.13)$$

In what follows let us fix numbers $\varepsilon > 0$ and $T > 0$. Next, choose a number $\delta > 0$ according to condition (3.13). Then, for arbitrary $t, s \in [0, T]$ such that $|t - s| \leq \delta$ and for arbitrary $x \in X$ we get

$$\begin{aligned} |(Fx)(t) - (Fx)(s)| &= |f(t, x(t)) - f(s, x(s))| \\ &\leq |f(t, x(t)) - f(t, x(s))| + |f(t, x(s)) - f(s, x(s))|. \end{aligned} \quad (3.14)$$

Further, let us observe that in view of our assumptions, for any $t \in \mathbb{R}_+$ we have that $-M \leq x(t) \leq M$, $-M \leq y(t) \leq M$ for arbitrary functions $x, y \in X$.

Now, let us define the function $\omega(\varepsilon) = \omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by putting

$$\omega(\varepsilon) = \sup \{ |f(t, x) - f(t, y)| : t \in \mathbb{R}_+, x, y \in [-M, M], |x - y| \leq \varepsilon \}.$$

Then, taking into account the assumptions imposed on the set X we have that $|x(t) - x(s)| \leq \varepsilon$ for $t, s \in [0, T]$, $|t - s| \leq \delta$. This implies the estimate

$$|f(t, x(t)) - f(t, x(s))| \leq \omega(\varepsilon).$$

Next, consider the function $\bar{\omega}_T : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ defined as follows

$$\bar{\omega}_T(\delta) = \sup \{ |f(t, x) - f(s, x)| : t, s \in [0, T], |t - s| \leq \delta, x \in [-M, M] \}.$$

Notice that in view of assumption (i) we infer that the function $f = f(t, x)$ is uniformly continuous on the set $[0, T] \times [-M, M]$. This implies that $\bar{\omega}_T(\delta) \rightarrow 0$ as $\delta \rightarrow 0$.

In what follows observe that from (3.14) we infer that for an arbitrary function $x \in X$ and for arbitrarily fixed $T > 0$, $\varepsilon > 0$ as well as for arbitrary numbers $t, s \in [0, T]$ such that $|t - s| \leq \delta$ (where δ is chosen according to (3.13)), we obtain

$$|(Fx)(t) - (Fx)(s)| \leq \omega(\varepsilon) + \bar{\omega}_T(\delta). \quad (3.15)$$

Taking into account assumption (ii) we conclude that $\omega(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Obviously, in this case we have that also $\delta \rightarrow 0$. Combining this fact with (3.15) we infer that the set $F(X)$ is locally equicontinuous on $\mathbb{R}_+$.

The proof is complete. $\square$

Our next result is closely related to the uniform Cauchy condition at infinity. That concept was introduced previously in Definition 2.3.

Theorem 3.8. *Let $f(t, x)$ satisfy assumptions (i)–(iii). Moreover, assume that X is a bounded subset of the space $CC(\mathbb{R}_+)$ such that functions of the set X satisfy uniformly the Cauchy condition at infinity. Then functions from the image $F(X)$ of the set X by the superposition operator F generated by the function $f(t, x)$ satisfy also uniformly the Cauchy condition at infinity.*

Proof. Since the set X is bounded in the space $CC(\mathbb{R}_+)$, there exists a constant $M > 0$ such that $|x(t)| \leq M$ for each $x \in X$ and for any $t \in \mathbb{R}_+$.

Now, fix arbitrarily $\varepsilon > 0$ and choose a number $T_1 > 0$ to the number $\frac{\varepsilon}{2}$ according to assumption (iii). Then, for an arbitrary function $x = x(u)$ belonging to the set X and for $t, s \geq T_1$ we infer that the following inequality is satisfied

$$|f(t, x(u)) - f(s, x(u))| \leq \frac{\varepsilon}{2} \quad (3.16)$$

for any $u \in \mathbb{R}_+$.

Further, fix an arbitrary function $x \in X$ and take arbitrary numbers t, s such that $t, s \geq T_1$. Then, we have

$$\begin{aligned} |(Fx)(t) - (Fx)(s)| &= |f(t, x(t)) - f(s, x(s))| \\ &\leq |f(t, x(t)) - f(t, x(s))| + |f(t, x(s)) - f(s, x(s))|. \end{aligned} \quad (3.17)$$

In view of (3.16) we obtain

$$|f(t, x(s)) - f(s, x(s))| \leq \frac{\varepsilon}{2}. \quad (3.18)$$

In what follows we will assume additionally that functions of the set X satisfy uniformly the Cauchy condition at infinity. This allows us to infer that for an arbitrary number $\delta_1 > 0$ we can choose a number $T_2 > 0$ such that for $x \in X$ and for arbitrary numbers $t, s \geq T_2$ we have the following inequality

$$|x(t) - x(s)| \leq \delta_1. \quad (3.19)$$

Next, utilizing assumption (ii), we choose a number $\delta_2 > 0$ to the number $\frac{\varepsilon}{2}$ in such a way that the following implication is true

$$|x - y| \leq \delta_2 \Rightarrow |f(t, x) - f(t, y)| \leq \frac{\varepsilon}{2}. \quad (3.20)$$

Now, take the number $\delta = \min \{ \delta_1, \delta_2 \}$. Then, on the basis of (3.19) and (3.20) we obtain

$$|f(t, x(t)) - f(t, x(s))| \leq \frac{\varepsilon}{2}. \quad (3.21)$$

Further, take the number $T = \max \{T_1, T_2\}$. Then, in view of (3.21) for $t, s \geq T$ we get

$$|f(t, x(t)) - f(t, x(s))| \leq \frac{\varepsilon}{2}. \quad (3.22)$$

Finally, keeping in mind (3.17), (3.18) and (3.22), for $t, s \geq T$ we derive the following estimate

$$|(Fx)(t) - (Fx)(s)| \leq \varepsilon.$$

This shows that functions of the set $F(X)$ satisfy uniformly the Cauchy condition at infinity.

The proof is complete. $\square$

In what follows linking the results contained in Theorems 3.7 and 3.8 and taking into account Theorem 2.4 we obtain the main result of this section.

Theorem 3.9. *Assume that the function $f(t, x) = f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies assumptions (i)–(iii). Then the superposition operator F generated by the function f transforms an arbitrary subset of the space $CC(\mathbb{R}_+)$ being relatively compact in the sense of the condition (A) into a set which is also relatively compact in the sense of the condition (A).*

Remark 3.10. The meaning of the relative compactness in the sense of the condition (A) is precisely explained in Theorem 2.4 and after Theorem 2.4.

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