Research Article

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Numerical Approximations for the Variable Coefficient Fractional Diffusion Equations with Non-smooth Data

<https://doi.org/10.1515/cmam-2019-0038>

Received February 24, 2019; revised October 17, 2019; accepted November 4, 2019

Abstract: In this article, we study the numerical approximation of a variable coefficient fractional diffusion equation. Using a change of variable, the variable coefficient fractional diffusion equation is transformed into a constant coefficient fractional diffusion equation of the same order. The transformed equation retains the desirable stability property of being an elliptic equation. A spectral approximation scheme is proposed and analyzed for the transformed equation, with error estimates for the approximated solution derived. An approximation to the unknown of the variable coefficient fractional diffusion equation is then obtained by post-processing the computed approximation to the transformed equation. Error estimates are also presented for the approximation to the unknown of the variable coefficient equation with both smooth and non-smooth diffusivity coefficient and right-hand side. Numerical experiments are presented to test the performance of the proposed method.

Keywords: Fractional Diffusion Equation, Jacobi Polynomials, Spectral Method

MSC 2010: 65N30, 35B65, 41A10, 33C45

1 Introduction

It has been shown that fractional partial differential equations (PDEs) can accurately model challenging phenomena including anomalous transport, long-range time memory and spatial interactions [1, 20]. Extensive research has been conducted on fractional PDEs in terms of their modeling, analysis, numerical approximations and applications. Space fractional diffusion process may stem from the so-called Lévy–Feller diffusion process, a Markovian process governed by a stable probability density function evolving in time, with a specific spatial Fourier transform expressed in terms of the Feller operator [6, 8, 9]. The explicit form of the Feller operator is the two-sided Riemann–Liouville space-fractional differential operator [21], which motivates the studies of the corresponding space fractional diffusion equation (FDE) models. In a representative piece of work, Ervin and Roop [4] proved the well-posedness of the Galerkin weak formulation of linear elliptic space FDEs of order $1 < \alpha < 2$ on the Sobolev space $H_0^{\alpha/2}$. They also proved optimal-order error estimates of its finite element approximations in the energy and L^2 norms, assuming that the true solution and the solution to the dual problem for an L^2 right-hand side have full regularity. However, it was later realized that the smoothness

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of the coefficients and source term for these space fractional differential equations cannot ensure the smoothness of their solution [3, 15, 26, 27]. This is in sharp contrast to integer-order linear elliptic PDEs [5, 7]. For this reason, the usual smoothness assumptions on the true solutions to fractional PDEs in the analysis of the numerical approximations are inappropriate.

It turns out that the spectral methods are particularly well suited for the accurate approximation of FDEs as they provide a clean expression of the true solution to FDEs for the convenience of analysis [3, 18] and ordinarily lead to a diagonal stiffness matrices (at least for constant-coefficient FDEs). This is in contrast to the dense stiffness matrices generated from the finite element, finite difference or finite volume approximations. Mao, Chen and Shen [17] analyzed the regularity of the solution to a symmetric case of the FDE and developed corresponding spectral methods. The solution structure to the general case was resolved completely in [3], in which a spectral method utilizing the weighted Jacobi polynomial was studied and a priori error estimates derived. The two-sided FDE with constant coefficient and Riemann–Liouville fractional derivative was investigated in [18], by employing a Petrov–Galerkin projection in a properly weighted Sobolev space using two-sided Jacobi polyfractionomials as test and trial functions. In [11, 13], the regularity of the two-sided fractional reaction-diffusion and advection-diffusion-reaction equations are analyzed in the weighted Sobolev spaces, based on which the optimal (or sub-optimal) convergence rates of the spectral Galerkin or Petrov–Galerkin method are proved.

The variable diffusivity K presents another bottleneck of FDEs. It was shown in [25] that the Galerkin weak formulation may lose its coercivity for a smooth $K(x)$ with positive lower and upper bounds, which increases the difficulties for the stability and convergence analysis and accurate simulations. To circumvent this issue, an indirect Legendre spectral Galerkin method was developed for the FDE in [27], in which the high-order convergence rates of numerical approximations were proved only under the regularity assumptions of coefficients and right-hand side term. In [16], with the introduction of an auxiliary variable, a mixed approximation was developed for an FDE, and the corresponding error estimates were proved. In [19], a spectral Galerkin method for a different variable coefficient FDE was analyzed, in which the outside and inside fractional derivatives are chosen carefully so that the corresponding Galerkin weak formulation are self-adjoint and coercive. Recently, a finite element approximation for the variable-coefficient two-sided FDEs in one space dimension was presented [12]. By splitting the variable-coefficient fractional diffusion operator into a constant-coefficient fractional diffusion operator added by a low-order term, the coercivity of the corresponding bilinear form was proved under certain constraint on the magnitude of the first-order derivatives of the diffusivity coefficients. A least-squared mixed finite element method for a one-sided variable-coefficient FDE was established in [28] via the direct-sum decomposition of a fractional derivative spaces. Optimal-order convergence rates were proved concerning the true regularity of the solutions.

Recently, the well-posedness of the variable coefficient FDE (2.1) was investigated in [30], in which the existence and uniqueness of the solution to the proposed model were proven for any $f \in L^2_{\omega^{(\beta-1, \alpha-\beta-1)}}$, with the space defined by (2.3). A spectral approximation method was then studied, and several error estimates were derived based on the regularity of the right-hand side term. In this paper, we continue to investigate model (2.1) using a different approach than that used in [30]. We prove in this paper that the model is well-posed for f belonging to a larger space $L^2_{\omega^{(\beta, \alpha-\beta)}}$, which extends the results in [30] and makes the well-posedness and regularity theorems for both the constant-coefficient FDEs proved in [3] and the variable-coefficient FDEs consistent. A spectral approximation scheme is proposed, and the error estimates are proved to be dependent on the weaker norms of f without loss of accuracy. Compared with the standard finite difference or finite element methods referred above, the implementation of the proposed spectral method is simpler as no linear system is introduced. Furthermore, the exponential convergence of the proposed method can be expected for a sufficiently smooth f (cf. Example 5 in Section 4), rather than the algebraic convergence in the finite difference/element methods. Compared with [30], numerical experiments assigned with low-regularity (and even discontinuous) coefficients or sufficiently smooth or non-smooth right-hand side terms are presented to test the performance of the proposed method from different aspects. In addition, we follow the idea of the K -method of interpolation to determine the range of the index of the weighted Sobolev space that the power function belongs to, which provides the theoretical complementation for the conjectures in [30] on estimating the convergence rate of the proposed method.

This paper is organized as follows. In Section 2, we present the formulation of the model and introduce notation and key lemmas used in the analysis. The well-posedness of the model and the regularity of its solution are studied in Section 3, based on which the spectral approximation method is formulated and a detailed analysis of its convergence is proved. Numerical experiments are presented in Section 4, whose results demonstrate the sharpness of the derived error estimates. Some concluding remarks are addressed in Section 5.

2 Model Problem and Preliminaries

In this paper, we consider the following homogeneous Dirichlet boundary-value problem of a two-sided Caputo flux FDE, which is obtained by incorporating a fractional Fick's law into a conventional local mass balance law with a variable fractional diffusivity [2, 29]:

$$-D((r {}_0I_x^{2-\alpha} + (1-r) {}_xI_1^{2-\alpha})K(x)Du(x)) = f(x), \quad x \in I := (0, 1), \quad (2.1)$$

$$u(0) = u(1) = 0. \quad (2.2)$$

Here $1 < \alpha < 2$, D is the first-order differential operator, $K(x)$ is the fractional diffusivity with

$$0 < K_m \leq K(x) \leq K_M < \infty$$

and $0 \leq r \leq 1$ indicates the relative weight of forward versus backward transition probability and $f(x)$ the source or sink term. The left and right fractional integrals of order $0 < \sigma < 1$ are defined as [22]

$${}_0I_x^\sigma w(x) := \frac{1}{\Gamma(\sigma)} \int_0^x w(s)(x-s)^{\sigma-1} ds,$$

$${}_xI_1^\sigma w(x) := \frac{1}{\Gamma(\sigma)} \int_x^1 w(s)(s-x)^{\sigma-1} ds,$$

where $\Gamma(\cdot)$ is the Gamma function.

We introduce notation and properties used subsequently in our discussion of the approximation scheme and in its error analysis.

Let Ω be a bounded open interval, and let $\omega(x) > 0$ (for $x \in \Omega$) be a smooth function. We define the weighted L^2 space $L_\omega^2(\Omega)$ and L^2 weighted inner product as

$$L_\omega^2(\Omega) := \left\{ f(x) : \|f\|_\omega^2 := \int_\Omega \omega(x)f(x)^2 dx < \infty \right\}, \quad (f, g)_\omega := \int_\Omega \omega(x)f(x)g(x) dx. \quad (2.3)$$

In addition, let $\mathbb{N}_0 := \mathbb{N} \cup 0$, and let $\omega^{(a,b)}$ be a weighting function defined on Ω and indexed by a and b for some $a, b > -1$. For any $m \in \mathbb{N}$, we introduce the weighted Sobolev spaces [10, 23]

$$H_{\omega^{(a,b)}}^m(\Omega) := \left\{ v : \|v\|_{m, \omega^{(a,b)}}^2 := \sum_{j=0}^m |v|_{j, \omega^{(a,b)}}^2 = \sum_{j=0}^m \|D^j v\|_{\omega^{(a+j, b+j)}}^2 < \infty \right\}.$$

For $t \in \mathbb{R}^+ \setminus \mathbb{N}_0$, $H_{\omega^{(a,b)}}^t(0, 1)$ is defined by the K -method of interpolation.

The Jacobi polynomials $P_n^{(a,b)}(x)$ are defined by [23, 24]

$$P_n^{(a,b)}(x) := \sum_{k=0}^n p_{n,k}(x-1)^{n-k}(x+1)^k, \quad x \in (-1, 1), \quad p_{n,k} := \frac{1}{2^n} \binom{n+a}{k} \binom{n+b}{n-k}.$$

Let $G_n^{(a,b)}(x)$ denote the translated and dilated Jacobi polynomials to the interval $[0, 1]$:

$$G_n^{(a,b)}(x) := P_n^{(a,b)}(2x-1), \quad x \in [0, 1].$$

We summarize the properties of $G_n^{a,b}$ in the following lemma.

Lemma 2.1. The polynomials $G_n^{(a,b)}$ have the following orthogonality and norm properties:

$$\int_0^1 \omega^{(a,b)}(x) G_j^{(a,b)}(x) G_k^{(a,b)}(x) dx = \delta_{j,k} \|G_j^{(a,b)}\|^2, \quad \omega^{(a,b)}(x) := (1-x)^a x^b, \quad (2.4)$$

where $\delta_{j,k} = 1$ if $j = k$ and 0 otherwise, and

$$\|G_j^{(a,b)}\| := \left(\frac{1}{(2j+a+b+1)} \frac{\Gamma(j+a+1)\Gamma(j+b+1)}{\Gamma(j+1)\Gamma(j+a+b+1)} \right)^{1/2} = \|G_j^{(b,a)}\|. \quad (2.5)$$

In addition, $G_n^{(a,b)}$ satisfies

$$\begin{aligned} D^k G_n^{(a,b)}(x) &= \frac{\Gamma(n+k+a+b+1)}{\Gamma(n+a+b+1)} G_{n-k}^{(a+k,b+k)}(x), \quad 0 \leq k \leq n, \\ D^k (\omega^{(a+k,b+k)}(x) G_{n-k}^{(a+k,b+k)}(x)) &= \frac{(-1)^k n!}{(n-k)!} \omega^{(a,b)}(x) G_n^{(a,b)}(x), \quad 0 \leq k \leq n. \end{aligned} \quad (2.6)$$

Finally, $G_n^{(a,b)}$ have the norm relation

$$\frac{1}{2} \leq \frac{\|G_j^{(a-b,b)}\|^2}{\|G_{j+1}^{(b-1,a-b-1)}\|^2} = \frac{j+1}{j+a} \leq 1, \quad j \geq 0. \quad (2.7)$$

Proof. The orthogonality property (2.4) of $G_n^{(a,b)}$ is a direct consequence of the orthogonality relation of $P_n^{(a,b)}$ (see [23, 24])

$$\int_{-1}^1 \tilde{\omega}^{(a,b)}(x) P_j^{(a,b)}(x) P_k^{(a,b)}(x) dx = \delta_{j,k} \|P_j^{(a,b)}\|_{\tilde{\omega}^{(a,b)}}^2,$$

where $\tilde{\omega}^{(a,b)}(x) := (1-x)^a (1+x)^b$. Using

$$\|P_j^{(a,b)}\|_{\tilde{\omega}^{(a,b)}} = \left(\frac{2^{(a+b+1)}}{(2j+a+b+1)} \frac{\Gamma(j+a+1)\Gamma(j+b+1)}{\Gamma(j+1)\Gamma(j+a+b+1)} \right)^{1/2} = \|P_j^{(b,a)}\|_{\tilde{\omega}^{(b,a)}}$$

and the following relation between $G_n^{(a,b)}$ and $P_n^{(a,b)}$ leads to (2.5):

$$\int_{-1}^1 \tilde{\omega}^{(a,b)}(x) P_j^{(a,b)}(x) P_k^{(a,b)}(x) dx = 2^{a+b+1} \int_0^1 \omega^{(a,b)}(x) G_j^{(a,b)}(x) G_k^{(a,b)}(x) dx.$$

The two equations in (2.6) are direct consequences of the following equations for $P_n^{(a,b)}(x)$ (see [17, equations (2.15) and (2.19)])

$$\begin{aligned} D^k P_n^{(a,b)}(x) &= \frac{\Gamma(n+k+a+b+1)}{2^k \Gamma(n+a+b+1)} P_{n-k}^{(a+k,b+k)}(x), \\ D^k (\tilde{\omega}^{a+k,b+k}(x) P_{n-k}^{(a+k,b+k)}(x)) &= \frac{(-1)^k 2^k n!}{(n-k)!} \tilde{\omega}^{a,b}(x) P_n^{(a,b)}(x). \end{aligned}$$

The norm relation (2.7) is derived in [30]. □

Let $\mathcal{S}_N$ denote the space of polynomials of degree $\leq N$. We define the weighted L^2 orthogonal projection $P_{N,a,b} : L_{\omega^{(a,b)}}^2(0,1) \rightarrow \mathcal{S}_N$ by the condition

$$(v - P_{N,a,b} v, \phi_N)_{\omega^{(a,b)}} = 0 \quad \text{for all } \phi_N \in \mathcal{S}_N.$$

Lemma 2.2 ([10, Theorem 2.1]). For $\mu \in \mathbb{N}_0$ and $v \in H_{\omega^{(a,b)}}^t(0,1)$, with $0 \leq \mu \leq t$, there exists a constant C , independent of v , N , a and b such that

$$\|v - P_{N,a,b} v\|_{\mu, \omega^{(a,b)}} \leq C(N(N+a+b))^{\frac{\mu-t}{2}} |v|_{t, \omega^{(a,b)}}. \quad (2.8)$$

Remark. In [10], (2.8) is stated for $t \in \mathbb{N}_0$. The result extends to $t \in \mathbb{R}^+$ using interpolation.

3 Approximation Scheme

3.1 Motivation for the Approximation Scheme

Introduce $D^{-1}: L^1(I) \rightarrow H^1(I)$, defined by $D^{-1}g(x) := \int_0^x g(s) ds$. Rewrite (2.1) as

$$-D((r_0 I_x^{2-\alpha} + (1-r) {}_x I_1^{2-\alpha}) D \underbrace{D^{-1}K(x)Du(x)}_{:=\widetilde{w}(x)}) = f(x).$$

Consider $\widetilde{w}(x) = D^{-1}K(x)Du(x) = \int_0^x K(s)Du(s) ds$. Note that $\widetilde{w}(0) = 0$, and we apply $u(0) = 0$ to get

$$u(x) = \int_0^x \frac{D\widetilde{w}(s)}{K(s)} ds. \quad (3.1)$$

Then the condition $u(1) = 0$ yields

$$\int_0^1 \frac{D\widetilde{w}(s)}{K(s)} ds = 0. \quad (3.2)$$

Hence if we could determine $\widetilde{w}(x)$ such that

$$\mathcal{L}_r^\alpha \widetilde{w}(x) := -D((r_0 I_x^{2-\alpha} + (1-r) {}_x I_1^{2-\alpha}) D\widetilde{w}(x)) = f(x), \quad x \in I,$$

$$\text{subject to } \widetilde{w}(0) = 0 \quad \text{and} \quad \int_0^1 \frac{D\widetilde{w}(s)}{K(s)} ds = 0,$$

then our solution to (2.1), (2.2) would be given by (3.1). With this in mind, consider the problem: determine $w(x)$ satisfying

$$\mathcal{L}_r^\alpha w(x) = f(x), \quad x \in I, \quad (3.3)$$

$$\text{subject to } w(0) = w(1) = 0. \quad (3.4)$$

Let $\beta \in [\alpha - 1, 1]$ be determined by $(1-r) \sin(\pi\beta) = r \sin(\pi(\alpha - \beta))$. The following theorem ensures the well-posedness of this problem.

Theorem 3.1 ([3, 14]). *Let $f(x) \in L^2_{\omega^{(\beta, \alpha-\beta)}}(I)$. Then there exists a unique solution $w(x) \in L^2_{\omega^{(-(\alpha-\beta), -\beta)}}(I)$ satisfying (3.3), (3.4). In addition, there exists $C > 0$ such that*

$$\|w\|_{\omega^{(-(\alpha-\beta), -\beta)}} + \|Dw\|_{\omega^{(-(\alpha-\beta)+1, -\beta+1)}} \leq C \|f\|_{\omega^{(\beta, \alpha-\beta)}} \quad \square$$

From [3], we have $k_1(x) := \int_0^x (1-s)^{\alpha-\beta-1} s^{\beta-1} ds \in \ker(\mathcal{L}_r^\alpha)$. Let

$$\widetilde{w}(x) = C_1 k_1(x) + w(x). \quad (3.5)$$

We have $\mathcal{L}_r^\alpha \widetilde{w}(x) = f(x)$ and $\widetilde{w}(0) = 0$. Condition (3.2) combined with (3.5) implies

$$\int_0^1 \frac{1}{K(s)} (C_1 (1-s)^{\alpha-\beta-1} s^{\beta-1} + Dw(s)) ds = 0,$$

which leads to

$$C_1 = \frac{-\int_0^1 \frac{Dw(s)}{K(s)} ds}{\int_0^1 \frac{(1-s)^{\alpha-\beta-1} s^{\beta-1}}{K(s)} ds} \quad (3.6)$$

In particular, if $D(1/K) \in L^2_{\omega^{(\alpha-\beta, \beta)}}$, we have

$$C_1 = \frac{\int_0^1 w(s) \frac{d}{ds} \left(\frac{1}{K(s)} \right) ds}{\int_0^1 \frac{(1-s)^{\alpha-\beta-1} s^{\beta-1}}{K(s)} ds} = \frac{-\int_0^1 w(s) \frac{K'(s)}{K(s)^2} ds}{\int_0^1 \frac{(1-s)^{\alpha-\beta-1} s^{\beta-1}}{K(s)} ds}.$$

Let

$$\text{den} := \int_0^1 \frac{(1-s)^{\alpha-\beta-1} s^{\beta-1}}{K(s)} ds, \quad c_1 := \int_0^1 \frac{-Dw(s)}{K(s)} ds. \quad (3.7)$$

Then $C_1 = \frac{c_1}{\text{den}}$ can be bounded by

$$|C_1| = \frac{1}{\text{den}} \left| \int_0^1 Dw(s) \frac{1}{K(s)} ds \right| \leq \frac{1}{\text{den}} \int_0^1 |Dw(s)| \frac{1}{K(s)} ds \quad (3.8)$$

$$\begin{aligned} &= \frac{1}{\text{den}} \int_0^1 \omega^{((- \alpha - \beta + 1)/2, (-\beta + 1)/2)}(s) |Dw(s)| \omega^{((\alpha - \beta - 1)/2, (\beta - 1)/2)}(s) \frac{1}{K(s)} ds \\ &\leq \frac{1}{\text{den}} \|Dw\|_{\omega^{(-(\alpha - \beta) + 1, -\beta + 1)}} \left\| \frac{1}{K(s)} \right\|_{\omega^{((\alpha - \beta - 1), (\beta - 1))}} \\ &\leq C \|f\|_{\omega^{(\beta, \alpha - \beta)}} \quad (\text{using Theorem 3.1}). \end{aligned} \quad (3.9)$$

Combining (3.1), (3.5), (3.6), (3.9) and Theorem 3.1, we have the following.

Theorem 3.2. For $f(x) \in L^2_{\omega^{(\beta, \alpha - \beta)}}(I)$, there exists a unique solution $u(x) \in L^\infty(I)$ to (2.1), (2.2), given by

$$u(x) = C_1 \int_0^x \frac{(1-s)^{\alpha-\beta-1} s^{\beta-1}}{K(s)} ds + \int_0^x \frac{Dw(s)}{K(s)} ds, \quad (3.10)$$

where $w(x)$ is determined by (3.3), (3.4) and C_1 by (3.6).

Additionally, for $\epsilon_1, \epsilon_2 > 0$, there exists $C > 0$ such that

$$\|u\|_{L^\infty} + \|u\|_{\omega^{(-1+\epsilon_1, -1+\epsilon_2)}} \leq C \|f\|_{\omega^{(\beta, \alpha - \beta)}}. \quad (3.11)$$

Proof. It is straightforward to show that $u(x)$ given by (3.10) satisfies (2.1), (2.2). Next we show that there exists a unique solution to (2.1), (2.2).

Assume that $u_1(x)$ and $u_2(x)$ are solutions of (2.1), (2.2). Let $z_1(x)$ and $z_2(x)$ be defined by

$$z_1(x) = \int_0^x K(s) Du_1(s) ds, \quad z_2(x) = \int_0^x K(s) Du_2(s) ds,$$

which implies

$$Dz_1(x) = K(x) Du_1(x), \quad z_1(0) = 0, \quad \text{and} \quad Dz_2(x) = K(x) Du_2(x), \quad z_2(0) = 0.$$

Note that $\mathcal{L}_r^\alpha(z_1 - z_2) = 0$. Hence $(z_1 - z_2) \in \ker(\mathcal{L}_r^\alpha)$. Thus, from [3], for constants A and B ,

$$(z_1 - z_2)(x) = A + B \int_0^x (1-s)^{\alpha-\beta-1} s^{\beta-1} ds.$$

It is clear that $(z_1 - z_2)(0) = 0$ implies $A = 0$, and $D(z_1 - z_2)(x) = B(1-x)^{\alpha-\beta-1} x^{\beta-1}$ leads to

$$D(u_1 - u_2)(x) = B \frac{1}{K(x)} (1-x)^{\alpha-\beta-1} x^{\beta-1}.$$

Using $(u_1 - u_2)(0) = 0$, we obtain

$$(u_1 - u_2)(x) = B \int_0^x \frac{1}{K(s)} (1-s)^{\alpha-\beta-1} s^{\beta-1} ds.$$

As the integrand is non-negative, $(u_1 - u_2)(1) = 0$ implies $B = 0$, which in turn gives $u_1(x) = u_2(x)$.

Using (3.8) and (3.9),

$$\begin{aligned} |u(x)| &= \left| C_1 \int_0^x \frac{(1-s)^{\alpha-\beta-1} s^{\beta-1}}{K(s)} ds + \int_0^x \frac{Dw(s)}{K(s)} ds \right| \\ &\leq C|C_1| + \int_0^1 |Dw(s)| \frac{1}{K(s)} ds \leq C\|f\|_{\omega^{(\beta, \alpha-\beta)}}. \end{aligned} \quad (3.12)$$

Consequently, we obtain

$$\begin{aligned} \|u\|_{\omega^{(-1+\epsilon_1, -1+\epsilon_2)}}^2 &= \int_0^1 u^2(x) \omega^{(-1+\epsilon_1, -1+\epsilon_2)}(x) dx \\ &\leq C\|f\|_{\omega^{(\beta, \alpha-\beta)}}^2 \int_0^1 \omega^{(-1+\epsilon_1, -1+\epsilon_2)}(x) dx \leq C\|f\|_{\omega^{(\beta, \alpha-\beta)}}^2. \end{aligned} \quad (3.13)$$

Estimate (3.11) then follows from (3.12) and (3.13). $\square$

3.2 Approximation Scheme

To compute an approximation to $u(x)$, $u_N(x)$, we firstly compute an approximation to $w(x)$, $w_N(x)$, satisfying (3.3), (3.4), and then use $w_N(x)$ in place of $w(x)$ in (3.6) and (3.10) to obtain $u_N(x)$.

3.2.1 Approximation of $w(x)$ Satisfying (3.3), (3.4)

Proceeding as in [3], $f(x) \in L_{\omega^{(\beta, \alpha-\beta)}}^2(\mathbb{I})$ may be expressed as $f(x) = \sum_{i=0}^{\infty} \frac{f_i}{\|G_i^{(\beta, \alpha-\beta)}\|^2} G_i^{(\beta, \alpha-\beta)}(x)$, where f_i is given by

$$f_i := \int_0^1 \omega^{(\beta, \alpha-\beta)}(x) f(x) G_i^{(\beta, \alpha-\beta)}(x) dx. \quad (3.14)$$

With f_i defined in (3.14), let

$$f_N(x) = \sum_{i=0}^N \frac{f_i}{\|G_i^{(\beta, \alpha-\beta)}\|^2} G_i^{(\beta, \alpha-\beta)}(x) \quad \text{and} \quad w_N(x) = \omega^{(\alpha-\beta, \beta)}(x) \sum_{i=0}^N c_i G_i^{(\alpha-\beta, \beta)}(x), \quad (3.15)$$

where

$$\lambda_i = \frac{-\sin(\pi\alpha)}{\sin(\pi(\alpha-\beta)) + \sin(\pi\beta)} \frac{\Gamma(i+1+\alpha)}{\Gamma(i+1)} \quad \text{and} \quad c_i = \frac{1}{\lambda_i \|G_i^{(\beta, \alpha-\beta)}\|^2} f_i.$$

Using Stirling's formula, we have

$$\lim_{n \rightarrow \infty} \frac{\Gamma(n+\mu)}{\Gamma(n)n^\mu} = 1 \quad \text{for } \mu \in \mathbb{R}; \quad \text{thus} \quad \lambda_i \sim (i+1)^\alpha. \quad (3.16)$$

Theorem 3.3 ([3, 14]). *Let $f(x) \in L_{\omega^{(\beta, \alpha-\beta)}}^2(\mathbb{I})$, and let $w_N(x)$ be as defined in (3.15). Then*

$$w(x) := \lim_{N \rightarrow \infty} w_N(x) = \omega^{(\alpha-\beta, \beta)}(x) \sum_{j=0}^{\infty} c_j G_j^{(\alpha-\beta, \beta)}(x) \in L_{\omega^{(-(\alpha-\beta), -\beta)}}^2(\mathbb{I}),$$

and it satisfies (3.3), (3.4). $\square$

Theorem 3.4. *For $f(x) \in H_{\omega^{(\beta, \alpha-\beta)}}^t(\mathbb{I})$, $t \geq 0$, and $w_N(x)$ given by (3.15), there exists $C > 0$ such that*

$$\|w - w_N\|_{\omega^{(-(\alpha-\beta), -\beta)}} \leq C(N+2)^{-\alpha} (N(N+\alpha))^{-t/2} \|f\|_{t, \omega^{(\beta, \alpha-\beta)}}, \quad (3.17)$$

$$\|D(w - w_N)\|_{\omega^{(-(\alpha-\beta)+1, -\beta+1)}} \leq C(N+2)^{-(\alpha-1)} (N(N+\alpha))^{-t/2} \|f\|_{t, \omega^{(\beta, \alpha-\beta)}}. \quad (3.18)$$

Proof. Using the definition of the $\|\cdot\|_{\omega^{(-(\alpha-\beta), -\beta)}}$ norm,

$$\begin{aligned}\|w - w_N\|_{\omega^{(-(\alpha-\beta), -\beta)}}^2 &= \int_0^1 \omega^{(-(\alpha-\beta), -\beta)}(x) \left(\omega^{(\alpha-\beta, \beta)}(x) \sum_{i=N+1}^{\infty} \frac{G_i^{(\alpha-\beta, \beta)}(x)}{(\lambda_i \|G_i^{(\beta, \alpha-\beta)}\|_2^2)} f_i \right)^2 dx \\ &\leq \max_{i \geq N+1} \left(\frac{1}{\lambda_i^2} \right) \sum_{i=N+1}^{\infty} \frac{f_i^2}{\|G_i^{(\beta, \alpha-\beta)}\|_2^2} \\ &= \frac{1}{\lambda_{N+1}^2} \int_0^1 \omega^{(\beta, \alpha-\beta)}(x) \left(\sum_{i=N+1}^{\infty} \frac{G_i^{(\beta, \alpha-\beta)}(x)}{\|G_i^{(\beta, \alpha-\beta)}\|_2^2} f_i \right)^2 dx \\ &= \frac{1}{\lambda_{N+1}^2} \int_0^1 \omega^{(\beta, \alpha-\beta)}(x) (f(x) - P_{N, \beta, \alpha-\beta} f(x))^2 dx \\ &\leq C(N+2)^{-2\alpha} \|f - P_{N, \beta, \alpha-\beta} f\|_{\omega^{(\beta, \alpha-\beta)}}^2 \quad (\text{using } |\lambda_{N+1}| \sim (N+2)^\alpha) \\ &\leq C(N+2)^{-2\alpha} (N(N+\alpha))^{-t} \|f\|_{t, \omega^{(\beta, \alpha-\beta)}}^2 \quad (\text{using (2.8)}).\end{aligned}$$

Similarly, using (2.6),

$$\begin{aligned}\|D(w - w_N)\|_{\omega^{(-(\alpha-\beta)+1, -\beta+1)}}^2 &= \int_0^1 \omega^{(-(\alpha-\beta)+1, -\beta+1)}(x) \left(D \left(\omega^{(\alpha-\beta, \beta)}(x) \sum_{i=N+1}^{\infty} \frac{G_i^{(\alpha-\beta, \beta)}(x)}{(\lambda_i \|G_i^{(\beta, \alpha-\beta)}\|_2^2)} f_i \right) \right)^2 dx \\ &= \int_0^1 \omega^{(-(\alpha-\beta)+1, -\beta+1)}(x) \left(\omega^{(\alpha-\beta-1, \beta-1)}(x) \sum_{i=N+1}^{\infty} \frac{(i+1) G_{i+1}^{(\alpha-\beta-1, \beta-1)}(x)}{(\lambda_i \|G_i^{(\beta, \alpha-\beta)}\|_2^2)} f_i \right)^2 dx \\ &= \sum_{i=N+1}^{\infty} \frac{(i+1)^2}{\lambda_i^2} \frac{f_i^2}{\|G_i^{(\beta, \alpha-\beta)}\|_4^4} \|G_{i+1}^{(\alpha-\beta-1, \beta-1)}\|_2^2 \\ &\leq \sum_{i=N+1}^{\infty} \frac{(i+1)^2}{\lambda_i^2} \left(\frac{i+\alpha}{i+1} \right)^2 \frac{f_i^2}{\|G_i^{(\beta, \alpha-\beta)}\|_2^2} \quad (\text{using (2.7) and (2.5)}).\end{aligned} \quad (3.19)$$

Using (3.16),

$$\frac{(i+\alpha)^2}{\lambda_i^2} \sim (i+\alpha)^2 (i+1)^{-2\alpha} \sim (i+1)^{-2(\alpha-1)}. \quad (3.20)$$

Combining (3.19) and (3.20),

$$\begin{aligned}\|D(w - w_N)\|_{\omega^{(-(\alpha-\beta)+1, -\beta+1)}}^2 &\leq \frac{C}{(N+2)^{2(\alpha-1)}} \sum_{i=N+1}^{\infty} \frac{f_i^2}{\|G_i^{(\beta, \alpha-\beta)}\|_2^2} \\ &= \frac{C}{(N+2)^{2(\alpha-1)}} \int_0^1 \omega^{(\beta, \alpha-\beta)}(x) \left(\sum_{i=N+1}^{\infty} \frac{G_i^{(\beta, \alpha-\beta)}(x)}{\|G_i^{(\beta, \alpha-\beta)}\|_2^2} f_i \right)^2 dx \\ &= \frac{C}{(N+2)^{2(\alpha-1)}} \int_0^1 \omega^{(\beta, \alpha-\beta)}(x) (f(x) - P_{N, \beta, \alpha-\beta} f(x))^2 dx \\ &= C(N+2)^{-2(\alpha-1)} \|f - P_{N, \beta, \alpha-\beta} f\|_{\omega^{(\beta, \alpha-\beta)}}^2 \\ &\leq C(N+2)^{-2(\alpha-1)} (N(N+\alpha))^{-t} \|f\|_{t, \omega^{(\beta, \alpha-\beta)}}^2.\end{aligned}$$

□

3.2.2 Approximation of $u(x)$ Satisfying (3.10)

The approximation $u_N(x)$ of $u(x)$ is obtained by substituting $w_N(x)$ in place of $w(x)$ in (3.6) and (3.10). With den defined in (3.7), let

$$c_{1,N} := \int_0^1 \frac{-Dw_N(s)}{K(s)} ds \quad \text{and} \quad C_{1,N} := \frac{c_{1,N}}{\text{den}}.$$

Note that $|C_1 - C_{1,N}| = \frac{|c_1 - c_{1,N}|}{\text{den}}$. Hence the rate of convergence as $N \rightarrow \infty$ of $|C_1 - C_{1,N}|$ is equal to the rate of convergence of $|c_1 - c_{1,N}|$. Now,

$$|c_1 - c_{1,N}| = \left| \int_0^1 D(w - w_N)(s) \frac{1}{K(s)} ds \right| \leq \int_0^1 |D(w - w_N)(s)| \frac{1}{K(s)} ds \quad (3.21)$$

$$\begin{aligned} &= \int_0^1 \omega^{((- \alpha - \beta) + 1)/2, (-\beta + 1)/2}(s) |D(w - w_N)(s)| \omega^{((\alpha - \beta - 1)/2, (\beta - 1)/2)}(s) \frac{1}{K(s)} ds \\ &\leq \|D(w - w_N)\|_{\omega^{(-(\alpha - \beta) + 1, -\beta + 1)}} \left\| \frac{1}{K} \right\|_{\omega^{((\alpha - \beta - 1), (\beta - 1))}} \\ &\leq C(N + 2)^{-(\alpha - 1)} (N(N + \alpha))^{-t/2} \|f\|_{t, \omega^{(\beta, \alpha - \beta)}} \quad (\text{using (3.18)}). \end{aligned} \quad (3.22)$$

In case $D(\frac{1}{K}) \in L^2_{\omega^{(\alpha - \beta, \beta)}}(\mathbb{I})$,

$$\begin{aligned} |c_1 - c_{1,N}| &= \left| \int_0^1 (w - w_N)(s) D\left(\frac{1}{K(s)}\right) ds \right| \\ &= \left| \int_0^1 \omega^{(-(\alpha - \beta)/2, -\beta/2)}(s) (w - w_N)(s) \omega^{((\alpha - \beta)/2, \beta/2)}(s) D\left(\frac{1}{K(s)}\right) ds \right| \\ &\leq \|w - w_N\|_{\omega^{(-(\alpha - \beta), -\beta)}} \left\| D\left(\frac{1}{K}\right) \right\|_{\omega^{((\alpha - \beta), \beta)}} \\ &\leq C(N + 2)^{-\alpha} (N(N + \alpha))^{-t/2} \|f\|_{t, \omega^{(\beta, \alpha - \beta)}} \quad (\text{using (3.17)}). \end{aligned} \quad (3.23)$$

We have the following error estimates for $u - u_N$.

Theorem 3.5. For $f \in H^t_{\omega^{(\beta, \alpha - \beta)}}(\mathbb{I})$, $t \geq 0$, and $\epsilon_1, \epsilon_2 > 0$, there exists $C > 0$ (independent of N and α) such that

$$\|u - u_N\|_{L^\infty} + \|u - u_N\|_{\omega^{(-1 + \epsilon_1, -1 + \epsilon_2)}} \leq C(N + 2)^{-(\alpha - 1)} (N(N + \alpha))^{-t/2} \|f\|_{t, \omega^{(\beta, \alpha - \beta)}}. \quad (3.24)$$

In particular, if $D(\frac{1}{K}) \in L^2_{\omega^{(\alpha - \beta, \beta)}}(\mathbb{I})$, we have

$$\|u - u_N\|_{\omega^{(-(\alpha - \beta), -\beta)}} \leq C(N + 2)^{-\alpha} (N(N + \alpha))^{-t/2} \|f\|_{t, \omega^{(\beta, \alpha - \beta)}}. \quad (3.25)$$

Proof. From (3.10), using (3.21) and (3.22), we have

$$u(x) - u_N(x) = (C_1 - C_{1,N}) \int_0^x \frac{(1 - s)^{\alpha - \beta - 1} s^{\beta - 1}}{K(s)} ds + \int_0^x \frac{D(w - w_N)(s)}{K(s)} ds \quad (3.26)$$

$$\begin{aligned} \Rightarrow \|u - u_N\|_{L^\infty} &\leq C|C_1 - C_{1,N}| + \int_0^1 |D(w - w_N)(s)| \frac{1}{K(s)} ds \\ &\leq C(N + 2)^{-(\alpha - 1)} (N(N + \alpha))^{-t/2} \|f\|_{t, \omega^{(\beta, \alpha - \beta)}}. \end{aligned} \quad (3.27)$$

Then, from (3.27) and

$$\begin{aligned} \|u - u_N\|_{\omega^{(-1 + \epsilon_1, -1 + \epsilon_2)}}^2 &= \int_0^1 \omega^{(-1 + \epsilon_1, -1 + \epsilon_2)}(x) (u - u_N)^2(x) dx \\ &\leq \|u - u_N\|_{L^\infty}^2 \int_0^1 \omega^{(-1 + \epsilon_1, -1 + \epsilon_2)}(x) dx \leq C \|u - u_N\|_{L^\infty}^2, \end{aligned}$$

we obtain (3.24).

For $D(\frac{1}{K}) \in L^2_{\omega^{(\alpha - \beta, \beta)}}(\mathbb{I})$, we apply integration by parts to (3.26) to obtain

$$u(x) - u_N(x) = (C_1 - C_{1,N}) \int_0^x \frac{(1 - s)^{\alpha - \beta - 1} s^{\beta - 1}}{K(s)} ds + \frac{w(x) - w_N(x)}{K(x)} + \int_0^x (w(s) - w_N(s)) \frac{K'(s)}{K^2(s)} ds.$$

Therefore,

$$\|u - u_N\|_{\omega^{(-(\alpha-\beta), -\beta)}}^2 \leq I_1 + I_2 + I_3, \quad (3.28)$$

where

$$I_1 = 3(C_1 - C_{1,N})^2 \int_0^1 \omega^{(-(\alpha-\beta), -\beta)}(x) \left(\int_0^1 \frac{(1-s)^{\alpha-\beta-1} s^{\beta-1}}{K(s)} ds \right)^2 dx \leq C(C_1 - C_{1,N})^2, \quad (3.29)$$

$$I_2 = 3 \frac{1}{K_m^2} \|w - w_N\|_{\omega^{(-(\alpha-\beta), -\beta)}}^2, \quad (3.30)$$

$$\begin{aligned} I_3 &= 3 \int_0^1 \omega^{(-(\alpha-\beta), -\beta)}(x) \left(\int_0^1 \omega^{(-\frac{\alpha-\beta}{2}, -\frac{\beta}{2})}(s) (w(s) - w_N(s)) \omega^{(\frac{\alpha-\beta}{2}, \frac{\beta}{2})}(s) \frac{K'(s)}{K^2(s)} ds \right)^2 dx \\ &\leq \frac{3}{K_m^2} \int_0^1 \omega^{(-(\alpha-\beta), -\beta)}(x) \left(\int_0^1 \omega^{(-(\alpha-\beta), -\beta)}(s) (w(s) - w_N(s))^2 ds \right) \\ &\quad \times \left(\int_0^1 \omega^{(\alpha-\beta, \beta)}(s) \frac{K'(s)}{K^2(s)} ds \right) dx \leq C \|w - w_N\|_{\omega^{(-(\alpha-\beta), -\beta)}}^2. \end{aligned} \quad (3.31)$$

Combining (3.28)–(3.31) with (3.23) and (3.17), we obtain (3.25). $\square$

We conclude this section with an error bound for $D(u - u_N)$.

Lemma 3.1. For $f \in H_{\omega^{(\beta, \alpha-\beta)}}^t(\mathbb{I})$ and $t \geq 0$, there exists $C > 0$ (independent of N and α) such that

$$\|D(u - u_N)\|_{\omega^{(-(\alpha-\beta)+1, -\beta+1)}} \leq C(N+2)^{-(\alpha-1)} (N(N+\alpha))^{-t/2} \|f\|_{t, \omega^{(\beta, \alpha-\beta)}}. \quad (3.32)$$

Proof. From (3.26), it follows that

$$D(u - u_N) = \frac{1}{K(x)} ((C_1 - C_{1,N})(1-x)^{\alpha-\beta-1} x^{\beta-1} + D(w - w_N)).$$

Thus,

$$\begin{aligned} \|D(u - u_N)\|_{\omega^{(-(\alpha-\beta)+1, -\beta+1)}} &\leq \frac{1}{K_m} (|C_1 - C_{1,N}| \|(1-x)^{(\alpha-\beta-1)/2} x^{(\beta-1)/2}\| + \|D(w - w_N)\|_{\omega^{(-(\alpha-\beta)+1, -\beta+1)}}) \\ &\leq C(N+2)^{-(\alpha-1)} (N(N+\alpha))^{-t/2} \|f\|_{t, \omega^{(\beta, \alpha-\beta)}}, \end{aligned}$$

where, in the last step, we have used (3.22) and (3.18)) and the fact that $(\alpha - \beta - 1)$ and $(\beta - 1) > -1$. $\square$

4 Numerical Experiments

In this section, we present numerical experiments to demonstrate our approximation scheme and to compare the experimental rate of convergence of the approximation with the theoretically predicated rate.

Experiment 1. Let $K(x) = \frac{1}{1+x^\gamma}$ and

$$f(x) = -r \frac{x^{1-\alpha}}{\Gamma(2-\alpha)} + (1-r) \frac{(1-x)^{1-\alpha}}{\Gamma(2-\alpha)}. \quad (4.1)$$

Then the solution $u(x)$ is given by (3.10), where

$$\begin{aligned} w(x) &= x - Cx^\beta {}_2F_1(-(\alpha-\beta-1), \beta; \beta+1, x), \\ C &= {}_2F_1(-(\alpha-\beta-1), \beta; \beta+1, 1)^{-1}, \end{aligned}$$

and ${}_2F_1(a, b; c, x)$ denotes the Gaussian three-parameter hypergeometric function.

In order to determine the theoretical rate of convergence for

$$\|u - u_N\|_{\omega^{(-(\alpha-\beta), -\beta)}}, \quad \|u - u_N\|_{L^\infty} \quad \text{and} \quad \|D(u - u_N)\|_{\omega^{(-(\alpha-\beta)+1, -\beta+1)}}$$

from (3.24), (3.25) and (3.32), respectively, we need to determine the largest value for t such that

$$f(x) \in H_{\omega^{(\beta, \alpha-\beta)}}^t(I).$$

The most singular terms for $f(x)$ in (4.1) are $x^{1-\alpha}$ and $(1-x)^{1-\alpha}$. Using Lemma A.1 (in the appendix), we have $x^{1-\alpha} \in H_{\omega^{(\beta, \alpha-\beta)}}^t(I)$ for $t < 3 - \alpha - \beta$, and $(1-x)^{1-\alpha} \in H_{\omega^{(\beta, \alpha-\beta)}}^t(I)$ for $t < 3 - \alpha - (\alpha - \beta)$.

Then, for Example 1 ($\alpha = 1.60$, $\beta = 0.80$), $f(x) \in H_{\omega^{(\beta, \alpha-\beta)}}^t(I)$ for $t < 3 - \alpha - \max\{\alpha - \beta, \beta\} = 0.60$, which leads to theoretical asymptotic convergence rates of

- $\|u - u_N\|_{\omega^{(-(\alpha-\beta), -\beta)}} \sim N^{-2.20}$ (using (3.25)),
- $\|u - u_N\|_{L^\infty} \sim N^{-1.20}$ (using (3.24)),
- $\|D(u - u_N)\|_{\omega^{(-(\alpha-\beta)+1, -\beta+1)}} \sim N^{-1.20}$ (using (3.32)).

Assuming that $\|\xi - \xi_N\|_{L_\rho} \sim N^{-\kappa}$, the experimental convergence rate is calculated using

$$\kappa \approx \frac{\log\left(\frac{\|\xi - \xi_{N_1}\|_{L_\rho}}{\|\xi - \xi_{N_2}\|_{L_\rho}}\right)}{\log\left(\frac{N_2}{N_1}\right)}.$$

Example 1. In this example, we select $\alpha = 1.60$, $r = 0.50$, $\beta = 0.80$, $\gamma = 0.80$, which leads to $f(x) \in H_{\omega^{(\beta, \alpha-\beta)}}^t(I)$ for $t < 0.60$.

Example 2. In this example, we take $\alpha = 1.30$, $r = 0.63$, $\beta = 0.50$, $\gamma = 0.80$. The above analysis gives that $f(x) \in H_{\omega^{(\beta, \alpha-\beta)}}^t(I)$ for $t < 0.90$.

Example 3. In this example, we select $\alpha = 1.30$, $r = 0.63$, $\beta = 0.50$, $\gamma = 0.10$, which leads to $f(x) \in H_{\omega^{(\beta, \alpha-\beta)}}^t(I)$ for $t < 0.90$. However, in this case, $D(\frac{1}{K(x)}) \notin L_{\omega^{(\alpha-\beta, \beta)}}^2(I)$ due to the relatively strong singularity of $\frac{1}{K(x)}$ at $x = 0$, which means that (3.23) is not applicable. Hence we can only apply bound (3.22) of $c_1 - c_{1,N}$, which leads to estimate (3.24) for $u - u_N$ instead of (3.25), and consequently an estimate for the convergence rate for $\|u - u_N\|_{\omega^{(-(\alpha-\beta), -\beta)}}$ of 1.20 by (3.24), instead of 2.20 if using (3.25).

We observe from Tables 1–3 that the experimental convergence rates for

$$\|u - u_N\|_{\omega^{(-(\alpha-\beta), -\beta)}} \quad \text{and} \quad \|D(u - u_N)\|_{\omega^{(-(\alpha-\beta)+1, -\beta+1)}}$$

are in strong agreement with the theoretically predicted rates for the first two examples. For Example 3, we note that the numerical convergence rate of $\|u - u_N\|_{\omega^{(-(\alpha-\beta), -\beta)}}$ is 2.20, which corresponds to the case for $D(\frac{1}{K}) \in L_{\omega^{(\alpha-\beta, \beta)}}^2(I)$ (even though this is not the case in this experiment). The convergence rates are in general not very high due to the low regularity of the data (e.g., the right-hand side term). Higher convergence rates will be observed in the following experiments. Additionally, we remark that the error in the L^∞ norm is difficult to measure accurately due to the oscillatory nature of the error function caused by the polynomial approximation, as illustrated by Figure 1.

N	$\ u - u_N\ _{\omega^{(-(\alpha-\beta), -\beta)}}$	κ	$\ D(u - u_N)\ _{\omega^{(-(\alpha-\beta)+1, -\beta+1)}}$	κ	$\ u - u_N\ _{L^\infty}$	κ
16	4.87E-04		1.15E-02		4.41E-04	
20	3.09E-04	2.15	8.93E-03	1.18	2.95E-04	1.89
24	2.12E-04	2.16	7.26E-03	1.18	2.00E-04	2.23
28	1.54E-04	2.16	6.09E-03	1.19	1.54E-04	1.78
32	1.16E-04	2.16	5.22E-03	1.19	1.21E-04	1.86
36	9.08E-05	2.16	4.56E-03	1.19	9.46E-05	2.14
Pred.		2.20		1.20		1.20

Table 1: Convergence properties of Example 1.

N	$\ u - u_N\ _{\omega^{(-(\alpha-\beta), -\beta)}} \kappa$	$\ D(u - u_N)\ _{\omega^{(-(\alpha-\beta)+1, -\beta+1)}} \kappa$	$\ u - u_N\ _{L^\infty} \kappa$
16	4.57E-04	1.07E-02	4.41E-04
20	2.88E-04	2.20	2.95E-04
24	1.96E-04	2.19	2.00E-04
28	1.42E-04	2.19	1.53E-04
32	1.07E-04	2.19	1.20E-04
36	8.33E-05	2.19	9.42E-05
Pred.	2.20	1.20	1.20

Table 2: Convergence properties of Example 2.

N	$\ u - u_N\ _{\omega^{(-(\alpha-\beta), -\beta)}} \kappa$	$\ D(u - u_N)\ _{\omega^{(-(\alpha-\beta)+1, -\beta+1)}} \kappa$	$\ u - u_N\ _{L^\infty} \kappa$
16	4.99E-04	1.14E-02	5.73E-04
20	3.11E-04	2.24	3.71E-04
24	2.10E-04	2.24	2.63E-04
28	1.51E-04	2.23	1.88E-04
32	1.13E-04	2.22	1.35E-04
36	8.80E-05	2.22	1.07E-04
Pred.	1.20	1.20	1.20

Table 3: Convergence properties of Example 3.

Figure 1: Plots of (left) $u(x)$ and (right) $u(x) - u_N(x)$.

Experiment 2. Let $K(x) = \frac{1}{1+x}$ and

$$f(x) = -r \frac{x^{1-\sigma}}{\Gamma(2-\sigma)} + (1-r) \frac{(1-x)^{1-\sigma}}{\Gamma(2-\sigma)}$$

for some constant σ . As the closed form of the corresponding solution u is not available, the approximation u_N with $N = 36$ serves as the reference solution.

Example 4. In this example, we select $\alpha = 1.60$, $r = 0.41$, $\beta = 0.84$, $\sigma = 0.10$, which leads to $f(x) \in H_{\omega^{(\beta, \alpha-\beta)}}^t(I)$ for $t < 3.56$.

Example 5. In this example, we take $\alpha = 1.60$, $r = 0.68$, $\beta = 0.72$, $\sigma = 1$. In this case, $f(x) \in H_{\omega^{(\beta, \alpha-\beta)}}^t(I)$ for any $t > 0$, which in turn gives the exponential convergence.

We observe from Table 4 that, compared with the numerical results in Experiment 1, the smoother right-hand side term gives the higher convergence rates. In particular, a sufficiently smooth right-hand side term yields

N	$\ u - u_N\ _{\omega^{(-(\alpha-\beta), -\beta)}}$	κ	$\ D(u - u_N)\ _{\omega^{(-(\alpha-\beta)+1, -\beta+1)}}$	κ	$\ u - u_N\ _{L^\infty}$	κ
8	8.08E-07		8.66E-06		6.38E-07	
12	1.29E-07	4.99	1.95E-06	4.06	1.10E-07	4.78
16	3.33E-08	5.05	6.48E-07	4.10	2.98E-08	4.88
20	1.13E-08	5.10	2.70E-07	4.15	1.03E-08	5.05
Pred.		5.16		4.16		4.16

Table 4: Convergence properties of Example 4.

N	$\ u - u_N\ _{\omega^{(-(\alpha-\beta), -\beta)}}$	$\ D(u - u_N)\ _{\omega^{(-(\alpha-\beta)+1, -\beta+1)}}$	$\ u - u_N\ _{L^\infty}$
4	1.06E-14	2.19E-13	1.08E-14
6	9.92E-15	2.17E-13	1.04E-14
8	9.21E-15	2.15E-13	9.67E-15
10	8.62E-15	2.12E-13	8.98E-15
12	8.09E-15	2.08E-13	8.34E-15

Table 5: Convergence properties of Example 5.

the exponential convergence such that the errors decay rapidly to the magnitude of the round-off errors for a quite small N (cf. Table 5).

Experiment 3. Let $r = 1$,

$$K(x) = \begin{cases} \hat{K}, & x \in [0, \frac{1}{2}], \\ 1, & x \in (\frac{1}{2}, 1], \end{cases}$$

for some positive constant $\hat{K}$ and

$$f(x) = \begin{cases} -\frac{\hat{K}x^{1-\alpha}}{\Gamma(2-\alpha)}, & x \in [0, \frac{1}{2}], \\ -\frac{\hat{K}(x^{1-\alpha} - (x - \frac{1}{2})^{1-\alpha})}{\Gamma(2-\alpha)} - \frac{(x - \frac{1}{2})^{1-\alpha}}{\Gamma(2-\alpha)}, & x \in (\frac{1}{2}, 1]. \end{cases}$$

Then the solution $u(x)$ is given by

$$u(x) = \begin{cases} x, & x \in [0, \frac{1}{2}], \\ 1 - x, & x \in (\frac{1}{2}, 1], \end{cases}$$

which is continuous and belongs to $H_0^1(I)$ (and thus $H_0^{\alpha/2}(I)$) [25, Lemma 3.2]. Direct calculation yields

$$(-D((r_0 I_x^{2-\alpha} + (1-r) {}_x I_1^{2-\alpha})K(x)Du(x))u, u) = \frac{\hat{K}(3 \times (\frac{1}{2})^{3-\alpha} - 1) + (\frac{1}{2})^{3-\alpha}}{\Gamma(4-\alpha)}. \quad (4.2)$$

If we set either $\alpha = 1.3$, $\hat{K} = 5$ or $\alpha = 1.2$, $\hat{K} = 3$, the right-hand side of (4.2) is negative, that is, the bilinear form of model (2.1), (2.2) is not coercive in these cases. Therefore, the corresponding finite difference or finite element approximations may be unstable and even diverge. We test the performance of the proposed spectral method for these two cases in the following two examples. To determine the largest value of t such that $f(x) \in H_{\omega^{(\beta, \alpha-\beta)}}^t(I)$, we follow the proof of Lemma A.1 to obtain $t < \frac{3}{2} - \alpha$. Then, concerning the fact that K is not differentiable at $x = \frac{1}{2}$, the convergence rates of $\|u - u_N\|_{\omega^{(-(\alpha-\beta), -\beta)}}$, $\|u - u_N\|_{L^\infty}$ and $\|D(u - u_N)\|_{\omega^{(-(\alpha-\beta)+1, -\beta+1)}}$ are anticipated to be $\frac{1}{2}$ by (3.24) and (3.32).

Example 6. In this example, we take $\alpha = 1.3$ and $\hat{K} = 5$.

Example 7. In this example, we take $\alpha = 1.2$ and $\hat{K} = 3$.

From Figure 2 and Tables 6 and 7, we observe that the spectral approximation works well and the experimental convergence rates for $\|D(u - u_N)\|_{\omega^{(-(\alpha-\beta)+1, -\beta+1)}}$ (which is about 0.50) are in strong agreement with the

N	$\ u - u_N\ _{\omega^{(-(\alpha-\beta), -\beta)}} \kappa$	$\ D(u - u_N)\ _{\omega^{(-(\alpha-\beta)+1, -\beta+1)}} \kappa$	$\ u - u_N\ _{L^\infty} \kappa$
16	2.46E-02	3.02E-01	3.60E-02
20	1.99E-02	1.01	0.43
24	1.67E-02	1.01	0.44
28	1.43E-02	1.01	0.45
32	1.26E-02	1.01	0.46
36	1.12E-02	1.01	0.46
Pred.	0.50	0.50	0.50

Table 6: Convergence properties of Example 6.

N	$\ u - u_N\ _{\omega^{(-(\alpha-\beta), -\beta)}} \kappa$	$\ D(u - u_N)\ _{\omega^{(-(\alpha-\beta)+1, -\beta+1)}} \kappa$	$\ u - u_N\ _{L^\infty} \kappa$
16	1.58E-02	2.04E-01	2.51E-02
20	1.27E-02	1.03	0.45
24	1.06E-02	1.03	0.45
28	9.14E-03	1.03	0.46
32	8.00E-03	1.03	0.46
36	7.11E-03	1.03	0.47
Pred.	0.50	0.50	0.50

Table 7: Convergence properties of Example 7.

Figure 2: Plots of $u(x)$ and $u_N(x)$ with $N = 36$ for Example 7.

theoretically predicted rates, while the numerical convergence rates of $\|u - u_N\|_{\omega^{(-(\alpha-\beta), -\beta)}}$ and $\|u - u_N\|_{\omega^{(-(\alpha-\beta), -\beta)}}$ are higher than the predictions. Further possible improvements will be attempted in the near future.

5 Concluding Remarks

In this paper, a spectral approximation of a variable coefficient FDE was proposed and analyzed via the change of variable that transformed the variable coefficient FDE to its constant-coefficient analogue. Error estimates were presented for the approximation concerning both smooth and non-smooth diffusivity coefficient and right-hand side. Numerical experiments were given to demonstrate the sharpness of the theoretically derived estimates.

For the FDE (2.1) equipped with non-homogeneous Dirichlet boundary conditions, i.e., the boundary condition (2.2) is replaced by $u(0) = a$ and $u(1) = b$ for some real numbers a and b , the corresponding solution u can still be expressed in terms of the solution w to the model (3.3), (3.4) like (3.10),

$$u(x) := \int_0^x \frac{Dw(s)}{K(s)} ds + \frac{b - a - \int_0^1 \frac{Dw(s)}{K(s)} ds}{\int_0^1 \frac{\omega^{(\alpha-\beta-1, \beta-1)}(s)}{K(s)} ds} \int_0^x \frac{\omega^{(\alpha-\beta-1, \beta-1)}(s)}{K(s)} ds + a.$$

Consequently, similar derivations for the numerical approximations can be performed. In [14], the well-posedness and regularity of the constant-coefficient FDE (2.1) equipped with either Dirichlet and Neumann mixed boundary conditions or Neumann boundary conditions were proved by constructing the series solutions like (3.15), which could be employed to analyze the variable-coefficient problems and to develop and analyze corresponding numerical methods. For more general boundary conditions, further explorations will be carried out in the near future.

A Appendix

In this section, we investigate which $H_{\omega^{(a,b)}}^s(\mathbb{I})$ space $u(x) = x^\mu$ lies in. For brevity of notation, in this section, we use $H_{(a,b)}^s(\mathbb{I}) \equiv H_{\omega^{(a,b)}}^s(\mathbb{I})$.

Lemma A.1. *Let $u(x) = x^\mu$ and $a, b > 0$. Then $u \in H_{(a,b)}^s(\mathbb{I})$ for $s \geq 0$ satisfying $s < 2\mu + b + 1$.*

Proof. Let $\chi(x) \in C^\infty[0, \infty)$ denote the cutoff function satisfying

$$\chi(x) = \begin{cases} 1 & \text{for } 0 < x \leq \frac{1}{4}, \\ 0 & \text{for } x \geq \frac{3}{4}, \end{cases}$$

and let $\chi_\delta(x) := \chi(\frac{x}{\delta})$ for $\delta > 0$. Note that

$$\chi_\delta(x) = \begin{cases} 1 & \text{for } 0 < x \leq \frac{\delta}{4}, \\ 0 & \text{for } x \geq \frac{3\delta}{4}, \end{cases} \quad \text{and} \quad \frac{d^m}{dx^m} \chi_\delta(x) = \begin{cases} 0 & \text{for } 0 < x < \frac{\delta}{4}, \\ 0 & \text{for } x > \frac{3\delta}{4}, \end{cases} \quad \text{for } m \in \mathbb{N}.$$

For δ to be determined, let $u = v + w$, where $v(x) = \chi_\delta(x)u(x)$ and $w(x) = (1 - \chi_\delta(x))u(x)$. We have

$$\left| \frac{d^m v(x)}{dx^m} \right| \leq C \sum_{j=0}^m \delta^{-(m-j)} x^{\mu-j}, \quad \text{and zero for } x > \frac{3\delta}{4}.$$

Thus,

$$\begin{aligned} \int_I (1-x)^{a+m} x^{b+m} \left| \frac{d^m v(x)}{dx^m} \right|^2 dx &\leq C \int_0^{\frac{3\delta}{4}} \sum_{j=0}^m \delta^{-2(m-j)} x^{2\mu-2j+b+m} dx \\ &\leq C \sum_{j=0}^m \delta^{-2(m-j)} \delta^{2\mu-2j+b+m+1} \quad (\text{provided } 2\mu - 2j + b + m > -1) \\ &\leq C \delta^{2\mu+b+1-m}, \end{aligned}$$

which implies that, for $m < 2\mu + b + 1$, $v \in H_{(a,b)}^m(\mathbb{I})$ and

$$\|v\|_{H_{(a,b)}^m}^2 \leq C \delta^{2\mu+b+1-m}. \quad (\text{A.1})$$

Next, consider $w(x)$.

$$\left| \frac{d^m w(x)}{dx^m} \right| \leq C \left((1 - \chi_\delta(x)) x^{\mu-m} + \sum_{j=0}^{m-1} x^{\mu-j} \frac{d^{m-j}}{dx^{m-j}} (1 - \chi_\delta(x)) \right). \quad (\text{A.2})$$

The first term on the right-hand side of (A.2) vanishes for $x < \frac{\delta}{4}$, and the second term vanishes for $x < \frac{\delta}{4}$ and $x > \frac{3\delta}{4}$. Using this,

$$\begin{aligned} \int_I (1-x)^{a+m} x^{b+m} \left| \frac{d^m w(x)}{dx^m} \right|^2 dx &\leq C \left(\int_{\delta/4}^1 x^{b+m} x^{2\mu-2m} dx + \int_{\delta/4}^{3\delta/4} x^{b+m} \sum_{j=0}^{m-1} x^{2\mu-2j} \delta^{-2(m-j)} dx \right) \\ &\leq C \left(\int_{\delta/4}^1 x^{2\mu+b-m} dx + \sum_{j=0}^{m-1} \delta^{-2m+2j} \int_{\delta/4}^{3\delta/4} x^{2\mu+b+m-2j} dx \right) \\ &\leq \begin{cases} C(1 + \delta^{2\mu+b-m+1}) & \text{if } 2\mu + b - m \neq -1, \\ C(1 + |\log \delta|) & \text{if } 2\mu + b - m = -1. \end{cases} \end{aligned} \quad (\text{A.3})$$

Hence, for $n > 2\mu + b + 1$,

$$\|w\|_{H_{(a,b)}^n}^2 \leq C\delta^{2\mu+b+1-n}. \quad (\text{A.4})$$

Remark. For $n > 2\mu + b + 1$, the exponent of δ in (A.4) is negative, so the “1”-term in (A.3) is bounded by the δ -term.

For $0 < t < 1$, from (A.1) and (A.4), we have

$$\begin{aligned} K(t, u) &= \inf_{u=u_1+u_2} (\|u_1\|_{H_{(a,b)}^m} + t\|u_2\|_{H_{(a,b)}^n}) \\ &\leq \|v\|_{H_{(a,b)}^m} + t\|w\|_{H_{(a,b)}^n} \leq C(\delta^{(2\mu+b+1-m)/2} + t\delta^{(2\mu+b+1-n)/2}). \end{aligned} \quad (\text{A.5})$$

Setting $\delta = t^{2/(n-m)}$ leads to $K(t, u) \leq Ct^{(2\mu+b+1-m)/(n-m)}$.

Recall that

$$\|u\|_{[H_{(a,b)}^m, H_{(a,b)}^n]_{\theta,2}}^2 = \int_0^\infty t^{-2\theta} (K(t, u))^2 \frac{dt}{t}. \quad (\text{A.6})$$

The larger the value of θ ($0 < \theta < 1$) in (A.6) such that the integral is finite, the “nicer” (i.e., more regular) is the function u . Hence, from (A.6), we are interested in the integrand about $t = 0$. We have trivially that, for $u_1 = u$, $u_2 = 0$ in (A.5) that $K(t, u) \leq \|u\|_{H_{(a,b)}^m} \leq C$. Hence it follows that

$$K(t, u) \leq \begin{cases} Ct^{(2\mu+b+1-m)/(n-m)} & \text{for } 0 < t < 1, \\ C & \text{for } t \geq 1. \end{cases}$$

Using (A.6),

$$\begin{aligned} \|u\|_{[H_{(a,b)}^m, H_{(a,b)}^n]_{\theta,2}}^2 &\leq \int_0^1 Ct^{-2\theta-1+2(2\mu+b+1-m)/(n-m)} dt + \int_1^\infty Ct^{-2\theta-1} dt < \infty \\ &\quad \text{if } \theta < \frac{2\mu+b+1-m}{n-m}. \end{aligned} \quad (\text{A.7})$$

For $s = (1 - \theta)m + \theta n = m + \theta(n - m)$, then $s < 2\mu + b + 1$ using (A.7).

Hence we can conclude that $u(x) = x^\mu \in H_{(a,b)}^s(I)$ for $s < 2\mu + b + 1$. $\square$

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