Research Article

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Time-Optimal Spacecraft Reorientation for the Observation of Multiple Asteroids

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Abstract: To observe multiple asteroids in a short time, the time-optimal reorientation is investigated here for the rest-to-rest reorientation of a generic rigid spacecraft. First, the problem is formulated and solved using the *hp*-adaptive pseudospectral method. It is found that there exist not merely bang-bang but also singular solutions for the problem. Then, the bang-bang and singular solutions are discussed by various cases. The results reveal that the optimal solution is essentially a balance between the larger torque, the shorter angular path and the less moment of inertia. For bang-bang solutions, the total number of switches varies with the reorientation angle and the moment of inertia. The number of switches is usually 5 or 6 and does not exceed 8. For singular solutions, either one and two controls singular are possible. The singular optimal solution is probably optimal for the slender rod-like spacecraft. Finally, an analytical estimation method is proposed for determining the range of the optimal time. The results of the simulations indicate that the proposed method is real-time and highly accurate. All formulas of this paper are derived in canonical units and therefore apply to any rigid spacecraft.

Keywords: three-axis reorientation, bang-bang control, singular control, analytical range

1 Introduction

Time-optimal reorientation of a spacecraft is an open and interesting problem for the observation of multiple asteroids. Previously, studies about asteroids mainly focused on an individual asteroid, such as 25143 Itokawa (Binzel *et al.* 2001), 433 Eros (Jiang and Baoyin 2016), 216 Kleopatra (Yu and Baoyin 2012; Jiang 2015). Consequently, the early exploration missions mainly aimed at a single asteroid. For instance, the Hayabusa spacecraft was specially designed to explore the asteroid 25143 Itokawa (Fujiwara *et al.* 2006). Recently, scientists have turned more and more attention to the research of a class of asteroids, such as near-Earth asteroids (Zeng *et al.* 2014; Bao *et al.* 2015), Trojan asteroids (Liu and Schmidt 2018), and so on. The research objects are multiple asteroids. The corresponding exploration missions have also been proposed. For example, projects SODA (Shugarov *et al.* 2018) and Universat-SOCRAT (Sadovnichiy

et al. 2018) have been planned to monitor near-Earth asteroids, and the project Lucy has been chosen to visit Jupiter's Trojan asteroids (A'Hearn 2017). These projects about multiple asteroids are essential and promising. However, when the spacecraft has to observe multiple asteroids in a short time, the real challenge is to reorientate the spacecraft as quickly as possible.

The goal of this paper is to investigate the time-optimal reorientation for a generic rigid spacecraft. During the mission for multiple asteroids investigation, the spacecraft need to observe two or more asteroids in a fix position, or in a short time, which can be considered to be approximately the same position in the process of observation. For an inertially symmetric rigid body, the solution structure and the overall nature have been well-researched in previous studies (Bilimoria and Wie 1993; Bai and Junkins 2009). But for an asymmetric body, the time-optimal solution has still remained an open problem for a long time. Due to its strong nonlinearity, no rigorous analytical solution has been found so far. It motivates us to answer a series of fundamental questions. What is the time-optimal solution for the asymmetric body? How can we get it? What is the structure of the optimal solution? Why? If we cannot obtain an analytical solution, can we have an analytical range? This paper is an attempt to answer these questions.

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The *hp*-adaptive pseudospectral method is chosen here to obtain the optimal solution. For the time-optimal reorientation of the symmetric rigid body, (Bilimoria and Wie 1993) used the indirect method (Jiang *et al.* 2012; Zeng and Liu 2017), which is to solve a two-point boundary value problem by a shooting algorithm. This method is mature enough, but the convergence radius is very small and the shooting process is extremely sensitive to the initial guess. To enlarge the convergence radius of the indirect method, Bai and Junkins (2009) applied a hybrid approach to solve the optimal solution. Their results shown this method is available. Similarly, to have a large convergence radius, the global pseudospectral method is extensively applied to the optimization problem in recent years (Li 2017). This method overcomes the problem of the initial value. But sometimes it is extremely slow for the use of a large degree polynomials. In order to improve the rapidity and accuracy, the *hp*-adaptive pseudospectral method (Darby *et al.* 2011) is chosen for the present problem, which combines the advantages of the pseudospectral method and *hp*-adaptive mesh refinement algorithm. In this method, a new result shows that both bang-bang and singular solutions exist for the problem.

Bang-bang solutions of the asymmetric body are distinguished from those of the symmetric body. For the inertially symmetric rigid body, Bilimoria and Wie (1993) found that the optimal solution was usually bang-bang in all three control components. Furthermore, their results indicated that the optimal control involved 7 switches and 5 switches for the reorientation angle intervals $[1, 72]$ deg and $[73, 180]$ deg, respectively. In the study by Bai and Junkins (2009), the 5-switch time-optimal solution was confirmed for reorientation angles greater than 72 deg. Meanwhile, they first reported that the 6-switch solution was time-optimal for the angles $[1, 72]$ deg. Recently, (Li 2017) used pseudospectral method to solved the problem again, and the results were consistent with Bai and Junkins'. On the contrary, no similar conclusions have been given for the asymmetric body. Even though Byers and Vadali (1993) assumed that time-optimal solution involved 5 switches, they had pointed out that the 5-switch solution is not always optimal. In this paper, we find that the total number of switches is usually 5 or 6. In general, the number of switches does not exceed 8.

To simplify the problem, researchers usually assumed that there isn't singular arcs for the problem (Li 2016). That is to neglect the singular optimal solution. The so-called singular arc means that the corresponding switching function is identically zero over a finite time interval. For the symmetric rigid body, the possibility of the singular optimal solution has been investigated and eliminated (Yin *et al.* 2018a,b). However, for the asymmetric body, the singular

solution has not been completely precluded so far. In this paper, we find an interesting result that there is indeed a singular optimal solution for some special cases. The singular solution is dissected in detail. We believe that the optimal solution is essentially a balance between the larger torque, the shorter angular path and the less moment of inertia.

An analytical range is also derived for engineering usage. Since no rigorous analytical solution is available, an accurate analytical range becomes necessary and practical. For example, when the available time is less than the lower bound, it is not necessary to solve the reorientation problem further. The reorientation task is apparently impossible to accomplish. Conversely, when the available time is more than the upper bound, it implies that there is great potential for optimizing the problem. Especially for observations of multiple asteroids, the time range contributes to scheduling the observations rapidly. In addition, it can be used to guess the initial value for the numerical optimization algorithms. Fleming *et al.* (2010) posed a simple estimation method for the three-axis reorientation. Their method is feasible and represents a good beginning for estimating the time range. Based on their method, we proposed a revised method for a more accurate estimation.

The present paper is organized as follows. Section 2 describes the reorientation problem for the asymmetric rigid spacecraft. Section 3 introduces the *hp*-adaptive pseudospectral method for solving the problem. Section 4 presents the bang-bang solutions and analyzes the time-optimal control, followed by Section 5 investigating the singular optimal solution. Then, Section 6 proposes an analytical range for the problem. Finally, the conclusions are drawn in Section 7.

2 Reorientation Problem Formulation

2.1 Problem Definition

Without loss of generality, we consider a rest-to-rest maneuver for an asymmetric rigid spacecraft. We define two coordinate frames here, O-xyz and O-XYZ. The x-, y-, and z-axes correspond to the principal axes of the moments of inertia. The X-, Y-, and Z-axes represent the three inertial axes which are coincident with x-, y-, and z-axes at the initial moment. Figure 1 shows the nondimensional angular velocity components $\omega_1, \omega_2, \omega_3$ and control components u_1, u_2, u_3 along with the three body principal axes (x, y, z). The Euler's rotational equation for the spacecraft can

Figure 1. Illustration of the control torques and angular velocity components.

easily be derived as

$$\begin{aligned} I_1 \dot{\omega}_1 - (I_2 - I_3) \omega_2 \omega_3 &= u_1 \\ I_2 \dot{\omega}_2 - (I_3 - I_1) \omega_3 \omega_1 &= u_2 \\ I_3 \dot{\omega}_3 - (I_1 - I_2) \omega_1 \omega_2 &= u_3 \end{aligned} \quad (1)$$

where I_1 , I_2 and I_3 are the moments of inertia about the x -, y -, and z - principal axes, respectively. It should be noticed that the moments of inertia satisfy the inequality constraints

$$\begin{aligned} I_1 + I_2 &\geq I_3 \\ I_1 + I_3 &\geq I_2 \\ I_2 + I_3 &\geq I_1 \end{aligned} \quad (2)$$

The normalized control components (Boyarko *et al.* 2011) are constrained such that

$$-1 \leq u_i \leq 1 \quad i = 1, 2, 3 \quad (3)$$

This constraint implies that the admissible control vector lies within a unit cube. Using quaternion $\mathbf{q} = [q_1, q_2, q_3, q_4]$, the kinematic equation can be described as

$$\dot{\mathbf{q}} = \frac{1}{2} \begin{bmatrix} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \frac{1}{2} \boldsymbol{\Omega} \mathbf{q} \quad (4)$$

where q_4 is the scalar part of the quaternion, and

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1 \quad (5)$$

The problem we study is to determine the controls which will drive the spacecraft from its initial conditions $\mathbf{q}_0, \boldsymbol{\omega}_0$, which points to one asteroid, to its final conditions $\mathbf{q}_f, \boldsymbol{\omega}_f$, which points to another asteroid, while minimizing the cost function:

$$J = \int_{t_0}^{t_f} dt \quad (6)$$

Like the previous research (Bilimoria and Wie 1993), we consider a reorientation about the inertial Z axis. The boundary conditions for the quaternion and the angular velocity vector are

$$\mathbf{q}_0 = [0 \quad 0 \quad 0 \quad 1]^T \quad (7)$$

$$\mathbf{q}_f = [0 \quad 0 \quad \sin \frac{\varphi}{2} \quad \cos \frac{\varphi}{2}]^T \quad (8)$$

$$\boldsymbol{\omega}_0 = \boldsymbol{\omega}_f = [0 \quad 0 \quad 0]^T \quad (9)$$

In Eq. (8), φ is the eigenaxis rotation angle. It also represents the reorientation angle here. We should emphasize that the optimal maneuver is not constrained to rotate around the inertial Z axis.

2.2 Optimal Controls

Let $\boldsymbol{\lambda}_\omega = [\lambda_{q1} \quad \lambda_{q2} \quad \lambda_{q3} \quad \lambda_{q4}]^T$ denote the adjoint vectors associated with $\boldsymbol{\omega}$ and $\mathbf{q}$, respectively. The Hamiltonian for minimizing the performance index is given by

$$H = 1 + \boldsymbol{\lambda}_\omega^T \dot{\boldsymbol{\omega}} + \boldsymbol{\lambda}_q^T \dot{\mathbf{q}} \quad (10)$$

Then the adjoint equations are obtained as

$$\dot{\boldsymbol{\lambda}}_\omega = -\frac{\partial H}{\partial \boldsymbol{\omega}} = -\frac{1}{2} \mathbf{M} \boldsymbol{\lambda}_q - \mathbf{N} \boldsymbol{\lambda}_\omega \quad (11)$$

$$\dot{\boldsymbol{\lambda}}_q = -\frac{\partial H}{\partial \mathbf{q}} = \frac{1}{2} \boldsymbol{\Omega} \boldsymbol{\lambda}_q \quad (12)$$

$$\mathbf{M} = \begin{bmatrix} q_4 & q_3 & -q_2 & -q_1 \\ -q_3 & q_4 & q_1 & -q_2 \\ q_2 & -q_1 & q_4 & -q_3 \end{bmatrix}$$

$$\mathbf{N} = \begin{bmatrix} 0 & \frac{(I_3 - I_1)\omega_3}{I_2} & \frac{(I_1 - I_2)\omega_2}{I_3} \\ \frac{(I_2 - I_3)\omega_3}{I_1} & 0 & \frac{(I_1 - I_2)\omega_1}{I_3} \\ \frac{(I_2 - I_3)\omega_2}{I_1} & \frac{(I_3 - I_1)\omega_1}{I_2} & 0 \end{bmatrix}$$

For the problem under consideration, the Hamiltonian is a constant of the motion. From the transversality condition, we have

$$H \equiv 0 \quad (13)$$

Note that all three controls appear linearly in the Hamiltonian, and $I_i > 0$. Hence, the optimal control law from Pontryagin's principle has the form:

$$u_i = \begin{cases} +1, & \text{if } \lambda_{\omega i} < 0 \\ -1, & \text{if } \lambda_{\omega i} > 0 \\ u_{is}, & \text{if } \lambda_{\omega i} \equiv 0 \end{cases} \quad i = 1, 2, 3 \quad (14)$$

where u_{is} represents a singular control component. Singular control means the associated switching function $\lambda_{\omega i}$ is identically zero over a finite time interval. For the optimal control, there are four cases possible at each given instant of time:

- 1) Bang-bang control without singular subarcs.
- 2) One control singular.
- 3) Two controls singular.
- 4) Three controls singular.

The “purely” bang-bang control has been reported in detail in previous research (Bilimoria and Wie 1991). However, the possibility of singular control does not get much attention in the literature. The following discussion shows that the case 4 is non-optimal. It implies that at least one control component must be saturated at any instant of time.

Using Eqs. (4) and (11), it can be verified that

$$\lambda_q^T \dot{q} = -\dot{\lambda}_\omega^T \omega - \lambda_\omega^T N^T \omega \quad (15)$$

Substituting Eq. (15) into Eq. (10), an expression can be obtained for the Hamiltonian.

$$H = 1 + \lambda_\omega^T (\dot{\omega} + N^T \omega) - \dot{\lambda}_\omega^T \omega \quad (16)$$

If all controls are singular on some nonzero time interval, then we obtain

$$\lambda_\omega = \dot{\lambda}_\omega = 0 \quad (17)$$

Thus, we can neglect the latter terms in Eq. (16) to yield $H = 1$. This contradicts Eq. (13) which requires the Hamiltonian must be zero along the total trajectory. Consequently, this control structure is precluded for time-optimal reorientation. Note that this conclusion remains valid for any set of boundary conditions, and for any set of inertias.

It is useful if we can dismiss the possibility of one or two controls singular, but a rigorous proof is not available so far. In sharp contrast, we discover a singular optimal solution in some cases, which has not been reported previously. It will be discussed further in the section 5.

3 Solving the Problem Using *hp*-Adaptive Pseudospectral Method

As discussed above, the possibility of singular conditions cannot be completely precluded. It indicates that we may miss the singular solution if using a conventional indirect method, since a bang-bang solution is usually assumed beforehand. In addition, the solution is difficult to obtain because of the small convergence domain. To avoid missing the singular optimal solution and to enhance the convergence towards the solution, we accomplish this task using the *hp*-adaptive pseudospectral method (Darby *et al.* 2011). It combines the advantages of the pseudospectral method and the *hp*-adaptive mesh refinement algorithm. Moreover, the optimality of the solution can be checked by the Karush-Kuhn-Tucker (KKT) multipliers. This is how the method is currently being used in recent research (Yu *et al.* 2014).

We summarize the procedure of solving this problem as Figure 2.

Firstly, initialize the problem with a global pseudospectral discretization, and solve the nonlinear programming problem (NLP) with the prescribed grid distribution. The NLP solution can be obtained by using the optimization software SNOPT (Gill *et al.* 2005).

Secondly, check each segment whether the constraints are satisfied to the tolerance at the midpoints of the collocation points. The segments should be updated until the required tolerance is reached.

Finally, according to the Pontryagin's principles, verify the optimality of the solution by an application of the costates. The accurate costates can be obtained from (KKT) multipliers.

This three-step procedure has been implemented in the software package GPOPS (Patterson and Rao 2014). The GPOPS package is open source and free availability. Thus, it was chosen for the present work. It should be noted that we have replicated the previous results (Bilimoria and Wie 1993; Bai and Junkins 2009) by using GPOPS. Extensive simulations illustrate that this package is efficient to generate an accurately converged result. It works for both bang-bang and singular controls. As mentioned before, its key idea is to convert the original optimal control problem into a NLP based on a *hp*-adaptive mesh refinement algorithm. The NLP is then solved using numerical optimization techniques. For the problem under consideration, after doing a lot of simulations, we find that both the two types of solution exist: bang-bang and singular controls.

Figure 2. Block diagram of solving this problem using the hp-adaptive pseudospectral method.

Table 1. The optimal solutions for some general cases

Case	I_1	I_2	I_3	Eigenaxis	Optimal	Switches	Improvement, %
Case 1	1.2	1	0.8	3.1707	3.0221	6	4.69%
Case 2	1.2	1.2	0.8	3.1707	3.0642	5	3.36%
Case 3	1.2	0.8	0.8	3.1707	2.9528	6	6.87%
Case 4	1.2	0.8	1	3.5449	3.2174	6	9.24%
Case 5	0.8	1.2	1	3.5449	3.2174	6	9.24%
Case 6	1	0.8	1.2	3.8833	3.3156	5	14.62%
Case 7	1	1	1.2	3.8833	3.4026	5	12.38%
Case 8	1	1.2	1.2	3.8833	3.4981	5	9.92%

4 Bang-Bang Control Solution

4.1 Numerical Simulation for General Cases

Similar to the previous work (Bilimoria and Wie 1993), we firstly focus on a 180-deg reorientation (*i.e.*, we set $\varphi = 180$ deg). That is to find the optimal solution for 180-deg reorientation as fast as possible. Different inertia weights are considered to investigate the structure of the optimal solu-

tions. The solutions are obtained by using the *hp*-Adaptive Pseudospectral Method. We also calculate the eigenaxis rotation time to compare with the optimal results. Table 1 summarizes the results for some general cases. The 5th column represents the minimum time if performing an eigenaxis rotation. The 6th column is the dimensionless time for the optimal solutions. The 7th column shows the total number of switches for the optimal solutions, and the last column presents the improvement with respect to the eigenaxis rotation time. Consider the boundary conditions in

Eqs. (7) and (8), the eigenaxis rotation of this problem is the rotation about the principal axes z . Thus, we classify the cases by I_3 . In the Cases 1, 2 and 3, I_3 is the minimum component among the moments of inertia. Next, I_3 is presented as the middle item in the Cases 4 and 5. It is clear that the results are the same for these two cases, because the problem remains essentially unchanged when I_1 and I_2 are interchanged. In other cases, something similar happens when interchanging I_1 and I_2 . This implies that, if I_3 is determined, it is not needed to consider all the permutations between I_1 and I_2 . Finally, the situations are discussed where I_3 is the maximum component. It should be noted that all the moments of inertia are constrained to the inequality constraints in Eq. (2).

From extensive numerical simulations, we find that in most cases, all control components show a bang-bang type structure. The optimal solutions for the Cases of Table 1 are all bang-bang controls. There is no singular arcs in the control components. Note that this does not imply that the possibility of singular arcs is dismissed for all such problems. Without loss of generality, we choose Case 1 to plot the optimal control histories and switching functions in Figure 3. All the control components are saturated along the optimal path. In the solution presented, the control components have 6 switches in all. u_1 has two switches. u_2 has three switches, and u_3 has a single switch. The switches are consistent with the signs of the switching function. That means the solution satisfies the necessary conditions for optimality. The corresponding nondimensional angular velocity and quaternion histories are shown in solid lines in Figures 4 and 5. To verify their reliability further, we propagate the state of the spacecraft again by using the Runge-Kutta solver, which is provided by the function `ode45` in MATLAB. To guarantee the accuracy of the propagated results, the tolerance should be enough for the Runge-Kutta solver. The propagated results are shown as circled markers in Figures 4 and 5. They lie on the solid lines exactly. It illustrates that the control solution does drive the spacecraft from the initial orientation to the final orientation. The feasibility of the *hp*-Adaptive Pseudospectral Method is demonstrated again. The other Cases of Table 1 exhibit the similar results and phenomenon. For the purpose of brevity, their solutions are not displayed.

4.2 Analyzing the Time-Optimal Solution

From Figures 3 and 4, it is clear that the time-optimal solution is not a eigenaxis rotation maneuver. The three control components always work. The optimized solution takes a less maneuver time than a simple eigenaxis rotation. It pro-

Figure 3. The optimal control histories and switching functions for Case 1.

Figure 4. Nondimensional angular velocity histories and the validation by propagation for Case 1.

vides about 4.69% improvement for Case 1. In other Cases of Table 1, the eigenaxis rotation is not time-optimal either. The improvement is even up to 14.62% for Case 6. Note that it does not imply the eigenaxis rotation cannot be an optimal result.

Why is the optimized solution faster than the classical eigenaxis maneuver? In Figure 4, the optimal maneuver

Figure 5. Quaternion histories and the validation by propagation for Case 1.

for Case 1 is shown in three dimensions (3D). The marker symbols “○”, “◇” and “—” represent the rotation trajectories of the body axis x , y , and z , respectively. We note that the body axis z is not static. Instead, it has an evident nutation. The maximum value of the nutation angle reaches 42.58 deg. In the research by Bilimoria and Wie (1993), the similar nutation was found for an inertially symmetric rigid spacecraft. They stated that, “its strong nutational component is able to provide more torque along the inertial Z axis and can therefore complete the desired maneuver in less time.” We attempt to explain the optimal maneuver for the asymmetric rigid body by using this viewpoint. However, it does not always work. The control component u_z along inertial Z axis for Case 1 is illustrated in Figure 7. As Bilimoria and Wie stated, there is indeed a small time interval where u_z is more than 1.0 (region A). Nevertheless, u_z is less than 1.0 over a third of the maneuver time interval (region B). In total, the region A is obviously smaller than region B. That is, the question here cannot be simply analyzed by the previous statement.

To answer the question above, more factors should be considered, including the control torque, the angular path and the moment of inertia.

Firstly, there is no doubt that the optimal solution tends to the larger control torque. The control torque is just 1.0 during the eigenaxis rotation about the axis Z . However, the maximum torque capability is up to $\sqrt{3}$ in the optimal maneuver.

Secondly, the optimal solution pursues an angular path as short as possible. That is, the optimal control is inclined to the eigenaxis, because the eigenaxis rotation provides the shortest angular path (Etter 1989; Steyn 1995). In fact, the recent research of Bai and Junkins (2009) has well sup-

Figure 6. The 3D representation of the time-optimal reorientation for Case 1.

Figure 7. The control component u_z along inertial Z axis for Case 1.

ported this most intuitive path again. They demonstrated that, for an inertially symmetric rigid spacecraft, when the control torque vector lies within a sphere, the eigenaxis maneuver is the time-optimal solution.

Finally, the optimal solution tends towards a less moment of inertia about the rotation axis. That is, the optimal control is inclined to the minimum principal axis of inertia. This point is significant but easily ignored. To verify the trend to the minimum axis, some numerical simulations are implemented. Based on the Case 1, we decreased the I_3 from 0.8 to 0.4 at a step of 0.1. The optimal solutions were also solved for a 180-deg reorientation. The nutation angle histories are presented in Figure 8. As I_3 decreased, the maximum nutation angle declined apparently. Furthermore, all the nutation angle curve decreased. This phenomenon illus-

Table 2. The total numbers of switches for various cases

Case	I_1	I_2	I_3	30°	60°	90°	120°	150°	180°
Case 1	1.2	1	0.8	6	6	5	5	5	6
Case 2	1.2	1.2	0.8	6	6	5	5	5	5
Case 3	1.2	0.8	0.8	6	6	5	7	8	6
Case 4	1.2	0.8	1	6	6	5	5	5	6
Case 5	0.8	1.2	1	6	6	5	5	5	6
Case 6	1	0.8	1.2	6	6	5	5	5	5
Case 7	1	1	1.2	6	6	5	5	5	5
Case 8	1	1.2	1.2	6	6	5	5	5	5

Figure 8. The nutation angle histories for the I_3 varying from 0.8 to 0.4.

trates that the optimal control prefers the minimum axis. At every instant, the effective moment of inertia is determined by the inherent moment of inertia I_1 , I_2 , I_3 and the unit vector about the instantaneous axis (King and Karpenko 2016). It is easily understood that the closer the rotation axis is to the minimum axis, the less the instantaneous moment of inertia is.

In sum, the optimal solution of Case 1 is essentially a balance between the larger torque, the shorter angular path and the less moment of inertia. The switches reflect the competition between these factors. The result illustrates that the switches aid the reorientation. Although the eigenaxis rotation has some advantages, the performance index benefits more from the bang-bang solution in Case 1. Actually, this explanation can be analogized from the classical single-axis rotation. From the research by King and Karpenko (2016), the minimum time for a single-axis rotation can be expressed as $\sqrt{4\theta I/\tau}$. The symbols τ , θ , and I represent the control torque, angular path and moment of inertia, respectively. We can find that the optimal performance index requires a larger τ , a shorter θ and a less I . It is consistent with the explanation for Case 1. Moreover, this

explanation is general and can be applied for all the rigid body.

4.3 The Total Number of Switches

In many optimization algorithms, the total number of switches should be known beforehand, such as the switch time optimization (STO) algorithm (Meier and Ryson 1990). Hence, it received much attention (Bilimoria and Wie 1993; Bai and Junkins 2009). For an inertially symmetric rigid spacecraft, the recent research indicated that the number of switches is a function of the reorientation angle (Li 2017). The 6-switch solution is time-optimal for the reorientation angles on $[1, 72]$, and the 5-switch solution is time-optimal on the interval $[73, 180]$ deg. Furthermore, it is of interest to verify whether the similar conclusion can be reached for the asymmetric rigid body or not.

Based on the Cases in Table 1, we study the total number of switches. The total numbers of switches for various reorientation angles are listed in Table 2. Hundreds of simulations have been carried out. For brevity, only some representative results are shown.

We firstly observed the switches for the same reorientation angle. When the reorientation angle is 30° deg, the numbers of switches are all 6 for the different moments of inertia. For the angles 60° and 90° deg, the numbers of switches are also the same for different moments of inertia. However, when the reorientation angle is larger, different switches come out. For the angle 120° and 150° deg, the number of switches in Case 3 is more than the others. For other Cases, the number of switches is 5, but the number of switches is 7 and 8 for Case 3. For the angle 180° deg, the distinction of the Cases is quite evident. Cases 1, 3, 4, and 5 have 6 switches, while Case 2, 6, 7, and 8 have 5 switches. From this, despite the same reorientation angle, we can see that the numbers of switches are not always the same for the different moments of inertia.

Then, we paid our attention to the switches of a same body. Take Case 1 for example, the time-optimal solutions

Figure 9. The total numbers of switches from 10 to 180 deg for Case 1.

were solved for the reorientation angles from 10 to 180 deg, incremented by 10 deg. The numbers of switches are shown in Figure 9. For the 10 deg angle, the optimal control has 6 switches. At first, although the angle increased, the number of switches remains the same. The 6-switch solution is time-optimal until the reorientation angle reaches 80 deg, and changes to 5-switch solution. Next, when the angle varies from 80 to 160 deg, the optimal control remains 5 switches. At last, the 6-switch solution becomes the optimal solution again, when the reorientation angle approaches 180 deg. To verify this trend and confirm the critical angles, we recalculated the switches at the step size of 1 deg. The result shows that the switch structure changes twice indeed. The critical angles are 79 and 163 deg. Compared with the inertially symmetric rigid spacecraft, the critical angle is no longer 72 deg for the asymmetric rigid body. For the inertially symmetric rigid spacecraft, the control structure changes one time, but it changes two times for the asymmetric rigid body. As seen from Table 2, there are even four changes for Case 3. In all, for the asymmetric rigid spacecraft, the pattern about the switches is distinguished from the symmetric one. For the different reorientation angles, the numbers of switches are not always the same in spite of a same body.

From the current investigation, we find that the total number of switches is usually 5 or 6. Especially for reorientation angles less than 60 deg, the time-optimal solution is almost 6 switches. Thus, when using the STO algorithm, it seems more reasonable to assume that the general optimal reorientation requires 6 switches, rather than 5 switches used by the previous researches. For the cases which have less than 6 switches, the results will have the zero intervals in time. These cases can be resolved by assuming fewer switches. In addition, the different switches exist such as

7 switches, 8 switches, but these cases are rare. Generally speaking, the number of switches does not exceed 8.

5 Singular Control Solution

5.1 Numerical Simulation for Special Cases

Does there exist a singular optimal solution for some special cases? In recent researches (Bai and Junkins 2009; Li 2017), to simplify the calculation, it is generally assumed that the singular solution is not optimal. However, a rigorous proof about the singular solution is not available so far. As noted earlier, the possibility for singular conditions has not been completely dismissed. Hence, this section focuses on the possibility of singular time optimal controls.

The optimal solutions for some special cases are listed in Table 3. The reorientation angle φ is 180 deg. Except the general cases, two types of special cases are considered here. Firstly, one of the three principal moments of inertia reaches its extremity with respect to the other two. That is, $I_1 + I_2 = I_3$ or $I_1 + I_3 = I_2$ or $I_2 + I_3 = I_1$. Note that no real body can have the moments of inertia like these. However, these configurations are researched here as the extreme cases, such as the Case 9, Case 10, etc. Secondly, two of the three principal moments of inertia reach their extremities with respect to the another one. Theoretically, the two moments of inertia are tending to infinity in spite of the constraint in Eq. (2), such as the Case 11, Case 17. Especially, it is possible that these two types of special configurations exist in a single case, such as the Case 12, Case 18. Similar to Table 1, the special cases are classified by I_3 in Table 3. I_3 is the minimum principal moment of inertia in the Cases 9-12. It is the middle one in the Cases 13 and 14, and it is the maximum one in the Cases 15-18.

As seen from Table 3, we find that in some cases the singular control is the optimal solution. The singular optimal solution is usually omitted in earlier research. However, we confirmed its existence here by numerical cases, such as the Case 11, Case 12. That means that the singular optimal solution must be considered in the optimization algorithms for the problem at hand. For example, if the indirect method is chosen to solve this problem, some techniques should be available beforehand for conducting the singular case.

5.2 Two Controls Singular

What is the rotation trajectory for the singular optimal control? Take Case 11 for example, the optimal control histories

Table 3. The optimal solutions for some special cases

Case	I_1	I_2	I_3	Eigenaxis	Optimal	Optimal rotation	Optimal control
Case 9	3	2	1	3.5449	3.4584	Non-Eigenaxis	Bang-Bang
Case 10	2	1	1	3.5449	3.3027	Non-Eigenaxis	Bang-Bang
Case 11	100	100	1	3.5449	3.5449	Eigenaxis	Singular
Case 12	101	100	1	3.5449	3.5449	Eigenaxis	Singular
Case 13	3	1	2	5.0133	4.4501	Non-Eigenaxis	Bang-Bang
Case 14	101	1	100	35.4491	29.9932	Non-Eigenaxis	Singular
Case 15	1	2	3	6.1400	4.7694	Non-Eigenaxis	Bang-Bang
Case 16	1	1	2	5.0133	3.6806	Non-Eigenaxis	Bang-Bang
Case 17	1	100	100	35.4491	29.8788	Non-Eigenaxis	Singular
Case 18	1	100	101	35.6259	29.9104	Non-Eigenaxis	Singular

Figure 10. The optimal control histories and switching functions for Case 11.

and switching functions are shown in Figure 11. u_1 and u_2 are obviously not the classical bang-bang control. They vary in the interval $[-1, 1]$. Meanwhile, the corresponding switching functions $\lambda_{\omega 1}$ and $\lambda_{\omega 2}$ are identically zero over all the time interval. According to the definition of singular control, u_1 and u_2 are both singular. That is, the time-optimal solution has two controls singular. Figure 11 presents the dimensionless angular velocity for Case 11. ω_3 has a full positive acceleration for half the total time, and then it decelerates for the remainder. However, ω_1 and ω_2 always remain zero. It is a typical eigenaxis rotation. u_1 and u_2 do

Figure 11. Nondimensional angular velocity histories for Case 11.

not aid the rotation at all. They are equivalent to zero. The three-dimensional representation of the time-optimal reorientation for Case 11 is depicted in Figure 12. The spacecraft performs an eigenaxis rotation about the inertial Z axis. The body axis z remains aligned with the inertial Z axis, and the body axes x and y trace out an arc of a circle. A more intuitive animation can be simulated by the technology of Satellite Tool Kit ActiveX (Yin *et al.* 2018a,b).

Why is the singular control the optimal solution in Case 11? Why is the eigenaxis rotation the optimal solution here? From the research by Bilimoria and Wie (1993), the eigenaxis rotation is generally not time optimal, because it cannot provide the largest torque. This conclusion is available for most cases. However, according to the analysis in Section 4, the optimal solution is influenced by multiple factors. In some special cases, the results may be very different. In Case 11, the moment of inertia about the principal axis z is far less than the others. The spacecraft has a slender rod-like body. Moreover, the eigenaxis rotation about the axis Z provides the shortest angular path. The time benefits from these two factors are considerable. Thus, the eigenaxis ro-

Figure 12. The 3D representation of the time-optimal reorientation for Case 11.

Figure 13. The optimal control histories and switching functions for Case 17.

tation, which is the singular control, becomes the optimal solution even though the control torque does not reach its extremity. The result of Case 12 is similar with Case 11. The optimal control has two singular components.

Figure 14. Nondimensional angular velocity histories for Case 17.

Figure 15. The 3D representation of the time-optimal reorientation for Case 17.

5.3 One Control Singular

The Case 11 has presented the solution with two controls being singular. Is one control singular possible? In fact, it is also confirmed in our investigation. The Case 17 of Table 3 is a representative sample. Figures 13-15 present its optimal solution. As evident from Figure 13, u_2 and u_3 are the classical bang-bang type structure. They have two switches and one switch respectively. The switching functions indicate that the Pontryagin's principle is satisfied. In sharp contrast, u_1 performs as a singular arc. The corresponding switching function λ_{w1} remains identically zero. In this sense, u_1 is a singular control component. This result verified the possibility of one control singular, which has not been discovered before.

From Figure 14, this solution is significantly different from the eigenaxis rotation. None of the angular velocity components stays at zero. Especially, ω_1 fluctuates frequently between positive and negative. It generates a non-eigenaxis rotation. The rotation of the spacecraft is intuitively illustrated in Figure 15. The body axis x traces out a smooth arc, but the trajectory of the axes x and z are irregular. A nutation is evident.

This singular optimal solution can also be explained by the balance between the torque, the angle path and the moment of inertia. Generally speaking, although the instantaneous rotation axis varies, the values of these three factors will not change in magnitude. For example, the torque capability about a principal axis is 1, and the maximum torque capability is $\sqrt{3}$. However, in Case 17, I_1 is far less than I_2 and I_3 . The difference between them is up to two orders of magnitude. That is, this special case is significantly different from the general cases. When rotating around the axis x , the instantaneous moment of inertia is smallest and little. This implies that the influence of the torque about the axis x is sensitive and powerful. It is highly useful for the time-optimal reorientation. However, the smallest moment of inertia conflicts with the shortest angle path and largest torque. To pursue for a minimum time, u_1 has to adjust its magnitude and direction frequently. That is why the singular solution is optimal for Case 17. Similar to Case 17, the optimal solutions of Cases 14 and 18 are singular. The optimal controls have one singular component.

Reviewing Table 3 again, there is a common feature between the singular Cases, including the Cases 11, 12, 14, 17 and 18. For these singular cases, there are two moments of inertia tending to infinity. In turn, one moment of inertia is close to zero relative to the other two. That means that the spacecraft is a long slender rod. For the slender rod-like body, the motion, which is around the minimum principal axis of inertia, is special and sensitive. A little control will lead to a significant angular acceleration. Thus, to benefit from this as much as possible, the singular solutions are generated. To verify this trend, some simulations are researched for the configuration of Case 11. Different final orientations are considered. As with Case 11, although the final state varies, the singular optimal solutions still exist. For brevity, the results are not shown.

By the numerical simulation above, it is demonstrated that the singular control solutions are optimal for some special cases. Although no rigorous proof has been given so far, for the slender rod-like spacecraft, the singular control solution is probably the optimal one.

6 Bounding the Optimal Solution

6.1 Classical Estimation Method

In the previous sections, the overall nature and structure have been discussed for the optimal solution. However, due to the strong nonlinearity of the problem, the analytical form of the optimal solution has still not been found so far. To determine the range of the optimal solution, the analytical expressions are derived in this section.

Referring to the optimal solution of the single-axis rotation, Fleming *et al.* (2010) proposed a simple estimation for the three-axis reorientation. Even though it is rough, it played a practical role in the previous researches. For a single-axis rotation, which is rest to rest, the minimum time can be expressed as

$$t_{\min} = \sqrt{\frac{4I\theta}{u_{\max}}} \quad (18)$$

Where $u_{\max}$, θ , and I denote the torque capability, angular path and moment of inertia, respectively. For the three-axis reorientation, the maximum torque capability becomes $\sqrt{3}u_{\max}$. The maximum principal moment of inertia is denoted as $I_{\max}$, and the minimum one is denoted as $I_{\min}$. Based on Eq. (18), Fleming *et al.* estimated the lower bound of the minimum time by using the maximum torque capability and the minimum moment of inertia. The lower bound can be described as

$$t_{\min}^L = \sqrt{\frac{4I_{\min}\theta}{\sqrt{3}u_{\max}}} \quad (19)$$

In the same way, using the minimum torque capability and the maximum moment of inertia, the upper bound of the minimum time was estimated as

$$t_{\min}^U = \sqrt{\frac{4I_{\max}\theta}{u_{\max}}} \quad (20)$$

On the one hand, the estimated lower bound is relatively accurate, because the expression of Eq. (19) is consistent with the overall nature of the optimal solution. According to our earlier analysis, the optimal solution is just to pursue the maximum torque, the shortest angle path and the minimum moment of inertia. In this sense, the optimal solution is essentially inclined to the result of Eq. (19) as close as possible. Especially, for the eigenaxis rotation of an inertially symmetric body, the result of Eq. (19) is just the optimal time.

On the other hand, due to the asymmetry of a spacecraft, the estimated upper bound is too large in many cases. To take a typical example, the optimal time is 3.5449 in

Case 11, but the estimated upper bound is 35.4491. The estimated value is ten times as much as the optimal value. The difference is up to an order of magnitude. It indicates that the accuracy cannot be guaranteed for the estimated upper bound. The classical estimation method is not available for all the asymmetric spacecraft.

6.2 Revised Estimation Method

To give a more accurate estimation, we proposed a revised method for the upper bound. As is known, a body can shift from one orientation to any other by three rotations about the principal axes. This concept actually provides a feasible solution to drive the spacecraft from its initial orientation to the final one. Based on this concept, the Euler angles are defined. There is no doubt that the maneuver time of the feasible solution is more than the optimal time. Hence, we chose the result of this feasible solution as the estimation of the upper bound.

The revised estimation method is implemented as follows. The maximum principal moment of inertia is

$$I_{\max} = \max \{I_1, I_2, I_3\} \quad (21)$$

Similarly, the minimum principal moment of inertia is given by

$$I_{\min} = \min \{I_1, I_2, I_3\} \quad (22)$$

We reorientate the spacecraft by three rotations, like the Euler angles. The three rotations are specified in the axis rotation sequence. The sequence is defined as

$$i-j-k, \quad i, j, k \in [1, 2, 3], i \neq j \neq k \quad (23)$$

where i, j, k represent the first, second and third rotation axis, respectively. The values 1, 2, 3 represent the body axes x, y, z , respectively. For example, the order 1-2-3 means that the spacecraft rotates around the axes x, y, z in sequence.

After assigning the sequence, we can obtain the corresponding rotation angles α, β, γ from

$$\begin{bmatrix} \alpha & \beta & \gamma \end{bmatrix} = \text{quat2eul} \left(\mathbf{q}_0^{-1} \otimes \mathbf{q}_f, 'i-j-k' \right) \quad (24)$$

where `quat2eul` is the function in Matlab, which is used to convert the quaternion to Euler angles. The symbol $\otimes$ denotes the quaternion multiplication operation. $\mathbf{q}_0$ and $\mathbf{q}_f$ are the initial and final quaternion. Knowing the angles α, β and γ , we can get the minimum time as the three rotations. For example, for the sequence 1-2-3, the maneuver time is

$$t_{1-2-3} = \sqrt{\frac{4I_1|\alpha|}{u_{\max}}} + \sqrt{\frac{4I_2|\beta|}{u_{\max}}} + \sqrt{\frac{4I_3|\gamma|}{u_{\max}}} \quad (25)$$

Considering all the permutations, no more than 12 permutations are possible. Hence, we can enumerate all the possible sequences. Then, we choose the minimum value as the upper bound of the optimal time, and it is given by

$$t_{\min}^U = \min_{\substack{i \neq j, j \neq k \\ i, j, k \in [1, 2, 3]}} \left\{ \sqrt{\frac{4I_i|\alpha|}{u_{\max}}} + \sqrt{\frac{4I_j|\beta|}{u_{\max}}} + \sqrt{\frac{4I_k|\gamma|}{u_{\max}}} \right\} \quad (26)$$

Consequently, the revised estimation of the minimum time is

$$t_{\min} \in \left[\sqrt{\frac{4I_{\min}\theta}{\sqrt{3}u_{\max}}}, \min_{\substack{i \neq j, j \neq k \\ i, j, k \in [1, 2, 3]}} \left\{ \sqrt{\frac{4I_i|\alpha|}{u_{\max}}} + \sqrt{\frac{4I_j|\beta|}{u_{\max}}} + \sqrt{\frac{4I_k|\gamma|}{u_{\max}}} \right\} \right] \quad (27)$$

To compare the two estimation methods, we summarized the estimated range for the Cases 1-10 in Figure 16. The problem is still a 180-deg reorientation about the inertial Z axis. For each Case, the symbol square represents the optimal time for the reorientation. The blue line (left) shows the estimation of the traditional method, and the red line (right) shows the estimation of the revised method. The bottom and top of the lines represent the estimated lower and upper bound. It implies that the shorter the line is, the more accurate the estimation is. We find in most cases that the revised method is significantly more accurate. Especially, the revised estimation provides about 75.30% improvement for Case 9.

Due to the analytical expressions, the lower and upper bound can be obtained instantly. The revised estimation method is efficient, and it is easy to understand for engineers. Moreover, it is suitable for any rest-to-rest reorientation.

Figure 16. The comparison of the traditional and revised estimation methods in Cases 1-10.

7 Conclusions

For the observation of multiple asteroids, the spacecraft needs to observe two or more asteroids in a fix position, or in a short time. So, the time-optimal reorientation problem has been researched here for an asymmetric rigid spacecraft. It is demonstrated that the optimal solution can be obtained by the *hp*-adaptive pseudospectral method. Application of this method to analyze the time-optimal solution reveals that the optimal control is essentially a balance between the larger torque, the shorter angular path and the less moment of inertia. Different from the usual assumption, there exists not merely bang-bang but also singular solutions for the problem. On the one hand, in most cases, the time-optimal solution is bang-bang. For the bang-bang solution, the total number of switches varies with the reorientation angle and the moment of inertia. We find that the total number of switches is usually 5 or 6. Generally speaking, the number of switches does not exceed 8. On the other hand, numerical simulations illustrate that the singular solution is time-optimal for some special cases. It implies that the possibility of the singular optimal solution cannot be neglected when solving the time-optimal reorientation. Although no rigorous proof has been given so far, for the slender rod-like spacecraft, the singular control solution is probably the optimal one. Finally, we proposed an analytical method for estimating the range of the minimum time. The results of the simulations indicate that the proposed method is real-time and more accurate than the classical method.

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