Research Article

Zhiqiang Zhang, Zeshan Saleem Mufti, Muhammad Faisal Nadeem*, Zaheer Ahmad, Muhammad Kamran Siddiqui, and Muhammad Reza Farahani

Computing Topological Indices for Para-Line Graphs of Anthracene

<https://doi.org/10.1515/chem-2019-0093>

Received Oct 14, 2018; accepted May 29, 2019

Abstract: Atoms displayed as vertices and bonds can be shown by edges on a molecular graph. For such graphs we can find the indices showing their bioactivity as well as their physio-chemical properties such as the molar refraction, molar volume, chromatographic behavior, heat of atomization, heat of vaporization, magnetic susceptibility, and the partition coefficient. Today, industry is flourishing because of the interdisciplinary study of different disciplines. This provides a way to understand the application of different disciplines. Chemical graph theory is a mixture of chemistry and mathematics, which plays an important role in chemical graph theory. Chemistry provides a chemical compound, and graph theory transforms this chemical compound into a molecular graph which further is studied by different aspects such as topological indices. We will investigate some indices of the line graph of the subdivided graph (para-line graph) of linear-[s] Anthracene and multiple Anthracene.

Keywords: Para-line graph; nanostructures; topological indices; linear-[s] Anthracene; multiple Anthracene

*Corresponding Author: **Muhammad Faisal Nadeem:** Department of Mathematics, COMSATS University Islamabad, Lahore Campus, 54000, Pakistan; Email: mfaisalnadeem@ymail.com

Zhiqiang Zhang: School of Information Science and Engineering, Chengdu University, Chengdu 610106, China; Email: zzq118@163.com

Zeshan Saleem Mufti: Department of Mathematics, COMSATS University Islamabad, Lahore Campus, 54000, Pakistan; Email: zeeshansaleem009@gmail.com

Zaheer Ahmad: Khwaja Fareed University of Engineering and Information Technology, Rahim Yar Khan, Pakistan; Email: zaheer@kfueit.edu.pk

Muhammad Kamran Siddiqui: Department of Mathematics, COMSATS University Islamabad, Lahore Campus, 54000, Pakistan; Email: kamransiddiqui75@gmail.com

Muhammad Reza Farahani: Department of Applied Mathematics, Iran University of Science and Technology, Tehran, Iran; Email: mrfarahani88@gmail.com

1 Introduction

Chemical graph theory is a branch of mathematical chemistry that is concerned with analyses of all consequences of a connectivity in a chemical graph. Some physical properties, e.g., breaking point, can be anticipated in view of the structure of the atoms. Numerical and computational systems are viably used to show and predict the structure at an atomic level. The structures of atoms, from a numerical perspective, are graphs. Graph theory is utilized as a part of nearly every field of science, and it is likewise utilized for training, both for recreation and design.

$V(G)$ and $E(G)$ compose the vertex set and edge set of a graph G respectively; $p, q \in V(G)$ are adjacent if p and q are end points of $u \in E(G)$ and u is an edge whose end vertices are p and q . The set of all neighbors of a vertex p represented by N_p , is called the neighborhood of p . The count of edges that occur on a vertex is called the degree of the vertex denoted by ξ_p and $S_p = \sum_{q \in N_p} \xi_q$, where $N_p = \{q \in V(G) : pq \in E(G)\}$. We can construct the line graph $L(G)$ of any graph G in such a way that the edges of the original graph will become the vertices of line graph i.e. two vertices p and q occur if and only when these vertices have a common end vertex in G . The para-line graph of G is the line graph of the subdivision of G i.e. $L(S(G))$, which will be represented as G^* . The subdivision graph is the graph attained from G by replacing each of its edges by a path of length 2. For instance, consider C_2H_6 as the hydrocarbon named Ethane which is characterized as a molecular structure. The graph of C_2H_6 is shown in Figure 1 (a) and (b), and the line graph of subdivision C_2H_6 is shown in Figure 1 (c).

Figure 1: (a) molecular construction of C_2H_6 , (b) molecular graph of C_2H_6 , (c) line graph of subdivision of C_2H_6 .

The topological index which is also referred to as the molecular descriptor, is a real number which describes the properties of a certain chemical compound. The study of topological indices on different chemical structures has been an area of research for all graph theorists. It is a bridge between mathematics and chemistry. The molecular descriptors which are separated into three groups depend on degree-based [16, 24], distance-based [8, 29] and spectrum-based [2, 18, 20, 21] indices. Some other topological indices have also been studied which are centred on both degrees and distances [4, 7, 11, 19].

The 1st general Zagreb index in [17] is the oldest and most used molecular descriptor and is defined as:

$$M_\alpha(G) = \sum_{p \in V(G)} (\xi_p)^\alpha. \quad (1)$$

The general sum-connectivity index $\chi_\alpha(G)$ is defined as [30]:

$$\chi_\alpha(G) = \sum_{pq \in E(G)} (\xi_p + \xi_q)^\alpha. \quad (2)$$

The *ABC* is specified by Estrada in [5]. The *ABC* index of graph *G* is defined as:

$$ABC(G) = \sum_{pq \in E(G)} \sqrt{\frac{\xi_p + \xi_q - 2}{\xi_p \xi_q}}. \quad (3)$$

*ABC*₄ index was presented by Ghorbani in [9] is explained as:

$$ABC_4(G) = \sum_{pq \in E(G)} \sqrt{\frac{S_p + S_q - 2}{S_p S_q}}. \quad (4)$$

*R*_α (general Randic connectivity index) of *G* is proposed as [1]:

$$R_\alpha(G) = \sum_{pq \in E(G)} (\xi_p \xi_q)^\alpha. \quad (5)$$

Where $\alpha \in \mathbb{R}$. If α is -0.5 , then *R*_{-0.5}(*G*) is called Randic connectivity index of *G*.

Vukicevic and Furtula presented the (*GA*) index in [28]. The *GA* index for graph *G* is defined as:

$$GA(G) = \sum_{pq \in E(G)} \frac{2\sqrt{\xi_p \xi_q}}{\xi_p + \xi_q}. \quad (6)$$

*GA*₅ was introduced by Graovac *et al.* in [10] is proposed as:

$$GA_5(G) = \sum_{pq \in E(G)} \frac{2\sqrt{S_p S_q}}{S_p + S_q}. \quad (7)$$

The hyper-Zagreb index is suggested as:

$$HM(G) = \sum_{pq \in E(G)} (\xi_p + \xi_q)^2 \quad (8)$$

In 2012, Ghorbani and Azimi introduced the Zagreb indices in different form as, the 1st and 2nd multiple Zagreb index *PM*₁(*G*) and *PM*₂(*G*), 1st and 2nd Zagreb polynomial *M*₁(*G*, *a*) and *M*₂(*G*, *a*) respectively, are suggested as:

$$PM_1(G) = \prod_{pq \in E(G)} (\xi_p + \xi_q) \quad (9)$$

$$PM_2(G) = \prod_{pq \in E(G)} (\xi_p \times \xi_q) \quad (10)$$

$$M_1(G, a) = \sum_{pq \in E(G)} a^{(\xi_p + \xi_q)} \quad (11)$$

$$M_2(G, a) = \sum_{pq \in E(G)} a^{(\xi_p \times \xi_q)} \quad (12)$$

2 Applications of Topological Indices

Randic observed the correlation between the Randic index and physio-chemical properties of alkane such as boiling point, enthalpies of formation, surface area and so on. The *ABC* index is a very effective index in heat formation [5]. *GA* index is a better predictive index than the Randic Index as *GA* has much chosen prophetic control on the prophetic energy of the *R*_α [3]. The 1st and 2nd Zagreb index were very useful in the calculation of the aggregate π -electron energy of the molecule [12]. These molecular descriptors were suggested for the approximation of stretched carbon-skeleton [13].

3 Topological indices of para-line graphs

Para-line graphs are an attractive field of study in chemical graph theory. Ranjini *et al.* calculated the explicit expression for the Schultz index and Zagreb index of the para-line graphs of the wheel, ladder, helm, tadpole [25, 26]. In 2015, Su *et al.* [27] investigated χ_α (general sum-connectivity index) and co-index of the above mentioned graph of wheel, tadpole. To study the para-line graphs

Figure 2: The molecular graph of linear [s]-Anthracene.

Table 1: The distribution of edges w.r.t. degree of end vertices of every edge.

Sr. No	$E_{(\xi_p, \xi_q)}$	Number of Edges
1	$E_{(2,2)}$	$6s + 10$
2	$E_{(2,3)}$	$12s - 4$
3	$E_{(3,3)}$	$30s - 16$

Table 2: The distribution of edges w.r.t. neighbor sum of every vertex the graph of Anthracene.

Sr. No	$S_{(p,q)}$	Number of Edges
1	$S_{(4,4)}$	10
2	$S_{(4,5)}$	4
3	$S_{(5,5)}$	$6s - 4$
4	$S_{(5,8)}$	$12s - 4$
5	$S_{(8,8)}$	$4s$
6	$S_{(8,9)}$	$16s - 8$
7	$S_{(9,9)}$	$10s - 8$

of ladder, tadpole and wheel, Nadeem *et al.* in [22] evaluated the ABC_4 , GA_5 index and calculated the generalized Randic index, general Zagreb index, χ_α , ABC index, GA index, ABC_4 index and GA_5 index of $TUC_4C_8[p, q]$ in [23]. Klein *et al.* [15] offered few applications and basic properties of the para-line graphs in chemical graph theory. Gutman also shed a light on the application of line graphs see [14]. Estrada showed the application of line graph in [6].

4 Results for para-line graph of linear [s]-Anthracene

In Figure 2, the graph of linear [s]-Anthracene is presented and it is denoted by T_s . T_s has $14s$ vertices and $18s - 2$ edges.

Suppose the graph of Anthracene contains three hexagon which are connected with a square expands vertically and horizontally. Let the edge $e_{(p,q)}$ represent the count of edges connecting the vertices of degree ξ_p and ξ_q . The graph of Anthracene holds the following edges shown in Table 2:

Theorem 1. Let G^* be the line graph of the subdivided graph (para-line graph) of T_s . Then

$$M_\alpha(G^*) = (12s + 8)2^{\alpha+2} + (24s - 12)3^{\alpha+1}$$

Proof. In G^* , the overall count of the vertices is $48s - 10$ which is the sum of degree two and degree three vertices $8 + 12s$, $24s - 12$ respectively. As

$$M_\alpha(G^*) = (12s + 8)2^{\alpha+2} + 3^{\alpha+1}(24s - 12).$$

Theorem 2. Let G^* be the line graph of the subdivided graph (para-line graph) of T_s . Then

- $R_\alpha(G^*) = (6s + 10)4^\alpha + (12s - 4)6^\alpha + (30s - 16)9^\alpha$;
- $\chi_\alpha(G^*) = (6s + 10)4^\alpha + (12s - 4)5^\alpha + (30s - 16)6^\alpha$;
- $ABC(G^*) = \left(9\sqrt{2} + 20\right)s + 3\sqrt{2} - \frac{32}{3}$;
- $GA(G^*) = \left(36 + \frac{24}{5}\sqrt{6}\right)s - 6 - 8/5\sqrt{6}$.

Proof. The overall count of edges of G^* is $74s - 10$. The set of edges of G^* is represented as $E(G^*)$ which is uniformly disjointed three edge sets established based on the degrees of the adjacent(end) vertices such as $E_1(G^*)$, $E_2(G^*)$ and $E_3(G^*)$. The first set $E_1(G^*)$ contains the edges such as $E_{(2,2)} = 6s + 10$, the second set $E_2(G^*)$ holds the edges $E_{(2,3)} = 12s - 4$, and the third set $E_3(G^*)$ includes the edges $E_{(3,3)} = 30s - 16$, where (2, 2), (2, 3) and (3, 3) represent the degree of end vertices respectively. From formulas (5), (2), (3) and (6) and using Table 1, the result is proved.

Theorem 3. Let G^* be the line graph of the subdivided graph (para-line graph) of T_s . Then

$$\begin{aligned}
 ABC_4(G^*) &= \left(\frac{3}{5}\sqrt{110} + \frac{12}{5}\sqrt{2} + \frac{4}{3}\sqrt{30} + \frac{40}{9}\right. \\
 &\quad \left. + \frac{1}{2}\sqrt{14}\right)s + \frac{5}{2}\sqrt{6} + \frac{2}{5}\sqrt{35} - \frac{8}{5}\sqrt{2} - \frac{2}{3}\sqrt{30} \\
 &\quad - \frac{1}{5}\sqrt{110} - \frac{32}{9} \\
 GA_5(G^*) &= \left(20 + \frac{48}{13}\sqrt{10} + \frac{192}{17}\sqrt{2}\right)s - 2 + \frac{16}{9}\sqrt{5} \\
 &\quad - \frac{16}{13}\sqrt{10} - \frac{96}{17}\sqrt{2}
 \end{aligned}$$

Figure 3: The Para-Line graph of linear [s]-Anthracene.

Proof. In Table 3, we have

$$\begin{aligned}
 ABC_4(G) &= \sum_{pq \in E(G)} \sqrt{\frac{S_p + S_q - 2}{S_p S_q}} \\
 ABC_4(G) &= S_{(4,4)} \times \sqrt{\frac{4+4-2}{4 \times 4}} + S_{(4,5)} \times \sqrt{\frac{4+5-2}{4 \times 5}} \\
 &\quad + S_{(5,5)} \times \sqrt{\frac{5+5-2}{5 \times 5}} + S_{(5,8)} \times \sqrt{\frac{5+8-2}{5 \times 8}} \\
 &\quad + S_{(8,9)} \times \sqrt{\frac{8+9-2}{8 \times 9}} + S_{(9,9)}
 \end{aligned}$$

After some simplification, we obtain the required results.

$$\begin{aligned}
 ABC_4(G^*) &= \left(\frac{3}{5} \sqrt{110} + \frac{12}{5} \sqrt{2} + \frac{4}{3} \sqrt{30} + \frac{40}{9} \right. \\
 &\quad \left. + \frac{1}{2} \sqrt{14} \right) s + \frac{5}{2} \sqrt{6} + \frac{2}{5} \sqrt{35} - \frac{8}{5} \sqrt{2} \\
 &\quad - \frac{2}{3} \sqrt{30} - \frac{1}{5} \sqrt{110} - \frac{32}{9}
 \end{aligned}$$

Similarly, by using Table 3, we have

$$\begin{aligned}
 GA_5(G) &= \sum_{pq \in E(G)} \frac{2\sqrt{S_p S_q}}{S_p + S_q} \\
 GA_5(G) &= S_{(4,4)} \times 2\sqrt{\frac{4 \times 4}{4+4}} + S_{(4,5)} \times 2\sqrt{\frac{4 \times 5}{4+5}} \\
 &\quad + S_{(5,5)} \times 2\sqrt{\frac{5 \times 5}{5+5}} + S_{(5,8)} \times 2\sqrt{\frac{5 \times 8}{5+8}} \\
 &\quad + S_{(8,8)} \times 2\sqrt{\frac{8 \times 8}{8+8}} + S_{(8,9)} \\
 &\quad + 2\sqrt{\frac{8 \times 9}{8+9}} + S_{(9,9)} \times 2\sqrt{\frac{9 \times 9}{9+9}} \\
 GA_5(G^*) &= \left(20 + \frac{48}{13} \sqrt{10} + \frac{192}{17} \sqrt{2} \right) s - 2 + \frac{16}{9} \sqrt{5} \\
 &\quad - \frac{16}{13} \sqrt{10} - \frac{96}{17} \sqrt{2}
 \end{aligned}$$

Theorem 4. Let G^* be the line graph of the subdivided graph (para-line graph) of T_s . Then

1. $HM(G^*) = 1404s - 636$
2. $PM_1(G^*) = 4^{6s+10} 5^{12s-4} 6^{30s-16}$
3. $PM_2(G^*) = 4^{6s+10} 6^{12s-4} 9^{30s-16}$

Proof. The edge set $E(G^*)$ established on the degrees of the adjacent(end) vertices is represented as the sum of the

cardinality of three sets such as $E_1(G^*)$, $E_2(G^*)$ and $E_3(G^*)$. The cardinality of first set $E_1(G^*)$, second set $E_2(G^*)$ and third set $E_3(G^*)$ is denoted as $E_{(2,2)}$, $E_{(2,3)}$ and $E_{(3,3)}$ respectively. The 1st partite edge set holds $E_{(2,2)} = 6s + 10$ edges, the 2nd partite set holds $E_{(2,3)} = 12s - 4$ edges and the 3rd partite set holds $E_{(3,3)} = 30s - 16$ edges. From Table 3, we obtain the results.

$$\begin{aligned}
 HM(G^*) &= \sum_{pq \in E(G)} (\xi_p + \xi_q)^2 \\
 HM(G^*) &= \sum_{pq \in E_1(G)} [\xi_p + \xi_q]^2 + \sum_{pq \in E_2(G)} [\xi_p + \xi_q]^2 \\
 &\quad + \sum_{pq \in E_3(G)} [\xi_p + \xi_q]^2 \\
 HM(G^*) &= 16 \times E_{(2,2)} + 25 \times E_{(2,3)} + 36 \times E_{(3,3)} \\
 HM(G^*) &= 16(6s + 10) + 25(12s - 4) + 36(30s - 16) \\
 HM(G^*) &= 1404s - 636 \\
 PM_1(G^*) &= \prod_{pq \in E(G)} (\xi_p + \xi_q) \\
 PM_1(G^*) &= \prod_{pq \in E_1(G)} (\xi_p + \xi_q) \times \prod_{pq \in E_2(G)} (\xi_p + \xi_q) \\
 &\quad \times \prod_{pq \in E_3(G)} (\xi_p + \xi_q) \\
 PM_1(G^*) &= 4^{E_{(2,2)}} \times 5^{E_{(2,3)}} \times 6^{E_{(3,3)}} \\
 PM_1(G^*) &= 4^{6s+10} \times 5^{12s-4} \times 6^{30s-16} \\
 PM_2(G^*) &= \prod_{pq \in E(G)} (\xi_p \times \xi_q) \\
 PM_2(G^*) &= \prod_{pq \in E_1(G)} (\xi_p \times \xi_q) \times \prod_{pq \in E_2(G)} (\xi_p \times \xi_q) \\
 &\quad \times \prod_{pq \in E_3(G)} (\xi_p \times \xi_q) \\
 PM_2(G^*) &= 4^{E_{(2,2)}} \times 6^{E_{(2,3)}} \times 9^{E_{(3,3)}} \\
 PM_2(G^*) &= 4^{6s+10} \times 6^{12s-4} \times 9^{30s-16}
 \end{aligned}$$

Theorem 5. Let G^* be the line graph of the subdivided graph (para-line graph) of T_s . Then

1. $M_1(G^*, a) = (6s + 10)a^4 + (12s - 4)a^5$

Table 3: The distribution of edges of multiple Anthracene w.r.t. degree of end vertices of every edge.

Sr. No	$E_{(\xi_p, \xi_q)}$	Number of Edges
1	$E_{(2,2)}$	$6r + 6s + 4$
2	$E_{(2,3)}$	$4r + 12s - 8$
3	$E_{(3,3)}$	$63rs - 20r - 33s + 4$

$$+ (30s - 16)a^6$$

$$2. \quad M_2(G^*, a) = (6s + 10)a^4 + (12s - 4)a^6 + (30s - 16)a^9$$

Proof. From Table 3, we obtain the results.

$$M_1(G^*, a) = \sum_{pq \in E(G)} a^{(\xi_p + \xi_q)}$$

$$M_1(G^*, a) = \sum_{pq \in E_1(G)} a^{(\xi_p + \xi_q)} + \sum_{pq \in E_2(G)} a^{(\xi_p + \xi_q)} + \sum_{pq \in E_3(G)} a^{(\xi_p + \xi_q)}$$

$$M_1(G^*, a) = \sum_{pq \in E_1(G)} a^4 + \sum_{pq \in E_2(G)} a^5 + \sum_{pq \in E_3(G)} a^6$$

$$M_1(G^*, a) = E_{(2,2)} \times a^4 + E_{(2,3)} \times a^5 + E_{(3,3)} \times a^6$$

$$M_1(G^*, a) = (6s + 10)a^4 + (12s - 4)a^5 + (30s - 16)a^6$$

$$M_2(G^*, a) = \sum_{pq \in E(G)} a^{(\xi_p \times \xi_q)}$$

$$M_2(G^*, a) = \sum_{pq \in E_1(G)} a^{(\xi_p \times \xi_q)} + \sum_{pq \in E_2(G)} a^{(\xi_p \times \xi_q)} + \sum_{pq \in E_3(G)} a^{(\xi_p \times \xi_q)}$$

$$M_2(G^*, a) = \sum_{pq \in E_1(G)} a^4 + \sum_{pq \in E_2(G)} a^6 + \sum_{pq \in E_3(G)} a^9$$

$$M_2(G^*, a) = E_{2,2} \times a^4 + E_{2,3} \times a^6 + E_{3,3} \times a^9$$

$$M_2(G^*, a) = (6s + 10)a^4 + (12s - 4)a^6 + (30s - 16)a^9$$

5 Results for para-line graph of multiple Anthracene

In Figure 4 we presented the graph G^* , and it is denoted by $T_{r,s}$. $T_{r,s}$ has $22rs$ vertices and $33rs - 2r - 5s$ edges. Also the para-line graph of multiple Anthracene are depicted in Figure 5.

Figure 4: The molecular graph of multiple Anthracene. **Figure 5:** The para-line graph of multiple Anthracene.**Table 4:** The distribution of edges w.r.t. neighbor sum of every vertex the graph G^* .

Sr. No	$S_{(p,q)}$	Number of Edges
1	$S_{(4,4)}$	$2r + 8$
2	$S_{(4,5)}$	$4r$
3	$S_{(5,5)}$	$6s - 4$
4	$S_{(5,8)}$	$12s - 4r - 8$
5	$S_{(8,8)}$	$4s$
6	$S_{(8,9)}$	$8r + 16s - 16$
7	$S_{(9,9)}$	$63rs - 28r - 53s + 20$

Theorem 6. Let G^* be the line graph of the subdivided graph (para-line graph) of $T_{r,s}$. Then

$$M_\alpha(G^*) = (2r + 3s)2^{\alpha+2} + 3^{\alpha+1}(14rs - 4r - 6s).$$

Proof. In Figure 5, we presented a graph G^* . In G^* , the overall count of the vertices is $63rs - 10r - 15s$ which is the sum of $8r + 12s$ (vertices of degree 2) and $42rs - 12r - 18s$ (vertices of degree 3). Hence proved.

Theorem 7. Let G^* be the line graph of the subdivided graph (para-line graph) of $T_{r,s}$. Then

$$1. \quad R_\alpha(G^*) = (6s + 6r + 4)4^\alpha + (4r + 12s - 8)6^\alpha + (63rs - 20r - 33s + 4)9^\alpha;$$

$$2. \quad \chi_\alpha(G^*) = (6s + 6r + 4)4^\alpha + (4r + 12s - 8)5^\alpha$$

$$\begin{aligned}
& + (63rs - 20r - 33s + 4)6^a; \\
3. \quad ABC(G^*) &= \left(9\sqrt{2} - 22\right)s + \left(5\sqrt{2} - \frac{40}{3}\right)r \\
& - 2\sqrt{2} + 42rs + 8/3; \\
4. \quad GA(G^*) &= \left(-27 + \frac{24}{5}\sqrt{6}\right)s + \left(8/5\sqrt{6} - 14\right)r \\
& + 63rs + 8 - \frac{16}{5}\sqrt{6}.
\end{aligned}$$

Proof. The total number of edges and vertices of subdivision graph $S(T_{r,s})$ are $63rs - 10r - 15s$ and $198rs - 20r - 50$ respectively. The vertices are subdivided in the following way: The count of degree 2 vertices is $8r + 12s$ and the count of degree 3 vertices is $42rs - 12r - 18s$. The overall count of edges of G^* are $63rs - 10r - 15s$. The set of edges $E(G^*)$ bifurcates into 3 partite sets established on degrees of the adjacent(end) vertices such as $E_1(G^*)$, $E_2(G^*)$ and $E_3(G^*)$. The edge partite set $E_1(G^*)$ contains $E_{(2,2)} = 6r + 6s + 4$ edges, where $E_{(2,2)}$ represents the edge whose both vertex has degree 2. The edge partite set $E_2(G^*)$ holds $E_{(2,3)} = 4r + 12s - 8$ edges, where $E_{(2,3)}$ represents the edge the vertex of which has degree 2 and second vertex has 3 and the edge partite set $E_3(G^*)$ contains $E_{(3,3)} = 63rs - 20r - 33s + 4$ edges, where $E_{(3,3)}$ represents the edge the vertex of which has degree 3. From formulas (5), (2), (3) and (6), the proof was established.

Theorem 8. Let G^* be the line graph of the subdivided graph (para-line graph) of $T_{r,s}$. Then

$$\begin{aligned}
1. \quad ABC_4(G^*) &= \left(28r + \frac{1}{2}\sqrt{14} + \frac{12}{5}\sqrt{2}\right)s \\
& + \left(\frac{3}{5}\sqrt{110} + \frac{4}{3}\sqrt{30} - \frac{212}{9}\right)s \\
& + \left(\frac{1}{2}\sqrt{6} + \frac{1}{5}\sqrt{110} + \frac{2}{5}\sqrt{35} - \frac{112}{9} + \frac{2}{3}\sqrt{30}\right)r \\
& + 2\sqrt{6} - \frac{8}{5}\sqrt{2} - 2/5\sqrt{110} - 4/3\sqrt{30} + \frac{80}{9} \\
2. \quad GA_5(G^*) &= \left(\frac{80}{13}\sqrt{10} + 99m + \frac{288}{17}\sqrt{2} - 69\right)s \\
& + \left(-26 + \frac{16}{13}\sqrt{10} + \frac{16}{9}\sqrt{5} + \frac{96}{17}\sqrt{2}\right)r \\
& - \frac{192}{17}\sqrt{2} - \frac{32}{13}\sqrt{10} + 24.
\end{aligned}$$

Proof. Let the degree sum of neighbors of end vertices p and q be represented as $S_{(p,q)}$. Suppose the collection of edge sets $E(G^*)$ is divided into 7 partite edge sets $E_4(G^*)$, $E_5(G^*)$, $E_6(G^*)$, $E_7(G^*)$, $E_8(G^*)$, $E_9(G^*)$, $E_{10}(G^*)$. The edge

partite set $E_4(G^*)$ contains $S_{(p,q)} = (4, 4) = 2r + 8$ edges, the edge partite set $E_5(G^*)$ contains $S_{(p,q)} = (4, 5) = 4r$ edges, the edge partite set $E_6(G^*)$ contains $S_{(p,q)} = (5, 5) = 6s - 4$ edges, the edge partite set $E_7(G^*)$ contains $S_{(p,q)} = (5, 8) = 12s + 4r - 8$ edges, the edge partite set $E_8(G^*)$ contains $S_{(p,q)} = (8, 8) = 4s$ edges, the edge partite set $E_9(G^*)$ contains $S_{(p,q)} = (8, 9) = 8r + 16s - 16$ edges, and the edge partite set $E_{10}(G^*)$ contains $S_{(p,q)} = (9, 9) = 63rs - 28r - 53s + 20$ edges. From Formulas (4) and (7), results were attained.

Theorem 9. Let G^* be the line graph of the subdivided graph (para-line graph) of $T_{r,s}$. Then

$$\begin{aligned}
1. \quad HM(G^*) &= 2268rs - 596r - 864s - 40; \\
2. \quad PM_1(G^*) &= 4^{6s+6r+4}5^{4r+12s-8}6^{63rs-20r-33s+4}; \\
3. \quad PM_2(G^*) &= 4^{6s+6r+4}6^{4r+12s-8}9^{63rs-20r-33s+4}; \\
4. \quad M_1(G^*, a) &= (6s + 6r + 4)a^4 + (4r + 12s - 8)a^5 \\
& + (63rs - 20r - 33s + 4)a^6; \\
5. \quad M_2(G^*, a) &= (6s + 6r + 4)a^4 + (4r + 12s - 8)a^6 \\
& + (63rs - 20r - 33s + 4)a^9;
\end{aligned}$$

Proof. The edge set $E(G^*)$ on the basis of the degree of end vertices is represented as the sum of the cardinality of three disjointed sets i.e. $E_1(G^*)$, $E_2(G^*)$ and $E_3(G^*)$. The cardinality of first set $E_1(G^*)$, second set $E_2(G^*)$ and third set $E_3(G^*)$ is denoted as $E_{(2,2)}$, $E_{(2,3)}$ and $E_{(3,3)}$ respectively. The first partite edge set holds $E_{(2,2)} = 8r + 12s$ edges, the second partite edge set holds $E_{(2,3)} = 42rs + 12r - 18s$ edges, and the third partite edge set holds $E_{(3,3)} = 63rs - 20r - 33s + 4$ edges.

Using Table 1, we obtained the following results.

$$\begin{aligned}
HM(G^*) &= \sum_{pq \in E(G)} (\xi_p + \xi_q)^2 \\
HM(G^*) &= \sum_{pq \in E_1(G)} [\xi_p + \xi_q]^2 + \sum_{pq \in E_2(G)} [\xi_p + \xi_q]^2 \\
& + \sum_{pq \in E_3(G)} [\xi_p + \xi_q]^2 \\
HM(G^*) &= 16 \times E_{(2,2)} + 25 \times E_{(2,3)} + 36 \times E_{(3,3)} \\
HM(G^*) &= 16(6s + 6r + 4) + 25(12s + 4r - 8) \\
HM(G^*) &= 2268rs - 596r - 864s - 40 \\
PM_1(G^*) &= \prod_{pq \in E_1(G)} (\xi_p + \xi_q) \times \prod_{pq \in E_2(G)} (\xi_p + \xi_q) \\
& \times \prod_{pq \in E_3(G)} (\xi_p + \xi_q)
\end{aligned}$$

$$\begin{aligned}
PM_1(G^*) &= 4^{E_{(2,2)}} \times 5^{E_{(2,3)}} \times 6^{E_{(3,3)}} \\
PM_1(G^*) &= 4^{6s+6r+4} \times 5^{12s+4r-8} \times 6^{63rs-20r-33s+4} \\
PM_2(G^*) &= \prod_{pq \in E(G)} (\xi_p \times \xi_q) \\
PM_2(G^*) &= \prod_{pq \in E_1(G)} (\xi_p \times \xi_q) \times \prod_{pq \in E_2(G)} (\xi_p \times \xi_q) \\
&\quad \times \prod_{pq \in E_3(G)} (\xi_p \times \xi_q) \\
PM_2(G^*) &= 4^{E_{(2,2)}} \times 6^{E_{(2,3)}} \times 9^{E_{(3,3)}} \\
PM_2(G^*) &= 4^{6s+6r+4} \times 6^{12s+4r-8} \times 9^{63rs-20r-33s+4} \\
M_1(G^*, a) &= \sum_{pq \in E(G)} a^{(\xi_p + \xi_q)} \\
M_1(G^*, a) &= \sum_{pq \in E_1(G)} a^{(\xi_p + \xi_q)} + \sum_{pq \in E_2(G)} a^{(\xi_p + \xi_q)} \\
&\quad + \sum_{pq \in E_3(G)} a^{(\xi_p + \xi_q)} \\
M_1(G^*, a) &= \sum_{pq \in E_1(G)} a^4 + \sum_{pq \in E_2(G)} a^5 + \sum_{pq \in E_3(G)} a^6 \\
M_1(G^*, a) &= E_{(2,2)} \times a^4 + E_{(2,3)} \times a^5 + E_{(3,3)} \times a^6 \\
M_1(G^*, a) &= (6s + 6r + 4) a^4 + (4r + 12s - 8) a^5 \\
&\quad + (63rs - 20r - 33s + 4) a^6 \\
M_2(G^*, a) &= \sum_{pq \in E(G)} a^{(\xi_p \times \xi_q)} \\
M_2(G^*, a) &= \sum_{pq \in E_1(G)} a^{(\xi_p \times \xi_q)} + \sum_{pq \in E_2(G)} a^{(\xi_p \times \xi_q)} \\
&\quad + \sum_{pq \in E_3(G)} a^{(\xi_p \times \xi_q)} \\
M_2(G^*, a) &= \sum_{pq \in E_1(G)} a^4 + \sum_{pq \in E_2(G)} a^6 + \sum_{pq \in E_3(G)} a^9 \\
M_2(G^*, a) &= E_{2,2} \times a^4 + E_{2,3} \times a^6 + E_{3,3} \times a^9 \\
M_2(G^*, a) &= (6s + 6r + 4) a^4 + (4r + 12s - 8) a^6 \\
&\quad + (63rs - 20r - 33s + 4) a^9
\end{aligned}$$

Hence proved.

6 Conclusion

In this paper, we have comprehended the work on these indices: R_a , M_a , χ_a , ABC , GA , ABC_4 , GA_5 , PM_1 , PM_2 , $M_1(G, a)$, and $M_1(G, a)$ of the line graph of the subdivided graph (para-line graph) of linear [s]-Anthracene and multiple Anthracene. Randic index R_a plays a vital role in the ex-

tension of the branching of the carbon-atom skeleton of hydrocarbons. $PM_1(G)$ and $PM_2(G)$ may be expressed in the QSPR study and shows a central role in the analysis of the boiling and melting point of drugs. Chemical graphs are currently being studied with the help of para-line graphs which are very important in the field of chemistry. Our upcoming work will be the emphasis on some new classes of the above mentioned graphs(para-line) of chemical structures with respect to the different topological indices.

Ethical approval: The conducted research is not related to either human or animal use.

Conflict of Interests: Authors declare no conflict of interest.

References

- [1] Bollobás B., Erdős P., Graphs of extremal weights, *Ars Combin.*, 1998, 50, 225–233.
- [2] Das K.C., Sorgun S., On Randic energy of graphs, *MATCH Commun. Math. Comput. Chem.*, 2014, 72, 227–238.
- [3] Das K.C., Gutman I., Furtula B., Survey on geometric-arithmetic indices of graphs. *MATCH Commun. Math. Comput. Chem.*, 2011, 65, 595–644.
- [4] Dehmer M., Emmert-Streib F., Grabner M., A computational approach to construct a multivariate complete graph invariant, *Inform. Sci.*, 2014, 260, 200–208.
- [5] Estrada E., Torres L., Rodríguez L., Gutman I., An atom-bond connectivity index: Modelling the enthalpy of formation of alkanes. *Indian J. Chem.*, 1998, 37A, 849–855.
- [6] Estrada E., "Edge adjacency relationships and a novel topological index related to molecular volume." *Journal of Chemical Information and Computer Sciences.*, 1995, 35, 31–33.
- [7] Feng L., Liu W., Ilic A., Yu G., The degree distance of unicyclic graphs with given matching number, *Graphs Comb.*, 2013, 29, 449–462.
- [8] Feng L., Liu W., Yu G., Li S., The hyper-Wiener index of graphs with given bipartition, *Utilitas Math.*, 2014, 95, 23–32.
- [9] Ghorbani M., Hosseinzadeh M. A., Computing ABC_4 index of nanostar dendrimers, *Optoelectron. Adv. Mater.-Rapid Commun.*, "2010, 4(9), 1419–1422.
- [10] Graovac A., Ghorbani M., Hosseinzadeh M. A., Computing fifth geometric-arithmetic index for nanostar dendrimers, *J. Math. Nanosci.*, 2011, 1, 33–42.
- [11] Gutman I., Selected properties of the Schultz molecular topological index, *J. Chem. Inf. Comput. Sci.*, 1994, 34, 1087–1089.
- [12] Gutman I., Ruscic B., Trinajstić N., Wilcox C.F., Graph theory and molecular orbitals, XII. Acyclic polyenes. *J. Chem. Phys.*, 1975, 62, 3399–3405.
- [13] Gutman I., Trinajstić N., Graph theory and molecular orbitals, Total π -electron energy of alternant hydrocarbons. *Chem. Phys. Lett.*, 1972, 17, 535–538.
- [14] Gutman I., Tomovic Z., "On the application of line graphs in quantitative structure-property studies." *J. Serb. Chem. Soc.*, 2000,

- 65, 577-580.
- [15] Klein D.J., Yi E., A comparison on metric dimension of graphs, line graphs, and line graphs of the subdivision graphs, *Eur. J. Pure Appl. Math.*, 2012, 5, 302–316.
 - [16] Li X., Shi Y., Wang L., An updated survey on the Randic index, in: B.F. I. Gutman (Ed.), *Recent Results in the Theory of Randic Index*, University of Kragujevac and Faculty of Science Kragujevac, (2008), 9-47.
 - [17] Li X., Zhao H., Trees with the first three smallest and largest generalized topological indices, *MATCH Commun. Math. Comput. Chem.*, 2004, 50, 57-62.
 - [18] Liu J. B., Pan X. F., Hu F. T., Hu F. F., Asymptotic Laplacian-energy-like invariant of lattices. *Appl. Math. Comput.*, 2015, 253, 205-214.
 - [19] Liu J. B., Wang W. R., Zhang Y. M., Pan X. F., On degree resistance distance of cacti. *Discret. Appl. Math.*, 2016, 203, 217-225.
 - [20] Liu J. B., Pan X. F., Asymptotic incidence energy of lattices. *Physica A Statistical Mechanics and its Applications*, 2015, 422, 193-202.
 - [21] Liu J. B., Cao J., Alofi A., Abdullah A. M., Elaiw A., Applications of Laplacian spectra for n-prism networks. *Neurocomputing*, 2016, 198, 69-73.
 - [22] Nadeem M.F., Sohail Z., Zohaib Z., On certain topological indices of the line graph of subdivision graphs. *Applied Mathematics and Computation*, 2015, 271, 790-794.
 - [23] Nadeem M.F., Sohail Z., Zohaib Z., On topological properties of the line graphs of subdivision graphs of certain nanostructures. *Applied Mathematics and Computation*, 2016, 273, 125-130.
 - [24] Rada J., Cruz R., Vertex-degree-based topological indices over graphs, *MATCH Commun. Math. Comput. Chem.*, 2014, 72, 603-616.
 - [25] Ranjini P. S., Loksha V., Cangül I. N., On the Zagreb indices of the line graphs of the subdivision graphs, *Appl. Math. Comput.*, 2011, 218, 699-702.
 - [26] Ranjini P. S., Loksha V., Rajan M.A., On the Shultz index of the subdivision graphs, *Adv. Stud. Contemp. Math.*, 2011, 21(3), 279-290.
 - [27] Su G., Xu L., Topological indices of the line graph of subdivision graphs and their Schur-bounds, *Appl. Math. Comput.*, 2015, 253, 395-401.
 - [28] Vukicevic D., Furtula, B., Topological index based on the ratios of geometrical and arithmetical means of end-vertex degrees of edges, *J. Math. Chem.*, 2009, 46, 1369-1376.
 - [29] Xu K., Liu M., Das K., Gutman I., Furtula B., A survey on graphs extremal with respect to distance-based topological indices, *MATCH Commun. Math. Comput. Chem.*, 2014, 71, 461-508.
 - [30] Zhou B., Trinajstić N., On general sum-connectivity index, *J. Math. Chem.*, 2010, 47, 210-218.