Research Article

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An analytical solution for static problems of curved composite beams

<https://doi.org/10.1515/cls-2019-0009>

Received Nov 08, 2018; accepted Dec 27, 2018

Abstract: An analytical solution is presented for the determination of deformation of curved composite beams. Each cross-section is assumed to be symmetrical and the applied loads are acted in the plane of symmetry of curved beam. In-plane deformations are considered of composite curved beams. Assumed form of the displacement field assures the fulfillment of the classical Bernoulli-Euler beam theory. The curvature of beam is constant and the internal forces in a cross-section is replaced by an equivalent force-couple system at the origin of the cylindrical coordinate system used. The internal forces are expressed in terms of two kinematical variables, which are the radial displacement and the rotation of the cross-sections. The determination of the analytical solutions of the considered static problems are based on the fundamental solutions. Linear combination of the fundamental solutions which are filling to the given loading and boundary conditions, gives the total solution. Closed form formulae are derived for the radial displacement, cross-sectional rotation, normal and shear forces and bending moments. The circumferential and radial normal stresses and shear stresses are obtained by the integration of equilibrium equations. Examples illustrate the developed method.

Keywords: Analytical solution; curved composite beam; in-plane deformation; fundamental solution

1 Introduction

The analysis of curved beam has been a topic of interest to research workers for over a century, it is a standard chapter in the most text-books of mechanics of solids [1–7]. This

topic is still concern at the present time because curved beam elements of constant curvature are important components in many modern engineering structures. In this paper an analytical solution is presented for the solution of static boundary value problems of non-homogeneous curved beams with constant curvature. For the solution static bending problems of layered curved beams Segura and Armengaud gave an analytical method [8]. Elasticity solutions are presented in [9] for curved beams with orthotropic functionally graded layers by means of Airy stress functions. The developed method is illustrated in curved cantilever beams with different types of loading conditions. Paper by Pydah and Sabale [10] presents an analytical model for the flexure of bidirectional functionally graded circular beam based on the Euler-Bernoulli beam theory. The material properties are simultaneously smooth functions of the polar angle φ and the radial coordinate r . The governing equations are solved for statically determinate circular cantilever beams under the action of tip loads. Static problems of curved composite beams were analysed by finite element methods in the papers of Ibrahimbegović and Frey [11], Ascione and Fraternali [12] and Bhimaraddi *et al.* [13]. Paper by Dorfi and Busby [14] presents a laminated curved beam finite element with six displacement degrees of freedom and three stress parameters. The classical lamination theory and the Timoshenko theory are used to formulate the finite element. In [15] a new highly accurate composite laminated hybrid-mixed curved beam element is presented by Kim. The formulation of the finite element is based on the Hellinger-Reissner variational principle and the first-order shear deformation theory. Paper [16] presents a simple one-dimensional mechanical model to analyse the static and dynamic features of non-homogeneous curved beams. The formulation of the analytical solution developed in this paper is based on paper [16]. In the present paper according to [16] the field equations and boundary conditions are formulated in cylindrical coordinate system $Or\varphi z$ and it is assumed that the plane $z = 0$ is the plane of symmetry including the material and geometrical properties and loading conditions. The material and geometrical properties do not depend on polar angle φ , moduli of elasticity may depend

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on the cross-sectional coordinates (r, z) , the curved beam in circumferential direction is homogeneous. This type of non-homogeneous curved beam is called φ -homogeneous curved beam [16]. The definition of φ -homogeneity includes those cases when the curved beam is a composite of different materials, so that moduli of elasticity are piecewise constants. Type of these beams are laminated and fibre reinforced beams. Their discontinuities in material properties should not affect the presented analysis. It must be mentioned that papers by Ecsedi and Lengyel [17, 18] provide analytical solution for layered curved composite beams with interlayer slip. In paper [17] the deformation of curved composite beam with partial shear interaction is determined under the action of concentrated radial load. Paper [18] gives numerical solution for in-plane deformation of two-layer composite beam with flexible shear connection by the application of principle of minimum of potential energy.

In this study φ -homogeneous curved beams are considered. Our aim is to develop an analytical solution to determine the deflection, cross-sectional rotation, internal forces and stresses. The presented method is based on the fundamental solutions. The fundamental solutions fulfil all the field equations and their initial values are zero except only one of them. Linear combination of the fundamental solutions which are fitted to given loading and boundary conditions provides the total solution of the considered static equilibrium problem.

2 Governing equations

Figure 1 shows the non-homogeneous curved beams in the cylindrical coordinate system $Or\varphi z$. The curved beam occupies the space domain $\bar{B}$ which is generated by the rotation of a plane figure, whose inner and boundary points are taken from the sets A and ∂A , respectively. The domain $\bar{A} = A \cup \partial A$ is bounded and called the cross-section of the

Figure 1: φ -homogeneous curved beam.

curved beam. The region $\bar{B} = B \cup \partial B$ is defined as $B = \{(r, \varphi, z) : (r, z) \in A, 0 < \varphi < \alpha\}$, $\partial B = \partial B_1 \cup \partial B_2 \cup \partial B_3$, $\partial B_i = \{(r, \varphi, z) : (r, z) \in A, \varphi = \varphi_i, i = 1, 2\}$, $\varphi_1 = 0$, $\varphi_2 = \alpha < 2\pi$, $\partial B_3 = \{(r, \varphi, z) : (r, z) \in \partial A, 0 \leq \varphi \leq \alpha\}$. The plane $z = 0$ is the plane of symmetry of the curved beam (Figure 1). Denote the unit vectors of the cylindrical coordinate system $Or\varphi z$ $\mathbf{e}_r$, $\mathbf{e}_\varphi$ and $\mathbf{e}_z$. Following paper [16] the next displacement field is used to describe the in-plane deformation of non-homogeneous curved beam

$$\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\varphi + w\mathbf{e}_z, \quad (1)$$

$$u = U(\varphi), \quad v = r\phi(\varphi) + \frac{dU}{d\varphi}, \quad w = 0, \quad (2)$$

$$(r, \varphi, z) \in B.$$

Application of the strain-displacement relationships of the linearised theory of elasticity provides the following results for the normal strains ε_r , ε_φ , ε_z and shearing strains $\gamma_{r\varphi}$, γ_{rz} , $\gamma_{\varphi z}$

$$\varepsilon_r = \varepsilon_z = \gamma_{r\varphi} = \gamma_{rz} = \gamma_{\varphi z} = 0, \quad (3)$$

$$\varepsilon_\varphi = \frac{1}{r} \left(\frac{d^2 U}{d\varphi^2} + U \right) + \frac{d\phi}{d\varphi}. \quad (4)$$

The strains given by Eqs. (3) and (4) satisfy the requirements of the Euler-Bernoulli beam theory, only the normal strain ε_φ is different from zero. According to the simplified form of Hooke's law and Eq. (4) we can write

$$\sigma_\varphi = E(r, z) \left\{ \frac{1}{r} \left(\frac{d^2 U}{d\varphi^2} + U \right) + \frac{d\phi}{d\varphi} \right\}. \quad (5)$$

In Eq. (5) σ_φ is the normal stress, $E = E(r, z)$ is the modulus of elasticity which may depend on the cross-sectional coordinate r, z . It is assumed that the plane of $z = 0$ is the plane of symmetry of material properties and loading conditions, so that we have $E(r, z) = E(r, -z)$, $(r, z) \in \bar{A}$. In order to formulate the stress resultants-displacement relationships we introduce the following cross-sectional and material properties [16]

$$AE_0 = \int_A E(r, z) dA, \quad r_c = \int_A rE(r, z) dA, \quad (6)$$

$$\frac{AE_0}{R} = \int_A \frac{E(r, z)}{r} dA, \quad e = r_c - R. \quad (7)$$

Here, we note that $e > 0$ independently of the shape of cross-section and the form of modulus of elasticity function which is always positive. The proof of this statement

follows from the next Schwarz's inequality

$$\int_A E(r, z)(\sqrt{r})^2 dA \int_A \frac{E(r, z)}{(\sqrt{r})^2} dA \geq \left(\int_A E(r, z) \frac{\sqrt{r}}{\sqrt{r}} dA \right)^2. \quad (8)$$

The stress resultants are defined according to paper [16] (Figure 2)

Figure 2: Illustration of stress resultants and couple stress resultant.

$$\begin{aligned} N(\varphi) &= \int_A \sigma_\varphi dA, \quad S(\varphi) = \int_A \tau_{r\varphi} dA, \quad M(\varphi) \\ &= \int_A r \sigma_\varphi dA. \end{aligned} \quad (9)$$

In Eq. (9) $\tau_{r\varphi} = \tau_{r\varphi}(r, \varphi, z)$ is the shearing stress, N is the normal force, S is the shear force and M is the bending moment. We note, in the Euler-Bernoulli beam theory the shear force does not effect to the deformation of non-homogeneous curved beam, its value is obtained by the application of the force equilibrium equation. Substitution of (5) into Eqs. (8)_{1,3} gives

$$\begin{aligned} N(\varphi) &= AE_0 \left(\frac{W}{R} + \frac{d\phi}{d\varphi} \right), \\ M(\varphi) &= AE_0 \left(W + r_c \frac{d\phi}{d\varphi} \right). \end{aligned} \quad (10)$$

Here, we introduce the designation

$$W(\varphi) = \frac{d^2 U}{d\varphi^2} + U. \quad (11)$$

In paper [16] the next equations of equilibrium are derived

$$\frac{\partial N}{\partial \varphi} + S + f_\varphi = 0, \quad \frac{\partial S}{\partial \varphi} - N + f_r = 0, \quad 0 < \varphi < \alpha, \quad (12)$$

$$\frac{\partial M}{\partial \varphi} + m = 0. \quad (13)$$

In Eqs. (12), (13) $f_r = f_r(\varphi)$ and $f_\varphi = f_\varphi(\varphi)$ are the applied external forces and $m = m(\varphi)$ is the applied moment obtained from the applied external forces [16]. Let $V = V(\varphi)$ be defined as

$$V = \frac{dU}{d\varphi}. \quad (14)$$

Figure 3 enumerates some possible boundary conditions.

Figure 3: In plane boundary conditions for φ -homogeneous curved beams: a) Clamped end: $U = 0, V = 0, \phi = 0$. b) Free end: $N = 0, S = 0, M = 0$. c) Simply supported in radial direction: $U = 0, M = 0, N = 0$. d) Pinned end: $U = 0, r_A \phi + V = 0, M - r_A N = 0$. e) Clamped-circumferentially guided end: $U = 0, \phi = 0, N = 0$. f) Clamped-radially guided end: $V = 0, \phi = 0, S = 0$. g) Loaded end cross section: $N = F_1, S = F_2, M = M_1$.

3 Equations of state of stresses

The cross-section A of the φ -homogeneous curved beam can be represented in terms of thickness function $t = t(r)$ (Figure 4) as

$$A = \{(r, z) : |z| \leq 0.5t(r), r_1 \leq r \leq r_2\}, \quad (15)$$

$$r_1 = \min_{(r,z) \in A} r, \quad r_2 = \max_{(r,z) \in A} r.$$

Figure 4: Illustration of r_1 and r_2 .

We introduce the new type of stress resultants, i.e. stresses integrated through the thickness, which are

$$n_\varphi(r, \varphi) = \int_{-0.5t(r)}^{0.5t(r)} \sigma_\varphi(r, \varphi, z) dz, \quad (16)$$

$$n_r(r, \varphi) = \int_{-0.5t(r)}^{0.5t(r)} \sigma_r(r, \varphi, z) dz, \quad (17)$$

$$n_{r\varphi}(r, \varphi) = \int_{-0.5t(r)}^{0.5t(r)} \tau_{r\varphi}(r, \varphi, z) dz. \quad (18)$$

Equation of mechanical equilibrium in terms of these stress resultants can be formulated as

$$\frac{\partial}{\partial r} (r^2 n_{r\varphi}) + r \frac{\partial n_\varphi}{\partial \varphi} = 0, \quad (19)$$

$$\frac{\partial}{\partial r} (r n_r) - n_\varphi + \frac{\partial n_{r\varphi}}{\partial \varphi} = 0.$$

Assuming that only the radial normal stress acts at $r = r_2$ then we have the next stress boundary conditions

$$n_{r\varphi}(r_1, \varphi) = n_{r\varphi}(r_2, \varphi) = 0, \quad (20)$$

$$n_r(r_1, \varphi) = 0, \quad n_r(r_2, \varphi) = \frac{f_r(r_2, \varphi)}{r_2}.$$

From Eqs. (5) and (11) it follows that

$$n_\varphi(r, \varphi) = \int_{-0.5t(r)}^{0.5t(r)} E(r, z) \left(\frac{W(\varphi)}{r} + \phi'(\varphi) \right) dz. \quad (21)$$

Knowing $n_\varphi = n_\varphi(r, \varphi)$ the shearing stress resultant $n_{r\varphi}$ is obtained from Eq. (19)₁ and Eq. (20)

$$n_{r\varphi}(r, \varphi) = -\frac{1}{r^2} \int_{r_1}^r \rho \frac{\partial n_\varphi(\rho, \varphi)}{\partial \varphi} d\rho. \quad (22)$$

Integration of Eq. (19)₂ gives the normal stress resultant n_r in the next form

$$n_r(r, \varphi) = \frac{1}{r} \int_{r_1}^r n(\rho, \varphi) d\rho, \quad (23)$$

$$n(\rho, \varphi) = n_\varphi(\rho, \varphi) - \frac{\partial n_{r\varphi}(\rho, \varphi)}{\partial \varphi}.$$

4 Betti-Rayleigh type reciprocity relation

Let us consider two equilibrium states 1 and 2 of the φ -homogeneous curved beam. These equilibrium states are denoted by upper one comma and upper two comma, respectively. We defined the mixed strain energy U_{12} for the states 1 and 2 and states 2 and 1 as

$$U_{12} = \int_0^\alpha \int_A \sigma'_\varphi \varepsilon''_\varphi r dA d\varphi, \quad (24)$$

$$U_{21} = \int_0^\alpha \int_A \sigma''_\varphi \varepsilon'_\varphi r dA d\varphi.$$

From the formulae

$$\varepsilon_\varphi = \frac{1}{r} \left(\frac{d^2 U}{d\varphi^2} + U \right) + \frac{d\phi}{d\varphi}, \quad (25)$$

$$\sigma_\varphi = E(r, z) \left[\frac{1}{r} \left(\frac{d^2 U}{d\varphi^2} + U \right) + \frac{d\phi}{d\varphi} \right]$$

and Eqs. (25) it follows that

$$U_{12} = U_{21}. \quad (26)$$

A detailed computation shows that U_{12} can be written in the form

$$\begin{aligned} U_{12} &= \int_0^\alpha \int_A \left[\sigma'_\varphi \left(\frac{d^2 U''}{d\varphi^2} + U'' \right) + r \sigma'_\varphi \frac{d\phi''}{d\varphi} \right] dA d\varphi \quad (27) \\ &= \int_0^\alpha \left[N' \left(\frac{d^2 U''}{d\varphi^2} + U'' \right) + M' \frac{d\phi''}{d\varphi} \right] d\varphi \\ &= \left\{ N' \frac{dU''}{d\varphi} \right\}_0^\alpha - \left\{ \frac{dN'}{d\varphi} U'' \right\}_0^\alpha + \int_0^\alpha \left[\frac{d^2 N'}{d\varphi^2} + N' \right] U'' d\varphi \\ &\quad + \left\{ M' \phi'' \right\}_0^\alpha - \int_0^\alpha \frac{dM'}{d\varphi} \phi'' d\varphi. \end{aligned}$$

In Eq. (27) the next designation has been introduced

$$\{F(\varphi)\}_0^\alpha = F(\alpha) - F(0). \quad (28)$$

According to equations of equilibrium (12) and (13) we have for $U_{12} = W_{12}$, where

$$\begin{aligned} W_{12} &= \{N' V'' + S' U'' + M' \phi''\}_0^\alpha \quad (29) \\ &\quad + \int_0^\alpha \left[\left(f'_r - \frac{df'_\varphi}{d\varphi} \right) U'' + m' \phi'' \right] d\varphi. \end{aligned}$$

The mechanical meaning of W_{12} is obvious, the work done by the applied forces and reactions of the first equilibrium state on the displacement field caused by the forces applied in second equilibrium state. Interchanging index 1 and 2 we get $U_{21} = W_{21}$, where

$$\begin{aligned} W_{21} &= \{N'' V' + S'' U' + M'' \phi'\}_0^\alpha \quad (30) \\ &\quad + \int_0^\alpha \left[\left(f''_r - \frac{df''_\varphi}{d\varphi} \right) U' + m'' \phi' \right] d\varphi. \end{aligned}$$

It is evident according to Eq. (26)

$$W_{12} = W_{21}. \quad (31)$$

Equation (31) formulates the Betti-Rayleigh type reciprocity relation for φ -homogeneous curved beam.

5 Fundamental solutions

In the following we are going to define the fundamental solutions. All fundamental solutions $U_i = U_i(\varphi)$, $V_i = V_i(\varphi)$, $\phi_i = \phi_i(\varphi)$, $N_i = N_i(\varphi)$, $S_i = S_i(\varphi)$, $M_i = M_i(\varphi)$ ($i = 1 \dots 9$) satisfy Eqs. (10), (12), (13) with the next initial and loading

conditions

$$\begin{aligned} U_1(0) &= 1, \quad V_1(0) = \phi_1(0) = N_1(0) = S_1(0) \quad (32) \\ &= M_1(0) = 0, \quad f_r(\varphi) = 0, \quad f_\varphi(\varphi) = 0, \quad m(\varphi) = 0, \\ &\varphi > 0, \end{aligned}$$

$$\begin{aligned} V_2(0) &= 1, \quad U_2(0) = \phi_2(0) = N_2(0) = S_2(0) \quad (33) \\ &= M_2(0) = 0, \quad f_r(\varphi) = 0, \quad f_\varphi(\varphi) = 0, \quad m(\varphi) = 0, \\ &\varphi > 0, \end{aligned}$$

$$\begin{aligned} \phi_3(0) &= 1, \quad U_3(0) = V_3(0) = N_3(0) = S_3(0) \quad (34) \\ &= M_3(0) = 0, \quad f_r(\varphi) = 0, \quad f_\varphi(\varphi) = 0, \quad m(\varphi) = 0, \\ &\varphi > 0, \end{aligned}$$

$$\begin{aligned} N_4(0) &= 1, \quad U_4(0) = V_4(0) = \phi_4(0) = S_4(0) \quad (35) \\ &= M_4(0) = 0, \quad f_r(\varphi) = 0, \quad f_\varphi(\varphi) = 0, \quad m(\varphi) = 0, \\ &\varphi > 0, \end{aligned}$$

$$\begin{aligned} S_5(0) &= 1, \quad U_5(0) = V_5(0) = \phi_5(0) = N_5(0) \quad (36) \\ &= M_5(0) = 0, \quad f_r(\varphi) = 0, \quad f_\varphi(\varphi) = 0, \quad m(\varphi) = 0, \\ &\varphi > 0, \end{aligned}$$

$$\begin{aligned} M_6(0) &= 1, \quad U_6(0) = V_6(0) = \phi_6(0) = N_6(0) \quad (37) \\ &= S_6(0) = 0, \quad f_r(\varphi) = 0, \quad f_\varphi(\varphi) = 0, \quad m(\varphi) = 0, \\ &\varphi > 0, \end{aligned}$$

$$\begin{aligned} U_7(0) &= V_7(0) = \phi_7(0) = N_7(0) = S_7(0) \quad (38) \\ &= M_7(0) = 0, \quad f_r(\varphi) = 1, \quad f_\varphi(\varphi) = 0, \quad m(\varphi) = 0, \\ &\varphi > 0, \end{aligned}$$

$$\begin{aligned} U_8(0) &= V_8(0) = \phi_8(0) = N_8(0) = S_8(0) \quad (39) \\ &= M_8(0) = 0, \quad f_r(\varphi) = 0, \quad f_\varphi(\varphi) = 1, \quad m(\varphi) = 0, \\ &\varphi > 0, \end{aligned}$$

$$\begin{aligned} U_9(0) &= V_9(0) = \phi_9(0) = N_9(0) = S_9(0) \quad (40) \\ &= M_9(0) = 0, \quad f_r(\varphi) = 0, \quad f_\varphi(\varphi) = 0, \quad m(\varphi) = 1, \\ &\varphi > 0. \end{aligned}$$

From the definitions of fundamental solutions we obtain

$$\begin{aligned} U_1(\varphi) &= \cos \varphi, \quad V_1(\varphi) = -\sin \varphi, \quad (41) \\ \phi_1(\varphi) &= N_1(\varphi) = S_1(\varphi) = M_1(\varphi) = 0, \quad \varphi \geq 0, \end{aligned}$$

$$\begin{aligned} U_2(\varphi) &= \sin \varphi, \quad V_2(\varphi) = \cos \varphi, \quad (42) \\ \phi_2(\varphi) &= N_2(\varphi) = S_2(\varphi) = M_2(\varphi) = 0, \quad \varphi \geq 0, \end{aligned}$$

$$\begin{aligned}\phi_3(\varphi) &= 1, \quad U_3(\varphi) = V_3(\varphi) = N_3(\varphi) \\ &= S_3(\varphi) = M_3(\varphi) = 0, \quad \varphi \geq 0,\end{aligned}\quad (43)$$

$$U_4(\varphi) = \frac{Rr_c}{2AE_0e} \varphi \sin \varphi, \quad (44)$$

$$V_4(\varphi) = \frac{Rr_c}{2AE_0e} (\sin \varphi + \varphi \cos \varphi),$$

$$\phi_4(\varphi) = -\frac{R}{AE_0e} \sin \varphi, \quad N_4(\varphi) = \cos \varphi$$

$$S_4(\varphi) = \sin \varphi, \quad M_4(\varphi) = 0, \quad \varphi \geq 0,$$

$$U_5(\varphi) = \frac{Rr_c}{2AE_0e} (\varphi \cos \varphi - \sin \varphi), \quad (45)$$

$$V_5(\varphi) = -\frac{Rr_c}{2AE_0e} \varphi \sin \varphi, \quad \phi_5(\varphi) = \frac{R}{AE_0e} (1 - \cos \varphi),$$

$$N_5(\varphi) = -\sin \varphi, \quad S_5(\varphi) = \cos \varphi, \quad M_5(\varphi) = 0, \quad \varphi \geq 0,$$

$$U_6(\varphi) = \frac{R}{AE_0e} (\cos \varphi - 1), \quad (46)$$

$$V_6(\varphi) = -\frac{R}{AE_0e} \sin \varphi,$$

$$\phi_6(\varphi) = \frac{1}{AE_0e} \varphi, \quad N_6(\varphi) = S_6(\varphi) = 0,$$

$$M_6(\varphi) = 1, \quad \varphi \geq 0,$$

$$U_7(\varphi) = \frac{Rr_c}{AE_0e} \left(1 - \cos \varphi - \frac{1}{2} \varphi \sin \varphi \right), \quad (47)$$

$$V_7(\varphi) = \frac{Rr_c}{2AE_0e} (\sin \varphi - \varphi \cos \varphi),$$

$$\phi_7(\varphi) = \frac{R}{AE_0e} (\sin \varphi - \varphi), \quad N_7(\varphi) = 1 - \cos \varphi,$$

$$S_7(\varphi) = -\sin \varphi, \quad M_7(\varphi) = 0, \quad \varphi \geq 0,$$

$$U_8(\varphi) = \frac{Rr_c}{2AE_0e} (\varphi \cos \varphi - \sin \varphi), \quad (48)$$

$$V_8(\varphi) = -\frac{Rr_c}{2AE_0e} \varphi \sin \varphi, \quad \phi_8(\varphi) = \frac{R}{AE_0e} (1 - \cos \varphi),$$

$$N_8(\varphi) = -\sin \varphi, \quad S_8(\varphi) = \cos \varphi - 1,$$

$$M_8(\varphi) = 0, \quad \varphi \geq 0,$$

$$U_9(\varphi) = \frac{R}{AE_0e} (\varphi - \sin \varphi), \quad (49)$$

$$V_9(\varphi) = \frac{R}{AE_0e} (1 - \cos \varphi), \quad \phi_9(\varphi) = -\frac{\varphi^2}{2AE_0e},$$

$$N_9(\varphi) = S_9(\varphi) = 0, \quad M_9(\varphi) = -\varphi, \quad \varphi \geq 0.$$

For intermediate loads such as in the case shown in Figure 5 the fundamental solutions can be expressed by the applications of Heaviside functions:

Figure 5: Intermediate applied loads.

$$\begin{aligned}X(\varphi) &= F_1 H(\varphi - \varphi_1) X_5(\varphi - \varphi_1) \\ &+ F_2 H(\varphi - \varphi_1) X_4(\varphi - \varphi_1) + M_1 H(\varphi - \varphi_1) X_6(\varphi - \varphi_1) \\ &+ H(\varphi - \varphi_1) X_7(\varphi - \varphi_1) - H(\varphi - \varphi_2) X_7(\varphi - \varphi_2),\end{aligned}\quad (50)$$

where $X(\varphi)$ may be $U(\varphi)$, $V(\varphi)$, $\phi(\varphi)$, $N(\varphi)$, $S(\varphi)$, $M(\varphi)$ and F_1 , F_2 , M_1 are applied at the cross section given by φ_1 and $H(\varphi) = 0$ for $\varphi \leq 0$, $H(\varphi) = 1$ for $\varphi > 0$.

6 Examples

6.1 Two-layer composite curved beam

The beam configuration and its loads are shown in Figure 6. The next numerical data are used: $a = 0.045$ m, $b = 0.025$ m, $c = 0.035$ m, $t_1 = 0.009$ m, $t_2 = 0.002$ m, $E_1 = 10^{11}$ Pa, $E_2 = 8 \times 10^9$ Pa, $f = f_1 = f_2 = -1000$ N, $\varphi_1 = \frac{\pi}{4}$. The boundary conditions of this problem are as follows

$$U(0) = V(0) = \phi(0) = 0, \quad (51)$$

$$U(\alpha) = V(\alpha) = \phi(\alpha) = 0, \quad (\alpha = \pi),$$

since both ends of the two-layered composite curved beam is fixed. The unknown initial values $N(0)$, $S(0)$ and $M(0)$ are obtained as a solution of next system of linear equations

$$\begin{aligned}U(\alpha) &= N(0)U_4(\alpha) + S(0)U_5(\alpha) + M(0)U_6(\alpha) \\ &+ f(U_7(\alpha) - U_7(\alpha - \varphi_1)) + fU_7(\alpha - \varphi_2) = 0,\end{aligned}\quad (52)$$

$$\begin{aligned}V(\alpha) &= N(0)V_4(\alpha) + S(0)V_5(\alpha) + M(0)V_6(\alpha) \\ &+ f(V_7(\alpha) - V_7(\alpha - \varphi_1)) + fV_7(\alpha - \varphi_2) = 0,\end{aligned}\quad (53)$$

Figure 6: Two-layer curved composite beam (a) and its cross section (b).

$$\begin{aligned} \phi(\alpha) &= N(0)\phi_4(\alpha) + S(0)\phi_5(\alpha) + M(0)\phi_6(\alpha) \\ &+ f(\phi_7(\alpha) - \phi_7(\alpha - \varphi_1)) + f\phi_7(\alpha - \varphi_2) = 0, \end{aligned} \quad (54)$$

where $\varphi_2 = \alpha - \varphi_1$. Knowing $N(0)$, $S(0)$ and $M(0)$ the solution for the layered curved composite beam with fixed ends can be given as

$$\begin{aligned} X(\varphi) &= N(0)X_4(\varphi) + S(0)X_5(\varphi) + M(0)X_6(\varphi) \\ &+ fX_7(\varphi) - fH(\varphi - \varphi_1)X_7(\varphi - \varphi_1) \\ &+ fH(\varphi - \varphi_2)X_7(\varphi - \varphi_2), \quad 0 \leq \varphi \leq \alpha = \pi, \end{aligned} \quad (55)$$

where $X(\varphi)$ may be $U(\varphi)$, $V(\varphi)$, $\phi(\varphi)$, $N(\varphi)$, $S(\varphi)$, $M(\varphi)$. The plots of $U = U(\varphi)$, $V = V(\varphi)$, $\phi = \phi(\varphi)$ are shown in Figures 7, 8 and 9. Figures 10, 11, 12 illustrate the graphs of $N = N(\varphi)$, $S = S(\varphi)$ and $M = M(\varphi)$. The plots of internal stress resultants $n_\varphi = n_\varphi(r, \varphi)$, $n_{r\varphi} = n_{r\varphi}(r, \varphi)$ and $n_r = n_r(r, \varphi)$ as a function of radial coordinate r for cross sections given by polar angles $\varphi_1 = 0$, $\varphi_2 = \frac{\pi}{8}$, $\varphi_3 = \frac{\pi}{6}$, $\varphi_4 = \frac{\pi}{4}$, $\varphi_5 = \frac{\pi}{2}$ can be seen in Figures 13, 14 and 15.

Figure 7: The plot of $U = U(\varphi)$ for Example 6.1.

Figure 8: The plot of $V = V(\varphi)$ for Example 6.1.

Figure 9: The plot of $\phi = \phi(\varphi)$ for Example 6.1.

Figure 10: The plot of $N = N(\varphi)$ for Example 6.1.

Figure 11: The plot of $S = S(\varphi)$ for Example 6.1.

Figure 12: The plot of $M = M(\varphi)$ for Example 6.1. Figure 13: The plot of $n_\varphi = n_\varphi(r, \varphi_i)$ for Example 6.1. Figure 14: The plot of $n_{r\varphi} = n_{r\varphi}(r, \varphi_i)$ for Example 6.1. Figure 15: The plot of $n_r = n_r(r, \varphi_i)$ for Example 6.1.

Figure 16: Layered composite ring (a) and its cross section (b).

6.2 Three-layer composite ring

The composite ring with uniform thickness and its cross section are shown in Figure 16. The three-layered ring is subjected to compressive point load in radial direction (Figure 16). Only a quarter of the ring has to be modelled because of the double symmetry of the geometry and loading conditions as shown in Figure 17. Geometric and material data are as follows (Figure 16): $a_1 = 0.15$ m, $a_2 = 0.09$ m, $a_3 = 0.06$ m, $a_4 = 0.03$ m, $t = 0.05$ m, $E_1 = 9 \times 10^{10}$ Pa, $E_2 = 2 \times 10^{11}$ Pa, $E_3 = 9 \times 10^{10}$ Pa, $F = 8000$ N, $\alpha = \frac{\pi}{2}$. The next boundary conditions are valid for the quarter of the ring shown in Figure 17:

Figure 17: A quarter of the composite ring ($0 \leq \varphi \leq \frac{\pi}{2}$).

$$\phi(0) = 0, \quad V(0) = 0, \quad N(0) = -\frac{F}{2}, \quad S(0) = 0, \quad (56)$$

$$\phi(\alpha) = 0, \quad V(\alpha) = 0, \quad N(\alpha) = 0, \quad S(\alpha) = -\frac{F}{2}.$$

The unknown initial values $U(0)$ and $M(0)$ can be computed from the next system of linear equations

$$V(\alpha) = U(0)V_1(\alpha) + M(0)V_6(\alpha) + N(0)V_4(\alpha) = 0, \quad (57)$$

$$\phi(\alpha) = U(0)\phi_1(\alpha) + M(0)\phi_6(\alpha) + N(0)\phi_4(\alpha) = 0. \quad (58)$$

Knowing all initial values the solution of the quarter of composite ring is represented as

$$X(\varphi) = U(0)X_1(\varphi) + N(0)X_4(\varphi) + M(0)X_6(\varphi), \quad (59)$$

where $X(\varphi)$ may be $U(\varphi)$, $V(\varphi)$, $\phi(\varphi)$, $N(\varphi)$, $S(\varphi)$, $M(\varphi)$. The plots of $U = U(\varphi)$, $V = V(\varphi)$ and $\phi = \phi(\varphi)$ are shown in Figures 18, 19 and 20. The graphs of $N = N(\varphi)$, $S = S(\varphi)$ and $M = M(\varphi)$ are illustrated in Figures 21, 22 and 23. The plots of average stresses $\bar{\sigma}_\varphi$, $\bar{\tau}_{r\varphi}$ and $\bar{\sigma}_r$ which are

$$\bar{\sigma}_\varphi(r, \varphi) = \frac{n_\varphi(r, \varphi)}{t}, \quad (60)$$

$$\bar{\tau}_{r\varphi}(r, \varphi) = \frac{n_{r\varphi}(r, \varphi)}{t}, \quad \bar{\sigma}_r(r, \varphi) = \frac{n_r(r, \varphi)}{t},$$

as functions of radial coordinate r for cross sections given by polar angles $\varphi_1 = 0$, $\varphi_2 = \frac{\pi}{8}$, $\varphi_3 = \frac{\pi}{6}$, $\varphi_4 = \frac{\pi}{4}$, $\varphi_5 = \frac{\pi}{2}$ are shown in Figures 24, 25 and 26.

Figure 18: The plot of $U = U(\varphi)$ for Example 6.2.

Figure 19: The plot of $V = V(\varphi)$ for Example 6.2.

Figure 20: The plot of $\phi = \phi(\varphi)$ for Example 6.2.

Figure 21: The graph of $N = N(\varphi)$ for Example 6.2.

Figure 22: The graph of $S = S(\varphi)$ for Example 6.2.

Figure 23: The graph of $M = M(\varphi)$ for Example 6.2.

Figure 24: The plot of $\bar{\sigma}_\varphi = \bar{\sigma}_\varphi(r, \varphi_i)$ for Example 6.2. Figure 25: The plot of $\bar{\sigma}_{r\varphi} = \bar{\sigma}_{r\varphi}(r, \varphi_i)$ for Example 6.2. Figure 26: The plot of $\bar{\sigma}_r = \bar{\sigma}_r(r, \varphi_i)$ for Example 6.2.

6.3 Functionally graded curved beam

The functionally graded beam and its cross section and the applied load can be seen in Figure 27. Geometric and material data are as follows (Figure 27): $a = 0.05$ m, $r_c = 0.5$ m, $Y = 2 \times 10^9$ Pa, $F = 40000$ N, $\alpha = \pi$,

$$E(\rho, \lambda) = Y \exp(\lambda \rho), \quad \rho = \sqrt{z^2 + (r - r_c)^2}. \quad (61)$$

The cross sectional properties E_0 and R can be computed

Figure 27: Functionally graded curved beam.

as

$$E_0(\lambda) = \frac{2}{a^2} \int_0^a E(\rho, \lambda) d\rho, \quad H_1(\lambda) = \int_0^a \rho E(\rho, \lambda) d\rho, \quad (62)$$

$$H_2(\lambda) = \int_0^a \frac{\rho E(\rho, \lambda)}{\sqrt{r_c^2 - \rho^2}} d\rho,$$

$$R(\lambda) = \frac{H_1(\lambda)}{H_2(\lambda)}, \quad e(\lambda) = r_c - R(\lambda). \quad (63)$$

The plots of $U = U(\varphi, \lambda)$, $V = V(\varphi, \lambda)$, $\phi = \phi(\varphi, \lambda)$ for $\lambda = -5$, $\lambda = 0$, $\lambda = 5$ are shown in Figures 28, 29 and 30. The internal forces $N = N(\varphi, \lambda)$, $S = S(\varphi, \lambda)$ and bending moment $M = M(\varphi, \lambda)$ does not depend on λ , since the considered beam configuration is statically determinate structure. The normal stress

$$\sigma_\varphi(r, \varphi, z, \lambda) = Y \exp(\lambda \sqrt{(r - r_c)^2 + z^2}) \left[\frac{1}{r} \left(\frac{\partial^2 U}{\partial \varphi^2} + U \right) + \frac{\partial \phi}{\partial \varphi} \right] \quad (64)$$

as a function of r for $\varphi = \frac{\pi}{2}$, $z = 0$, for $\lambda = -5$, $\lambda = 0$ and $\lambda = 5$ are illustrated in Figure 31.

Figure 28: The plot of $U(\varphi, \lambda)$ for Example 6.3.

Figure 29: The plot of $V(\varphi, \lambda)$ for Example 6.3. Figure 30: The plot of $\phi(\varphi, \lambda)$ for Example 6.3.

Figure 31: The plot of normal stress for Example 6.3.

6.4 Illustration of the application of Betti-Rayleigh reciprocity theorem

The aim is to obtain the displacement $V(\alpha)$ and cross sectional rotation $\phi(\alpha)$ for end loaded φ -homogeneous curved beam shown in Figure 32a. we define the second

Figure 32: Three different equilibrium states of curved beam.

and third equilibrium states according to Figures 32b and 32c. For the second equilibrium state we have

$$\begin{aligned} N(0) &= F_2 \cos \varphi, & S(0) &= -F_2 \sin \varphi, \\ M(0) &= 0, & F_2 &= |F_2|. \end{aligned} \quad (65)$$

From the Betti-Rayleigh reciprocity theorem it follows that

$$\begin{aligned} W_{12} &= F_1 F_2 (U_4(\alpha) \cos \alpha - U_5(\alpha) \sin \alpha) \\ &= W_{21} = F_2 V(\alpha) \end{aligned} \quad (66)$$

$$W_{13} = F_1 M_1 U_6(\alpha) = W_{31} = M_1 \phi(\alpha). \quad (67)$$

A simple computation gives the next result for $V(\alpha)$ and $\phi(\alpha)$

$$\begin{aligned} V(\alpha) &= F_1 \frac{R r_c}{2 A E_0 e} \sin^2 \alpha, \\ \phi(\alpha) &= F_1 \frac{R}{A E_0 e} (\cos \alpha - 1). \end{aligned} \quad (68)$$

7 Conclusions

An analytical solution is presented for non-homogeneous curved beam. In-plane deformation is considered and the classical Euler-Bernoulli beam theory is used. The solution of the considered equilibrium problems is based on the fundamental solutions. Paper presents the nine

types of fundamental solutions and provides a formulation of the Betti-Rayleigh type reciprocity relation for non-homogeneous curved beam. The presented numerical results can be used as benchmark solutions to check the validity of solutions obtained by other methods.

Acknowledgement: The described study was carried out as a part of the EFOP-3.6.1-16-2016-00011 "Younger and Renewing University – Innovative Knowledge City – institutional development of the University of Miskolc aiming at intelligent specialisation" project implemented in the framework of the Szechenyi 2020 program. The realization of this project is supported by the European Union, co-financed by the European Social Fund and supported by the National Research, Development and Innovation Office - NKFIH, K115701.

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