Research Article

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Analysis of concrete shallow funicular shells of rectangular plan

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Abstract: Analysis of concrete shallow funicular shells of rectangular plan with simply supported boundary conditions under static loads is performed using the Ritz method. Double Fourier series with the unknown constant coefficients are assumed for the displacement components of the shell and their unknown coefficients are determined such that the potential energy of the shell becomes minimum. The solution is presented in a simple form and is suitable for practical applications. The responses of rectangular-plan concrete shallow funicular shells including deflections, strains, internal forces, internal moments and stresses could be easily determined using the proposed semi-analytical method. The Ritz method results are verified against the finite element method results.

Keywords: Concrete shallow funicular shells; Ritz method; Finite element method

1 Introduction

Shells can be considered as curved plates and double curved shells resist forces efficiently. Both the material resistance and the actual building shape contribute to the strength of double curved shell structures. Design of funicular shells as a special type of shells differs from other shell structures. In the design of funicular shells, the type of stresses due to a selected load is chosen first and then the geometry of the shell is determined. The funicular state exists only for the selected load. Performance of concrete funicular shells under dead weight is chosen to be compressive. Thus, concrete funicular shells carry their dead weight by pure compression where no tensile and shear stresses are generated in the shell. Concrete funicular

shells as roof ceilings should be analyzed for concentrated and distributed static loads in addition to their dead weight. Therefore, presenting a semi-analytical method to analyze concrete funicular shells due to static loads could be useful.

The performance of rectangular-plan concrete shallow funicular shells has been considered in previous studies. [15] investigated the behavior of square-plan concrete shallow funicular shells by constructing ten funicular shell units and testing them under concentrated load. The experimental results were compared with the finite element method results. Good agreement was found between results of two methods at zones away from the supports while two methods had noticeable differences at zones closer to the supports. [1] studied the performance of shallow funicular shells of rectangular plan under a uniformly distributed load. The analysis was done by using a program which was developed by using the finite element method. An isoparametric element with eight nodes and with the assumption of five degrees of freedom at each node was used. Clamped boundary conditions were considered. [3] presented a solution to obtain the shape of deep funicular shells which is more difficult to define than the configuration of shallow funicular shells. The boundary integral element method was applied to solve the nonlinear governing equations. A very good match was found between the results of the boundary integral element method and the finite element method for deep funicular shells of circular plan.

[7] studied the performance of concrete shallow funicular shells of square plan with 0%, 2%, 4%, 6%, 8%, 10%, 12%, and 16% openings by using the finite element method. The ratio of span to rise for all models was equal to 12. It was found that an increase in the percentage of openings caused a drastic increase in the deflections up to 800%. Also, with the increase in the percentage of openings, the membrane and bending stresses increased up to a maximum of 6% and 60%, respectively. The effect of the span to rise ratio on the performance of rectangular-plan concrete shallow funicular shells was carried out by [2]. The analysis was performed via the finite element method. The best performance of funicular shell units was found

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for the ratio of span to rise ranged from 5 to 12.5. [11] considered the behavior of concrete shallow funicular shells under concentrated load with the finite element method. Six funicular shell units with a rectangular base of 1×0.7 m, the rise of 5 and 10 cm, and the thickness of 2, 4, and 5 cm were studied and each unit was compared with a slab of the same dimensions. It was observed that the deflection and the membrane stresses decrease when the rise and the thickness of the shell units increase. The deflection of the shell units founded 34% to 83% less than the deflection of the slab units. Also, some experimental studies considered the ultimate load of concrete shallow funicular shells of rectangular plan with different geometric parameters [8–10, 12–14].

Furthermore, [6] investigated the forced vibration analysis of rectangular-plan concrete shallow funicular shells for simply supported and clamped boundary conditions under impulse loads. The displacement components were expanded by double Fourier series and the kinetic energy, as well as the strain energy, was determined. The governing equations of motion were obtained using the Lagrange equations and were solved via the modal analysis technique. The results indicated that tensile stresses are generated in funicular shells under impulse loads. Moreover, the effect of the plan dimensions and the rise of shells on the performance of these shells were considered. Also, the geometrically nonlinear vibration of concrete shallow funicular shells of rectangular plan was considered by [5] using an approach similar to that used in the previous research. The fourth order Runge-Kutta method was used to solve the nonlinear equations of motion.

In this paper, the performance of rectangular-plan concrete shallow funicular shells under static loads for simply supported conditions is considered using the Ritz method. The displacement components of the funicular shell are predicted with double Fourier series which satisfies the considered boundary conditions. The strain energy and the work performed by external loads are determined in terms of the coefficients of the assumed Fourier series and are presented in a simplified form. Then, based on the principle of minimum potential energy, these coefficients are obtained in such a way that the potential energy of the shell becomes minimum. In this step, a system of linear equations in terms of the coefficients of the Fourier series is achieved which is solved and finally the displacement components and then the strains and the stresses are determined. The finite element method results are obtained close to the results of the Ritz method which approves the validity of the solution.

2 Ritz method

The potential energy of the shell (Π) consists of the potential energy of the external forces (Ω) and the strain energy (U) as the potential energy of the internal forces:

$$\Pi = \Omega + U \quad (1)$$

The strain energy of rectangular-plan shallow funicular shells could be expressed in terms of their middle surface displacement components and their middle surface geometry equation according to the shallow shell theory [6]:

$$\begin{aligned} U = & \frac{Eh}{2(1-\nu^2)} \iint_{00}^{ab} \left\{ \left(\frac{\partial u}{\partial x} \right)^2 - 2 \frac{\partial u}{\partial x} r w + r^2 w^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right. \\ & - 2 \frac{\partial v}{\partial y} t w + t^2 w^2 + 2 \nu \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - 2 \nu \frac{\partial u}{\partial x} t w - 2 \nu \frac{\partial v}{\partial y} r w \\ & + 2 \nu r t w^2 + \frac{1-\nu}{2} \left[\left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 + 4 s^2 w^2 + 2 \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right. \\ & \left. \left. - 4 \frac{\partial u}{\partial y} s w - 4 \frac{\partial v}{\partial x} s w \right] \right\} dx dy + \frac{Eh^3}{24(1-\nu^2)} \iint_{00}^{ab} \left[\left(\frac{\partial^2 w}{\partial x^2} \right)^2 \right. \\ & \left. + \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + 2 \nu \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + 2(1-\nu) \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] dx dy \end{aligned} \quad (2)$$

Where, E is the Young's modulus, h is the thickness of the shell, ν is the Poisson's ratio, a and b are the plan dimensions in x and y directions and u , v and w are the middle surface displacement components in the directions of x , y and z , respectively. Also, w is taken positive downwards. Furthermore,

$$r = \frac{\partial^2 Z}{\partial x^2}, \quad s = \frac{\partial^2 Z}{\partial x \partial y}, \quad t = \frac{\partial^2 Z}{\partial y^2}. \quad (3)$$

Where, $Z=f(x,y)$ is the middle surface equation of shallow funicular shells which is given for shells on rectangular plan as [4]:

$$Z = -H \left[1 - \frac{(x-a/2)^2}{(a/2)^2} \right] \left[1 - \frac{(y-b/2)^2}{(b/2)^2} \right] \quad (4)$$

Where, H is the rise of the shell which is the height of the shell at the middle point of the plan ($x=a/2$ and $y=b/2$). Figure 1 shows the surface that is depicted according to the Eq. (4).

The middle surface displacement components of the shell are approximated by the following double Fourier series:

$$u(x, y) = \sum_{m=1}^M \sum_{n=1}^N c u_{m,n} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \quad (5)$$

$$v(x, y) = \sum_{m=1}^M \sum_{n=1}^N c v_{m,n} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \quad (6)$$

$$w(x, y) = \sum_{m=1}^M \sum_{n=1}^N c w_{m,n} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \quad (7)$$

Where, $c u_{m,n}$, $c v_{m,n}$ and $c w_{m,n}$ are unknown constants or unknown coefficients which would be obtained via the minimum potential energy principle. Also, M and N are the numbers of expressions used in the Fourier series in the directions of x and y , respectively. Simply boundary conditions which are considered in this study (Eq. (8) and Eq. (9)), are satisfied by using the mentioned displacement components.

$$u = v = w = M_x = 0, \text{ at } x = 0, a \quad (8)$$

$$u = v = w = M_y = 0, \text{ at } y = 0, b \quad (9)$$

M_x and M_y are the bending moments per unit length. Using the double Fourier series form of the displacement components, the strain energy is obtained as:

$$\begin{aligned} U = & \sum_{m=1}^M \sum_{n=1}^N \left[\alpha_1 c u_{m,n}^2 + \alpha_2 c v_{m,n}^2 + \alpha_3 c w_{m,n}^2 \right] \\ & + \sum_{m=1}^M \sum_{n_a=1}^N \sum_{n_b=1}^N \left[\alpha_4 c w_{m,n_a} c w_{m,n_b} + \alpha_5 c v_{m,n_a} c w_{m,n_b} \right] \\ & + \sum_{m_i=1}^M \sum_{m_j=1}^M \sum_{n=1}^N \left[\alpha_6 c w_{m_i,n} c w_{m_j,n} + \alpha_7 c u_{m_i,n} c w_{m_j,n} \right] \\ & + \sum_{m_i=1}^M \sum_{m_j=1}^M \sum_{n_a=1}^N \sum_{n_b=1}^N \left[\alpha_8 c w_{m_i,n_a} c w_{m_j,n_b} + \alpha_9 c u_{m_i,n_a} c v_{m_j,n_b} \right. \\ & \left. + \alpha_{10} c u_{m_i,n_a} c w_{m_j,n_b} + \alpha_{11} c v_{m_i,n_a} c w_{m_j,n_b} \right] \end{aligned} \quad (10)$$

Where, the coefficients α_1 to α_{11} are presented by [6] and are reported in the Appendix.

The potential energy of the external forces is defined by:

$$\Omega = - \int_0^a \int_0^b (q_x u + q_y v + q_z w) dx dy \quad (11)$$

Where, q_x , q_y and q_z are the distributed forces per unit area applying in x , y and normal directions, respectively. According to Eq. (11) and Eq. (7), the potential energy of the normal central concentrated force (p) and the potential energy of the normal distributed force (q) which is distributed uniformly throughout the entire surface are defined as:

$$\Omega = -p \sum_{m=1}^M \sum_{n=1}^N c w_{m,n} \quad (12)$$

$$\begin{aligned} \Omega = & -q \sum_{m=1}^M \sum_{n=1}^N c w_{m,n} \frac{ab}{mn\pi^2} \\ & [\cos(m\pi) - 1][\cos(n\pi) - 1] \end{aligned} \quad (13)$$

Figure 1: Funicular surface on rectangular plan

Using Eqs. (10), (12) and (13), the potential energy of the shell is obtained as a function of the unknown coefficients. Now, the unknown coefficients are determined by the minimum potential energy principle. The unknown coefficients should be obtained such that the potential energy becomes minimum. Therefore,

$$\begin{aligned} \frac{\partial \Pi}{\partial c u_{1,1}} = 0, \quad \frac{\partial \Pi}{\partial c v_{1,1}} = 0, \quad \frac{\partial \Pi}{\partial c w_{1,1}} = 0, \dots, \\ \frac{\partial \Pi}{\partial c u_{m,n}} = 0, \quad \frac{\partial \Pi}{\partial c v_{m,n}} = 0, \quad \frac{\partial \Pi}{\partial c w_{m,n}} = 0, \dots \end{aligned} \quad (14)$$

Applying the above equations, a linear system of equations is obtained for the unknown coefficients that will be solved and the unknown coefficients will be determined. Then, the displacement components are obtained through Eqs. (5)-(7). Also, the strains and the stresses of the shell could be determined (see the governing equations in the Appendix).

3 Results and discussions

Table 1 presents the central deflection (the middle surface displacement component in the direction of z) of four funicular shell models due to a uniformly distributed load, which is equal to 85 kPa, obtained by the Ritz and the finite element methods. Table 1 indicates good agreement between the results of these two methods (the difference between the results of these two methods is less than 20%). It should be noted that the geometric and material nonlinearities are not considered in the Ritz method and its results are valid only for the linear elastic cases. The results of the Ritz method presented in Table 1 show that the deflections of funicular shells are small compared with the thickness of the shells and therefore, the large deflection theory which considers the geometric nonlinearity is

Table 1: Normal displacement at the center of four funicular shell models under a uniformly distributed load of 85 kPa

Shell Model Number	Normal Displacement (mm)		
	Ritz Method	FEM	
		Linear Analysis	Geometrically Nonlinear Analysis
1	15.67	17.48	16.2
2	10.95	11.3	14.3
3	7.4	9.4	8.8
4	6.74	6.74	6.8

Table 2: Compressive stresses at the center of four funicular shell models under a uniformly distributed load of 85 kPa

Shell Model Number	σ_x (MPa)		σ_y (MPa)	
	Ritz Method	FEM	Ritz Method	FEM
1	12.32	12.24	12.32	12.24
2	6.2	5.5	12.5	12.45
3	12.06	12.19	12.06	12.19
4	9.1	7.6	10.6	10.9

not necessary to be used. Also, on the other hand, Table 1 shows that the linear finite element analysis results are close to the geometrically nonlinear finite element analysis results for the considered load magnitude, which is a large considerable load and therefore, the effect of the geometric nonlinearity could be ignored.

For the funicular shells considered in Table 1, it is seen that stresses generated in these shells are mainly compressive. It is due to the fact that the considered distributed load has the same load pattern with the shells' dead weight for which the funicular state exists. Table 2 presents the compressive stress generated at the center of the considered funicular shell models determined with the Ritz and the finite element methods. Table 2 again shows that the proposed Ritz method predicts the performance of funicular shells close to the finite element results. Also, the results presented in Table 2 indicates that the assumption of the linear elastic behavior of the concrete material of the considered funicular shells for the considered load magnitude is acceptable because the maximum compressive stress generated in the considered funicular shells are approximately smaller than the half of the compressive strength of the concrete material, which is the known limit of the linear elastic behavior and is equal to 12 MPa in this study.

Table 3: Geometric characteristics of shell models

Shell Model Number	a (m)	b (m)	H (m)	h (cm)
1	20	20	1.8	10
2	20	10	0.9	10
3	20	20	3.6	5
4	20	10	1.8	5

Table 4: Central normal displacement of funicular shell models for two different cases

Shell Model Number	Normal Displacement (mm)	
	$M=N=20$	$M=N=30$
1	15.67	15.66
2	10.95	10.95
3	7.4	7.35
4	6.74	6.76

The geometric characteristics of the considered funicular shell models are presented in Table 3. Also, the elastic Young's modulus and the Poisson's ratio of the concrete material for all models are equal to 23.5 GPa and 0.2.

The Ritz method results presented in Tables 1 and 2 are obtained by considering twenty expressions in the Fourier series of displacement components in the directions of x and y ($M=20$, $N=20$). The convergence of the Fourier series with $M=20$ and $N=20$ is investigated. The deflection at the center of the funicular shell models is presented in Table 4 for two different cases. The number of expressions considered in both x and y directions for case 1 is twenty ($M=20$, $N=20$) while for case 2, thirty expressions ($M=30$, $N=30$) are considered. Good agreement between the results of two cases in Table 4 shows the convergence of the Fourier series with $M=20$ and $N=20$.

The response of rectangular-plan concrete shallow funicular shells due to arbitrary static loads could be obtained using the proposed Ritz method. Table 5 presents the comparison of the central deflection of the funicular shell models between the results of the Ritz and the finite element methods where good agreement between these two methods is obtained. The Results presented in Table 5 are determined due to a concentrated load, which is equal to 30 kN. Also, Table 6 shows the stresses at the center of the funicular shell models. Tensile stresses are smaller than the one-tenth of the compressive strength of the concrete material, which is the tensile strength of the concrete, except for the shell model number four.

Table 5: Normal displacement at the center of four funicular shell models under a central concentrated load of 30 kN

Shell Model Number	Normal Displacement (mm)	
	Ritz Method	FEM
1	1.4	1.43
2	1.23	1.22
3	2.28	2.2
4	2.47	2.17

Table 6: Stresses at the center of four funicular shell models under a central concentrated load of 30 kN

Shell Model Number	σ_x (MPa)		σ_y (MPa)	
	C*	T**	C*	T**
1	2.58	1.39	2.58	1.39
2	2.45	1.4	3.36	2.18
3	5.4	1.53	5.4	1.53
4	5.7	1.3	8.2	4.8

*Compressive

**Tensile

4 Summary and conclusions

A semi-analytical method is presented in order to conduct the analysis of rectangular-plan concrete shallow funicular shells under static loads. The results of the semi-analytical method are verified against the finite element method results. Defining the exact configuration of funicular shells in a finite element software is complicated and it is the difficulty of analysis of funicular shells with finite element method. Using the proposed semi-analytical method, one can easily determine the responses (including deflections, strains, internal forces, internal moments and stresses) of funicular shells of rectangular plan with different geometric dimensions due to static loads. It should be noted that the geometric and material nonlinearities are not considered in the proposed method and its results are valid only for the linear elastic cases. Of course, the results show that the behavior of the funicular shells could be considered linear elastic even due to large concentrated and distributed loads.

References

- [1] Elangovan, S. (1990). Analysis of funicular shells by the isoparametric finite element. *Computers & Structures*, 34, 303-311.

- [2] Lakshmikantham, K. N., Sivakumar, P., Theresa Jose, L., Sivasubramanian, K., Balasubramanian S. R., Saibabu, S. (2014). Parametric study on development, testing and evaluation of concrete funicular shells. *International Journal of Engineering and Innovative Technology (IJEIT)*, 3, 183-191.
- [3] Rajasekaran, S., Sujatha, P. (1992). Configuration of deep funicular shells by boundary integral element method. *Computers & Structures*, 44, 213-221.
- [4] Ramaswamy, G. S. (1968). *Design and construction of concrete shell roofs*, New York: McGraw-Hill.
- [5] Sabermahany, H., Mofid, M., Daneshmand, N. (in press). Geometrically non-linear vibration of concrete shallow funicular shells. *Proceedings of the Institution of Civil Engineers – Structures and Buildings*, <https://doi.org/10.1680/jstbu.18.00115>.
- [6] Sabermahany, H., Vafai, A., Mofid, M. (2018). Dynamic response of concrete funicular shells with rectangular base under impulse loads. *Scientia Iranica*, <https://doi.org/10.24200/SCI.2018.20643>.
- [7] Sachithanatham, P. (2012a). Study of shallow funicular concrete shells of plan to rise ratio 1:2. *International Journal of Biotech Trends and Technology (IJBT)*, 2, 53-64.
- [8] Sachithanatham, P., (2012b). Study of geo-grid reinforced shallow funicular concrete shells subjected to ultimate loads. *International Journal of Biotech Trends and Technology (IJBT)*, 2, 34-46.
- [9] Sachithanatham, P., Sankaran, S., Elavenil, S. (2014). Study on shallow funicular concrete shells over rectangular ground plan ratio 1:0.9. *International Journal of Emerging Technology and Advanced Engineering (IJETA)*, 4, 102-107.
- [10] Sachithanatham, P., Sankaran, S., Elavenil, S. (2015). Influence of rise on ultimate load of shallow funicular concrete shells over rectangular ground plan ratio 1:0.6. *International Journal of Scientific & Engineering Research*, 6, 1467-1471.
- [11] Siddesh, T. M., Harish, B. A., Manjunatha, K. (2016). Finite element analysis of funicular shells with rectangular plan ratio 1:0.7 under concentrated load using SAP2000. *International Research Journal of Engineering and Technology (IRJET)*, 3, 873-878.
- [12] Sivakumar, P., Manjunatha, K., Harish, B. A. (2015). Experimental and FE analysis of funicular shells. *International Journal of Engineering and Innovative Technology (IJEIT)*, 4, 178-186.
- [13] Tarunkumar, T., Sachithanatham, P., (2012). Study on shallow funicular concrete shells over rectangular ground plan ratio 1:0.8. *International Journal of Computer Trends and Technology (IJCTT)*, 3, 29-49.
- [14] Vafai, A., Estekanchi, H. E., Javidruz, M. (2005). Experimental study of prefabricated funicular shell units. *Proceedings of the Institution of Civil Engineers - Structures and Buildings*, 158, 167-174.
- [15] Vafai, A., Farshad, M. (1979). Theoretical and experimental study of prefabricated funicular shell units. *Building and Environment*, 14, 209-216.

Appendix

Definitions of α_1 to α_{11} :

$$\alpha_1 = \frac{Eh}{2(1-\nu^2)} \left[\frac{m^2 \pi^2 b}{4a} + (1-\nu) \frac{n^2 \pi^2 a}{8b} \right] \quad (\text{A.1})$$

$$\alpha_2 = \frac{Eh}{2(1-\nu^2)} \left[\frac{n^2 \pi^2 a}{4b} + (1-\nu) \frac{m^2 \pi^2 b}{8a} \right] \quad (\text{A.2})$$

$$\begin{aligned} \alpha_3 = & \frac{Eh^3}{24(1-\nu^2)} \left[\frac{m^4 \pi^4 b}{4a^3} + \frac{n^4 \pi^4 a}{4b^3} + \frac{m^2 n^2 \pi^4}{2ab} \right] \quad (\text{A.3}) \\ & + \frac{Eh}{2(1-\nu^2)} \left[\frac{512H^2}{a^3 b^3} \right] \left\{ b^4 \left[\frac{1}{60} + \frac{3}{4n^4 \pi^4} \right] + \right. \\ & a^4 \left[\frac{1}{60} + \frac{3}{4m^4 \pi^4} \right] + 4\nu a^2 b^2 \left[\frac{1}{12} + \frac{1}{4m^2 \pi^2} \right] \\ & \left[\frac{1}{12} + \frac{1}{4n^2 \pi^2} \right] + (1-\nu) a^2 b^2 \left[\frac{1}{6} - \frac{1}{m^2 \pi^2} \right] \\ & \left. \left[\frac{1}{6} - \frac{1}{n^2 \pi^2} \right] \right\} \end{aligned}$$

$$\begin{aligned} \alpha_4 = & \frac{Eh}{2(1-\nu^2)} \left[-\frac{1024H^2}{a^3 b^3} \right] \left\{ \frac{16\nu a^2 b^2 n_a n_b}{\pi^2 (n_a^2 - n_b^2)^2} \quad (\text{A.4}) \right. \\ & \left[\frac{1}{12} + \frac{1}{4m^2 \pi^2} \right] + \frac{96b^4 n_a n_b (n_a^2 + n_b^2)}{\pi^4 (n_a^2 - n_b^2)^4} \\ & \left. - (1-\nu) \frac{16a^2 b^2 n_a n_b}{\pi^2 (n_a^2 - n_b^2)^2} \left[\frac{1}{6} - \frac{1}{m^2 \pi^2} \right] \right\} \end{aligned}$$

In Eq. (A.4), both indices of n , a and b , should be even or odd. Otherwise, this equation will be equal to zero.

$$\begin{aligned} \alpha_5 = & \frac{Eh}{2(1-\nu^2)} \left[-\frac{128H}{a^2 b^2} \right] \quad (\text{A.5}) \\ & \left\{ \frac{\nu a b^2 n_a n_b (3n_a^2 + n_b^2)}{\pi^2 (n_a^2 - n_b^2)^3} + \frac{a^3 n_a n_b}{(n_a^2 - n_b^2)} \right. \\ & \left. \left[\frac{1}{12} + \frac{1}{4m^2 \pi^2} \right] + (1-\nu) \frac{a b^2 n_a n_b}{\pi^2 (n_a^2 - n_b^2)^2} \right\} \end{aligned}$$

In Eq. (A.5), one of the indices of n , a or b , should be even and the other one should be odd. Otherwise, this equation will be equal to zero.

$$\begin{aligned} \alpha_6 = & \frac{Eh}{2(1-\nu^2)} \left[-\frac{1024H^2}{a^3 b^3} \right] \left\{ \frac{16\nu a^2 b^2 m_i m_j}{\pi^2 (m_i^2 - m_j^2)^2} \quad (\text{A.6}) \right. \\ & \left[\frac{1}{12} + \frac{1}{4n^2 \pi^2} \right] + \frac{96a^4 m_i m_j (m_i^2 + m_j^2)}{\pi^4 (m_i^2 - m_j^2)^4} \\ & \left. - (1-\nu) \frac{16a^2 b^2 m_i m_j}{\pi^2 (m_i^2 - m_j^2)^2} \left[\frac{1}{6} - \frac{1}{n^2 \pi^2} \right] \right\} \end{aligned}$$

In Eq. (A.6), both indices of m , i and j , should be even or odd. Otherwise, this equation will be equal to zero.

$$\begin{aligned} \alpha_7 = & \frac{Eh}{2(1-\nu^2)} \left[-\frac{128H}{a^2 b^2} \right] \quad (\text{A.7}) \\ & \left\{ \frac{\nu b a^2 m_i m_j (3m_i^2 + m_j^2)}{\pi^2 (m_i^2 - m_j^2)^3} + \frac{b^3 m_i m_j}{(m_i^2 - m_j^2)} \right. \\ & \left. \left[\frac{1}{12} + \frac{1}{4n^2 \pi^2} \right] + (1-\nu) \frac{a^2 b m_i m_j}{\pi^2 (m_i^2 - m_j^2)^2} \right\} \end{aligned}$$

In Eq. (A.7), one of the indices of m , i or j , should be even and the other one should be odd. Otherwise, this equation will be equal to zero.

$$\begin{aligned} \alpha_8 = & \frac{Eh}{2(1-\nu^2)} \left[\frac{1024H^2}{25ab} \right] \quad (\text{A.8}) \\ & \left[\frac{1600\nu m_i m_j n_a n_b}{\pi^4 (m_i^2 - m_j^2)^2 (n_a^2 - n_b^2)^2} \right. \\ & \left. + \frac{6400(1-\nu) m_i m_j n_a n_b}{\pi^2 (m_i^2 - m_j^2)^2 (n_a^2 - n_b^2)^2} \right] \end{aligned}$$

In Eq. (A.8), both indices of m , i and j , should be even or odd and similarly both indices of n , a and b , should be even or odd. Otherwise, this equation will be equal to zero.

$$\alpha_9 = \frac{Eh}{2(1-\nu^2)} \left[-\frac{4(1+\nu) m_i m_j n_a n_b}{(m_i^2 - m_j^2) (n_a^2 - n_b^2)} \right] \quad (\text{A.9})$$

In Eq. (A.9), one of the indices of m , i or j , should be even and the other one should be odd, and similarly, one of the indices of n , a or b , should be even and the other one should be odd. Otherwise, this equation will be equal to zero.

$$\begin{aligned} \alpha_{10} = & \frac{Eh}{2(1-\nu^2)} \left[\frac{512H}{a^2 b^2} \right] \quad (\text{A.10}) \\ & \left[\frac{b^3 m_i m_j n_a n_b}{\pi^2 (n_a^2 - n_b^2)^2 (m_i^2 - m_j^2)} \right. \\ & \left. + \frac{(1-\nu) a^2 b m_i m_j n_a n_b}{\pi^2 (m_i^2 - m_j^2)^2 (n_a^2 - n_b^2)} \right] \end{aligned}$$

In Eq. (A.10), one of the indices of m , i or j , should be even and the other one should be odd and both indices of n , a and b , should be even or odd. Otherwise, this equation will

be equal to zero.

$$\alpha_{11} = \frac{Eh}{2(1-\nu^2)} \left[\frac{512H}{a^2b^2} \right] \quad (\text{A.11})$$

$$\left[\frac{a^3 m_i m_j n_a n_b}{\pi^2 (m_i^2 - m_j^2)^2 (n_a^2 - n_b^2)} + \frac{(1-\nu)b^2 a m_i m_j n_a n_b}{\pi^2 (n_a^2 - n_b^2)^2 (m_i^2 - m_j^2)} \right]$$

In Eq. (A.11), both indices of m , i and j , should be even or odd and one of the indices of n , a or b should be even and the other one should be odd. Otherwise, this equation will be equal to zero.

The Governing Equations

The relations between the middle surface strains (ϵ_x , ϵ_y and γ_{xy}) and the changes in curvatures (χ_x , χ_y and χ_{xy}) with the middle surface displacement components based on the shallow shell theory are expressed as:

$$\epsilon_x = \frac{\partial u}{\partial x} - rw \quad (\text{A.12})$$

$$\epsilon_y = \frac{\partial v}{\partial y} - tw \quad (\text{A.13})$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2sw \quad (\text{A.14})$$

$$\chi_x = \frac{\partial^2 w}{\partial x^2} \quad (\text{A.15})$$

$$\chi_y = \frac{\partial^2 w}{\partial y^2} \quad (\text{A.16})$$

$$\chi_{xy} = \frac{\partial^2 w}{\partial x \partial y} \quad (\text{A.17})$$

Also, the relations between the internal forces (N_x , N_y and N_{xy}) and the internal moments (M_x , M_y and M_{xy}) with the middle surface strains and curvatures are given by:

$$N_x = \frac{Eh}{1-\nu^2} (\epsilon_x + \nu\epsilon_y) \quad (\text{A.18})$$

$$N_y = \frac{Eh}{1-\nu^2} (\epsilon_y + \nu\epsilon_x) \quad (\text{A.19})$$

$$N_{xy} = N_{yx} = \frac{Eh}{2} \frac{1-\nu}{1-\nu^2} \gamma_{xy} \quad (\text{A.20})$$

$$M_x = -D (\chi_x + \nu\chi_y) \quad (\text{A.21})$$

$$M_y = -D (\chi_y + \nu\chi_x) \quad (\text{A.22})$$

$$M_{xy} = -D (1-\nu) \chi_{xy} \quad (\text{A.23})$$

Finally, the relations between stress components with the internal forces and moments are given by:

$$\sigma_x^z = \frac{N_x}{h} \pm \frac{12M_x}{h^3} z \quad (\text{A.24})$$

$$\sigma_y^z = \frac{N_y}{h} \pm \frac{12M_y}{h^3} z \quad (\text{A.25})$$

$$\tau_{xy}^z = \frac{1}{2h} (N_{xy} + N_{yx}) \pm \frac{6}{h^3} (M_{xy} + M_{yx}) z \quad (\text{A.26})$$

Where, z is the distance from the middle surface.