ON DISCONTINUOUS GALERKIN METHODS FOR ELLIPTIC PROBLEMS WITH DISCONTINUOUS COEFFICIENTS

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Dedicated to Raytcho Lazarov on the occasion of his 60th birthday.

Abstract — Discontinuous Galerkin methods for elliptic problems with discontinuous coefficients are discussed. First the error bound of the methods is analyzed. Then a multilevel additive Schwarz preconditioner for one of the discrete problems is designed and analyzed. Although the preconditioner is not optimal it is very well suited for parallel computations and its rate of convergence is independent of the jumps of coefficients.

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1. Introduction

In this paper, discontinuous Galerkin discretizations of an elliptic problem with discontinuous coefficients are discussed. The problem is considered in a polygonal region Ω which is a union of polygonal subregions Ω_i . The discontinuity of coefficients are across $\partial\Omega_i$. The problem is approximated by the finite element method on matching triangulation in each Ω_i and nonmatching triangulation across $\partial\Omega_i$. This kind of triangulation and composite discretizations are motivated first of all by the regularity of the solution of the problem being discussed. Discrete problems are formulated using a discontinuous Galerkin method, symmetric and with interior penalty terms across the interfaces of $\partial\Omega_i$.

In the first part of this paper, we analyze the error bound of the methods under consideration taking into account the discontinuity of coefficients. Two discretizations of the problem are considered. They differ in the form of coefficients at the interfaces. In the first discretization, they have natural forms which are obtained by integrating by parts on Ω_i while in the second one they are harmonic averages, which is more natural for discontinuous coefficients. It is proved that the error bound is optimal for the first discretization and not optimal for the second one. An advantage of the second discretization is the possibility of designing a very simple parallel algorithm. It is proposed in the second part of the paper. In this paper, a multilevel additive Schwarz method (ASM) is designed and analyzed for the second discrete problem. A preconditioner obtained in this way is very well suited for parallel

computations. The resulting local problems are defined on Ω_i , i.e., on matching triangulation with constant coefficients and are independent. The conjugate gradient method with this preconditioner is not optimal, its rate of convergence depends on the ratio of meshsize of fine triangulation and the diameter of Ω_i . On the other hand, it is independent of the jumps of coefficients.

Discontinuous Galerkin (DG) methods are becoming more and more popular for approximation of PDEs (see [1, 7] and literature in devoted to the forms of discretizations of elliptic problems and the analysis of the error bounds). There are also several papers devoted to methods for solving the resulting discrete problems. We mention [5] and [4] where composite discretizations similar to those discussed in this paper are considered (for continuous coefficients). In these papers, Neumann-Dirichlet and two level ASMs, respectively, are proposed and analyzed.

The paper is organized as follows. In Section 2, the differential problem and its DG discretization are formulated. Section 3 is devoted to the analysis of the discrete problem, it is proved that the problem is well-posed and an estimation of its error bound is made. In Section 3, the modified discrete problem is analyzed using a harmonic average of discontinuous coefficients at the interfaces of Ω_i . For this discrete problem, a multilevel ASM is designed and analyzed in Section 5.

2. Differential and discrete problems

In this paper, we use the standard notation for Sobolev spaces of functions defined in a polygonal region $\Omega \subset R^2$. We are looking for a solution of the following differential problem:

Find $u^* \in H_0^1(\Omega)$ such that

$$a(u^*, v) = f(v), \quad v \in H_0^1(\Omega), \quad (2.1)$$

where

$$a(u, v) = \sum_{i=1}^N \int_{\Omega_i} \varrho_i(x) \nabla u \nabla v dx, \quad f(v) = \int_{\Omega} f v dx.$$

We assume that $\bar{\Omega} = \bigcup_{i=1}^N \bar{\Omega}_i$ and Ω_i are polygonal subregions of diameter H_i . Thus $\Omega_i, i = 1, \dots, N$, form a partition of Ω which is not necessarily triangulation. The positive coefficients $\varrho_i(x)$ can be discontinuous across $\partial\Omega_i$ and, for simplicity of presentation, we assume that they are constants, ϱ_i on Ω_i . Assuming $f \in L^2(\Omega)$, we guarantee that problem (2.1) is well-posed, i.e., there is a unique solution $u^* \in H_0^1(\Omega)$.

The regularity of u^* , $u^* \in H^{1+s}$ for $s > 0$, global on Ω and local on Ω_i , depends on the geometry of Ω_i and the properties of ϱ_i (see for example [3, 6]).

Problem (2.1) is discretized by the discontinuous Galerkin (DG) method (see [1, 7]) across $\partial\Omega_i$ while in each Ω_i the discretization is given by the conforming finite element method (FEM) on triangular elements. The form of this discretization is determined by the regularity of the solution.

Let us introduce the shape regular triangulation in each Ω_i with a triangular element and parameter h_i . The resulting triangulation on Ω is nonmatching across $\partial\Omega_i$, in general. Let $X^{h_i}(\Omega_i)$ be a finite element space of piecewise linear continuous functions on Ω_i , vanishing on $\partial\Omega_i \cap \partial\Omega$ (see for example [2]). Let

$$X^h(\Omega) = X^{h_1}(\Omega_1) \times \dots \times X^{h_N}(\Omega_N).$$

Note that $X^h(\Omega)$ is a subspace of $L^2(\Omega)$ but not of $H^1(\Omega)$.

We now discretize problem (2.1) by the DG method on the introduced partition and triangulation (see [1, 7]).

Find $u_h^* \in X^h(\Omega)$ such that

$$a^h(u_h^*, v_h) = f(v_h), \quad v_h \in X^h(\Omega), \quad (2.2)$$

where

$$a^h(u, v) \equiv \sum_{i=1}^N a_i(u, v) + s_a(u, v) + s_p(u, v). \quad (2.3)$$

Here

$$\begin{aligned} a_i(u, v) &\equiv \int_{\Omega_i} \varrho_i \nabla u \nabla v dx; \\ s_a(u, v) &\equiv \sum_{i=1}^N \sum_{F_{ij} \subset \partial\Omega_i} 0.5 \left\{ \int_{F_{ij}} 0.5 \left(\varrho_i \frac{\partial u_i}{\partial n} + \varrho_j \frac{\partial u_j}{\partial n} \right) (v_i - v_j) ds + \right. \\ &\quad \left. + \int_{F_{ij}} 0.5 \left(\varrho_i \frac{\partial v_i}{\partial n} + \varrho_j \frac{\partial v_j}{\partial n} \right) (u_i - u_j) ds \right\}, \end{aligned} \quad (2.4)$$

where u_i is the restriction of $u \in X^h(\Omega)$ to Ω_i , $F_{ij} = \partial\Omega_i \cap \partial\Omega_j$ and $\frac{\partial u_i}{\partial n}$ is the normal derivative of u_i , outward to Ω_i (F_{ij} is an edge of Ω_i);

$$s_p(u, v) \equiv \sum_{i=1}^N \sum_{F_{ij} \subset \partial\Omega_i} 0.5 \int_{F_{ij}} \varrho_i \frac{\delta}{h_i} (u_i - u_j)(v_i - v_j) ds,$$

where δ is a penalty positive parameter which can be chosen below.

3. Analysis of the discrete problem

In this section, we first prove that problem (2.2) is well-posed and then its error bound, taking into account the jumps of ϱ_i .

Let us introduce the so-called broken norm in $X^h(\Omega)$ ($u = \{u_i\}_{i=1}^N \in X^h(\Omega)$):

$$\|u\|_{1,h}^2 \equiv \sum_{i=1}^N \left\{ \varrho_i \|\nabla u_i\|_{L^2(\Omega_i)}^2 + \sum_{F_{ij} \subset \partial\Omega_i} 0.5 \int_{F_{ij}} \frac{\varrho_i}{h_i} (u_i - u_j)^2 ds \right\}.$$

Theorem 3.1. *Problem (2.2) has a unique solution and is stable for sufficiently large δ , the penalty term.*

The proof of Theorem 3.1 follows from the following lemmas using Lax-Milgram lemma, (see, for example, [2]).

Lemma 3.1. *There exist $\delta_0 > 0$ and $\gamma > 0$ such that for $\delta > \delta_0$ and $u \in X^h(\Omega)$*

$$\gamma \|u\|_{1,h}^2 \leq a^h(u, u), \quad (3.1)$$

where γ is independent of h_i , N and ϱ_i .

Proof. Note that (see (2.3) and (2.4))

$$a^h(u, u) = \|u\|_{1,h}^2 + s_a(u, u) + (\delta - 1)s_p(u, u).$$

The proof reduces to show that

$$s_a(u, u) \leq 0.5 \sum_{i=1}^N a_i(u, u) + \delta_0 s_p(u, u). \quad (3.2)$$

Note that

$$\begin{aligned} I_{ij} &\equiv \int_{F_{ij}} \left(\varrho_i \frac{\partial u_i}{\partial n} + \varrho_j \frac{\partial u_j}{\partial n} \right) (u_i - u_j) ds \\ &\leq (\varrho_i \|\nabla u_i\|_{L^2(F_{ij})} + \varrho_j \|\nabla u_j\|_{L^2(F_{ij})}) \|u_i - u_j\|_{L^2(F_{ij})}. \end{aligned} \quad (3.3)$$

Using the following simple inequality for the finite element functions:

$$h_i \|\nabla u_i\|_{L^2(F_{ij})}^2 \leq C \|\nabla u_i\|_{L^2(\Omega_i)}^2,$$

and the ε - inequality, we have

$$\varrho_i \|\nabla u_i\|_{L^2(F_{ij})} \|u_i - u_j\|_{L^2(F_{ij})} \leq C \left\{ \varepsilon \varrho_i \|\nabla u_i\|_{L^2(\Omega_i)}^2 + \frac{1}{4\varepsilon} \frac{\varrho_i}{h_i} \|u_i - u_j\|_{L^2(F_{ij})}^2 \right\}. \quad (3.4)$$

Using this and a similar estimate for the term with ϱ_j in (3.3), we get

$$\begin{aligned} I_{ij} &\leq C \left\{ \varepsilon (\varrho_i \|\nabla u_i\|_{L^2(\Omega_i)}^2 + \varrho_j \|\nabla u_j\|_{L^2(\Omega_j)}^2) \right. \\ &\quad \left. + \frac{1}{4\varepsilon} \left(\frac{\varrho_i}{h_i} \|u_i - u_j\|_{L^2(F_{ij})}^2 + \frac{\varrho_j}{h_j} \|u_i - u_j\|_{L^2(F_{ij})}^2 \right) \right\}. \end{aligned}$$

Summing this with respect to $F_{ij} \subset \Omega_i$ and $i = 1, \dots, N$, and choosing, for example, $\varepsilon = 0.5/C$, we get (3.2) with $\delta_0 = C^2/2$. $\square$

Lemma 3.2. *There exists $M > 0$ such that for any $u, v \in X^h(\Omega)$*

$$|a^h(u, v)| \leq M \|u\|_{1,h} \|v\|_{1,h}, \quad (3.5)$$

where M is independent of h_i, N and ϱ_i .

Proof. We have (see (2.3))

$$a^h(u, v) = \sum_{i=1}^N a_i(u, v) + s_a(u, v) + s_p(u, v). \quad (3.6)$$

Using the Schwarz and Cauchy inequalities, we have

$$\sum_{i=1}^N a_i(u, v) \leq \sum_{i=1}^N \varrho_i \|\nabla u_i\|_{L^2(\Omega_i)} \|\nabla v_i\|_{L^2(\Omega_i)} \leq \|u\|_{1,h} \|v\|_{1,h} \quad (3.7)$$

and

$$s_p(u, v) \leq \sum_{i=1}^N \sum_{F_{ij} \subset \partial\Omega_i} 0.5 \frac{\delta}{h_i} \varrho_i \|u_i - u_j\|_{L^2(F_{ij})} \|v_i - v_j\|_{L^2(F_{ij})} \leq C \|u\|_{1,h} \|v\|_{1,h}. \quad (3.8)$$

It remains to show that

$$|s_a(u, v)| \leq C \|u\|_{1,h} \|v\|_{1,h}. \quad (3.9)$$

It follows to the proof of (3.2), see (3.3) and (3.4), where one of $u_i - u_j$ should be replaced by $v_i - v_j$. Using now the estimates (3.7), (3.8), (3.9) in (3.6), we get (3.5). $\square$

We now prove the error bound of the discussed method.

Theorem 3.2. *Let u^* and u_h^* be the solutions of (2.1) and (2.2). For $u^* \in H_0^1(\Omega)$ and $u_{|\Omega_i}^* \in H^{1+s}(\Omega_i)$, $s \in (0.5, 1]$*

$$\|u^* - u_h^*\|_{1,h}^2 \leq C \sum_{i=1}^N \left(h_i^{2s} + \sum_{F_{ij} \subset \partial\Omega_i} \frac{h_j^{1+2s}}{h_i} \right) \varrho_i |u^*|_{H^{1+s}(\Omega_i)}^2, \quad (3.10)$$

where $\bar{F}_{ij} = \bar{\Omega}_i \cap \bar{\Omega}_j$ and C is independent of h_i, N and the jumps of ϱ_i , holds.

Proof. Let $I_{h_i}u^*$ be an interpolant of u^* on Ω_i and $I_h u^*$ denote the interpolant on Ω such that $I_h u^* = I_{h_i}u^*$ on Ω_i . Note that $I_h u^*$ is discontinuous across $\partial\Omega_i$. By the triangle inequality we have

$$\|u^* - u_h^*\|_{1,h} \leq \|u^* - I_h u^*\|_{1,h} + \|I_h u^* - u_h^*\|_{1,h} \quad (3.11)$$

and

$$\begin{aligned} \|u^* - I_h u^*\|_{1,h}^2 &\leq \sum_{i=1}^N \left\{ \varrho_i \|u^* - I_{h_i}u^*\|_{H^1(\Omega_i)}^2 \right. \\ &\quad \left. + \sum_{F_{ij} \subset \partial\Omega_i} \frac{\varrho_i}{h_i} \|I_{h_i}u^* - I_{h_j}u^*\|_{L^2(F_{ij})}^2 \right\}. \end{aligned} \quad (3.12)$$

By the interpolation theorem in $H^1(\Omega_i)$ (see for example [2]), we have for $s \in (0.5, 1]$

$$\|u^* - I_{h_i}u^*\|_{H^1(\Omega_i)}^2 \leq C h_i^{2s} |u^*|_{H^{1+s}(\Omega_i)}^2. \quad (3.13)$$

To estimate the second term of (3.12), we use the interpolation theorem in $L^2(F_{ij})$ on the h_i - and h_j -triangulation. It gives

$$\|I_{h_i}u^* - I_{h_j}u^*\|_{L^2(F_{ij})}^2 \leq C (h_i^{2s+1} + h_j^{2s+1}) |u^*|_{H^{s+0.5}(F_{ij})}^2.$$

By the trace theorem, we get

$$\|I_{h_i}u^* - I_{h_j}u^*\|_{L^2(F_{ij})}^2 \leq C (h_i^{2s+1} + h_j^{2s+1}) |u^*|_{H^{1+s}(\Omega_i)}^2. \quad (3.14)$$

Using estimates (3.13) and (3.14) in (3.12), we obtain

$$\|u^* - I_h u^*\|_{1,h}^2 \leq C \sum_{i=1}^N \left(h_i^{2s} + \sum_{F_{ij} \subset \partial\Omega_i} h_j^{2s+1}/h_i \right) \varrho_i |u^*|_{H^{1+s}(\Omega_i)}^2. \quad (3.15)$$

We now estimate the second term of (3.11). By Lemma 3.1 we have

$$\| I_h u^* - u_h^* \|_{1,h}^2 \leq \gamma^{-1} a^h(I_h u^* - u_h^*, I_h u^* - u_h^*). \quad (3.16)$$

On the other hand,

$$a^h(I_h u^* - u_h^*, I_h u^* - u_h^*) = a^h(I_h u^*, I_h u^* - u_h^*) - f(I_h u^* - u_h^*) \quad (3.17)$$

by (2.2). Note that

$$f(I_h u^* - u_h^*) = a^h(u^*, I_h u^* - u_h^*).$$

To see that we integrate by parts the original problem, which corresponds to (2.1), over Ω_i and use the assumption on u^* . Note also that the terms with the penalty parameter vanish. Using this in (3.17) and the resulting equality in (3.16), we get

$$\| I_h u^* - u_h^* \|_{1,h}^2 \leq \gamma^{-1} a^h(I_h u^* - u^*, I_h u^* - u_h^*) \leq \gamma^{-1} M \| u^* - I_h u^* \|_{1,h} \| I_h u^* - u_h^* \|_{1,h}.$$

We have also used Lemma 3.2. The first term has already been estimated (see (3.15)). Using this in (3.11) and the estimate (3.15), we obtain (3.10). $\square$

4. Modified discrete problem

We modify the discrete problem (2.2) by using harmonic averages of the coefficients and meshsides on interfaces. In the case of jump coefficients across interfaces it is a natural way to use that. To get a discrete problem we replace ϱ_i on interfaces in (2.2) by $\varrho_{ij} = 2\varrho_i\varrho_j/(\varrho_i + \varrho_j)$, a harmonic average of ϱ_i and ϱ_j , and h_i by $h_{ij} = 2h_i h_j / (h_i + h_j)$.

Find $u_h^* \in X^h(\Omega)$ such that

$$\tilde{a}^h(u_h^*, v_h) = f(v_h), \quad v_h \in X^h(\Omega), \quad (4.1)$$

where

$$\tilde{a}^h(u, v) = \sum_{i=1}^N a_i(u, v) + \tilde{s}_a(u, v) + \tilde{s}_p(u, v)$$

with $a_i(.,.)$ as before and $\tilde{s}_a(.,.)$, $\tilde{s}_p(.,.)$ of the form:

$$\begin{aligned} \tilde{s}_a(u, v) \equiv & \sum_{i=1}^N \sum_{F_{ij} \subset \partial\Omega_i} 0.5 \left\{ \int_{F_{ij}} 0.5 \varrho_{ij} \left(\frac{\partial u_i}{\partial n} + \frac{\partial u_j}{\partial n} \right) (v_i - v_j) ds \right. \\ & \left. + \int_{F_{ij}} 0.5 \varrho_{ij} \left(\frac{\partial v_i}{\partial n} + \frac{\partial v_j}{\partial n} \right) (u_i - u_j) ds \right\} \end{aligned}$$

and

$$\tilde{s}_p(u, v) \equiv \sum_{i=1}^N \sum_{F_{ij} \subset \partial\Omega_i} 0.5 \int_{F_{ij}} \varrho_{ij} \frac{\delta}{h_{ij}} (u_i - u_j) (v_i - v_j) ds.$$

We have used the same notations as in Section 3.

Now the broken norm in $X^h(\Omega)$ is

$$\|u\|_{1,h}^2 \equiv \sum_{i=1}^N \left\{ \varrho_i \|\nabla u_i\|_{L^2(\Omega_i)}^2 + \sum_{F_{ij} \subset \partial\Omega_i} 0.5 \int_{F_{ij}} \frac{\varrho_{ij}}{h_{ij}} (u_i - u_j)^2 ds \right\}.$$

It is easy to see that Lemma 3.1 and Lemma 3.2 are valid for the form $\tilde{a}^h(.,.)$ for sufficiently large δ with constants γ and M different from the previous, in general (see (3.1) and (3.5)) but also independent of h_i and the jumps of ϱ_i . In the proof of these lemmas, we use additionally a simple observation that $\varrho_{ij} \leq 2\varrho_i$ and $\varrho_{ij} \leq 2\varrho_j$ and the same for h_{ij} . Using these lemmas, we get the following theorem.

Theorem 4.1. *The problem (4.1) has a unique solution and is stable for sufficiently large δ , the penalty term.*

We now analyze the error bound of the modified method. Unfortunately, the error bound is not optimal for $u_{|\Omega_i}^* \in H^2(\Omega_i)$, the solution of (2.1).

Theorem 4.2. *Let u^* and u_h^* be the solutions of (2.1) and (4.1). For $u^* \in H_0^1(\Omega)$ and $u_{|\Omega_i}^* \in H^{3/2+\alpha}(\Omega_i)$, $\alpha > 0$,*

$$\|u^* - u_h^*\|_{1,h}^2 \leq C \sum_{i=1}^N \left(h_i + \sum_{F_{ij} \subset \partial\Omega_i} \frac{h_j^2}{h_i} \right) |u^*|_{H^{3/2}(\Omega_i)}^2, \quad (4.2)$$

where C is independent of h_i and N , holds.

Proof. It follows to the proof of Theorem 3.2 with the following modification. We have (cf. (3.17))

$$\tilde{a}^h(I_h u^* - u_h^*, I_h u^* - u_h^*) = \tilde{a}^h(I_h u^*, I_h u^* - u_h^*) - f(I_h u^* - u_h^*).$$

Let $u_i^* = u_{|\Omega_i}^*$. Note that this time

$$f(I_h u^* - u_h^*) = \tilde{a}^h(u^*, I_h u^* - u_h^*) + b(u^*, I_h u^* - u_h^*),$$

where $(z \equiv I_h u^* - u_h^*)$

$$b(u^*, z) \equiv \sum_{i=1}^N 0.5 \left\{ \int_{F_{ij}} \left(0.5(\varrho_{ij} - \varrho_i) \frac{\partial u_i^*}{\partial n}(z_i - z_j) + 0.5(\varrho_{ij} - \varrho_j) \frac{\partial u_j^*}{\partial n}(z_i - z_j) \right) ds \right\}. \quad (4.3)$$

The first term is estimated as

$$\int_{F_{ij}} (\varrho_{ij} - \varrho_i) \frac{\partial u_i^*}{\partial n}(z_i - z_j) ds \leq C \left\{ \frac{1}{4\varepsilon} \varrho_i h_i \|u_i^*\|_{H^{3/2+\alpha}(\Omega_i)}^2 + \varepsilon \frac{\delta}{h_{ij}} \varrho_{ij} \|z_i - z_j\|_{L^2(F_{ij})}^2 \right\}$$

for any $\varepsilon > 0$. In the similar way is estimated the second term of (4.3). Using these estimates and proceeding as in the proof of Theorem 3.2, we get (4.2). $\square$

5. Multilevel ASM

We design and analyze a multilevel additive Schwarz method (ASM) for solving (4.1). For that we use the general framework of ASMs (see the book [8], pp. 153–160).

5.1. Decomposition

We associate with each Ω_i an initial shape regular triangulation $\tau_i^{(0)}$ and its parameter is denoted by $h_i^{(0)}$. For simplicity we assume that $h_i^{(0)}$ is of the order of H_i – the diameter of Ω_i . Note that the resulting triangulation $\tau^{(0)}$ on Ω is nonmatching at the common interfaces. Using uniform refinement techniques, we obtain a sequence of nested triangulation $\tau_i^{(k)}$, $k = 0, \dots, L$, in each Ω_i with the parameter at level k given by $h_i^{(k)} = 2^{-k}h_i^{(0)}$. The corresponding finite element spaces on the $\tau_i^{(k)}$ - triangulation are denoted by $X_i^k(\Omega_i) \equiv X_i^{h_i^k}(\Omega_i)$ – the spaces of piecewise linear continuous functions which vanish on $\partial\Omega_i \cap \partial\Omega$. We decompose $X^h(\Omega)$ as follows:

$$X^h(\Omega) \equiv X^L(\Omega) = X^0(\Omega) + X^1(\Omega) + \dots + X^L(\Omega),$$

where

$$X^k(\Omega) = X_1^k(\Omega_1) \times X^k(\Omega_2) \times \dots \times X_N^k(\Omega_N).$$

Note that $X_j^k(\Omega_j) \subset X_j^{k+1}(\Omega_j)$, therefore, $X^k(\Omega) \subset X^{k+1}(\Omega)$.

5.2. Inexact bilinear forms

$\tilde{a}^h(u, v)$ on $X^k(\Omega) \times X^k(\Omega)$ is approximated by

$$b^k(u, v) : X^k(\Omega) \times X^k(\Omega) \rightarrow R$$

of the form:

for $k = 0$

$$b^0(u, v) \equiv \tilde{a}^h(u, v);$$

for $k = 1, \dots, L$

$$b^k(u, v) \equiv \sum_{i=1}^N \sum_{x \in \bar{\Omega}_{i,k}} \left(1 + h_i^{(k)} h_{ij}^{-1}\right) \varrho_i u(x) v(x),$$

where $\bar{\Omega}_{i,k}$ is the set of nodal points of the $h_i^{(k)}$ - triangulation on $\bar{\Omega}_i$.

Let $T^k : X^L \rightarrow X^k$ be defined as

$$b^k(T^k u, v) = \tilde{a}^h(u, v), \quad v \in X^k(\Omega)$$

and

$$T \equiv T^0 + T^1 + \dots + T^L.$$

Problem (4.1) is replaced by

$$Tu_h^* = g_h, \quad g_h \equiv \sum_{k=0}^L g_k, \tag{5.1}$$

where g_k is the solution of

$$b^k(g_k, v) = f(v), \quad v \in X^k(\Omega).$$

5.3. Analysis

Theorem 5.1. *For $u \in X^h(\Omega)$*

$$C_0(L\beta)^{-1}\tilde{a}^h(u, u) \leq \tilde{a}^h(Tu, u) \leq C_1 L \tilde{a}^h(u, u)$$

holds, where C_0 and C_1 are positive constants independent of h_i, L and ϱ_i , and $\beta = \max_i \frac{H_i}{h_{ij}}$.

Proof. By the general theorem of ASMs (see [8], pp. 155–156) we need to check the three key assumptions.

Assumption(ii). By Lemma 3.2, which is also valid for $\tilde{a}^h(., .)$, we have

$$\tilde{a}^h(u, u) \leq M \|u\|_{1,\tilde{h}}^2, \quad u \in X^k, \quad k = 1, \dots, L.$$

Using the inverse inequality for $u \in X^k(\Omega)$, $k = 1, \dots, L$, $\varrho_{ij} \leq 2\varrho_i$ and $\varrho_{ij} \leq 2\varrho_j$, we have

$$\tilde{a}^h(u, u) \leq C b^k(u, u).$$

For $k = 0$ and $u \in X^0(\Omega)$

$$\tilde{a}^h(u, u) = b^0(u, u).$$

Thus $\omega_0 = 1$, $\omega_i = C$, $i = 1, \dots, L$, where C is independent of h_i, L and ϱ_i .

Assumption(i). We prove that for $u \in X^h(\Omega)$ there exists $u^k \in X^k(\Omega)$, $k = 0, \dots, L$, such that $u = \sum_{k=0}^L u^k$, $u^k \in X^k(\Omega)$, and

$$\sum_{k=0}^L b_k(u^k, u^k) \leq C_0 L \beta \tilde{a}^h(u, u). \quad (5.2)$$

Let us introduce Q_i^k – the L_2 -projection from $X_i^{h_i}(\Omega_i)$ to $X_i^k(\Omega_i)$ defined as follows: for $u_i \in X_i^{h_i}(\Omega_i)$, $Q_i^k u_i \in X_i^k(\Omega_i)$ is a solution of

$$(Q_i^k u_i, v_i)_{L^2(\Omega_i)} = (u_i, v_i)_{L^2(\Omega_i)}, \quad v_i \in X_i^k(\Omega_i).$$

Let for $u \in X^h(\Omega)$, $u = \{u_i\}_{i=1}^N$,

$$u_i = Q_i^0 u_i + (Q_i^1 u_i - Q_i^0 u_i) + \dots + (Q_i^k u_i - Q_i^{k-1} u_i) + \dots + (Q_i^L u_i - Q_i^{L-1} u_i) = \sum_{k=0}^L u_i^k,$$

where $u_i^0 \equiv Q_i^0 u_i$, $u_i^k \equiv Q_i^k u_i - Q_i^{k-1} u_i$, $k = 1, \dots, L$. Note that $u_i^k \in X_i^k(\Omega_i)$.

Let $u^k = \{u_i^k\}_{i=1}^N$. The term $b^k(u^k, u^k)$, $k = 1, \dots, L$ is estimated as follows:

$$b^k(u^k, u^k) \leq C \sum_{i=1}^N \frac{\varrho_i}{(h_i^{(k)})^2} \left\{ \|u_i^k\|_{L^2(\Omega_i)}^2 + h_i^{(k)} h_{ij}^{-1} \|u_i^k\|_{L^2(\Omega_i)}^2 \right\}. \quad (5.3)$$

Using the known properties of Q_i^k , we get

$$\|u_i^k\|_{L^2(\Omega_i)} \leq \|Q_i^k u_i - u_i\|_{L^2(\Omega_i)} + \|Q_i^{k-1} u_i - u_i\|_{L^2(\Omega_i)} \leq C h_i^{(k)} |u_i|_{H^1(\Omega_i)}.$$

Substituting this into (5.3), we obtain

$$b^k(u^k, u^k) \leq C \sum_{i=1}^N \frac{H_i}{h_{ij}} \varrho_i |\nabla u_i|_{H^1(\Omega_i)}^2. \quad (5.4)$$

We now estimate $b^0(u_i^0, u_i^0)$. We have

$$b^0(u^0, u^0) \leq C \sum_{i=1}^N \left\{ \varrho_i \|\nabla Q_i^0 u_i\|_{L^2(\Omega_i)}^2 + \sum_{F_{ij} \subset \partial\Omega_i} \frac{\varrho_{ij}}{h_{ij}} \|Q_i^0 u_i - Q_j^0 u_j\|_{L^2(F_{ij})}^2 \right\}.$$

Using again the properties of Q_i^0 , we get

$$b^0(u^0, u^0) \leq C \sum_{i=1}^N \frac{H_i}{h_{ij}} \varrho_i |\nabla u_i|_{L^2(\Omega_i)}^2. \quad (5.5)$$

Summing (5.4) with respect to $k = 1, \dots, L$ and adding (5.5) to the resulting inequality, we obtain (5.2).

Assumption (iii) Let $\varepsilon_{km} \in (0, 1]$ be the minimal values such that

$$\tilde{a}^h(u^k, u^m) \leq \varepsilon_{kl} (\tilde{a}^h(u^k, u^k))^{1/2} (\tilde{a}^h(u^m, u^m))^{1/2},$$

where $u^k \in X^k(\Omega)$ and $u^m \in X^m(\Omega)$. We need to prove that $\varrho(\varepsilon)$, the spectral radius of matrix ε , $\varepsilon = \{\varepsilon_{km}\}$, $k, m = 1, \dots, L$, is bounded. It straightforwardly follows that if $\varepsilon_{km} = 1$, then $\varrho(\varepsilon) \leq L$. $\square$

Remark 5.1. The suggested preconditioner is not optimal but very well suited for parallel computations. The local problems for $k = 1, \dots, L$ are defined on $\bar{\Omega}_i$, i.e., on matching triangulation with constant coefficients. These problems are independent and reduce to systems of algebraic equations with a diagonal matrix.

Remark 5.2. To solve (5.1), we can use a conjugate gradient method. The number of iterations of the method is proportional to $(H/h)^{1/2}L$ and is independent of the jumps of ϱ_i .

References

- [1] D. Arnold, *An interior penalty finite element method with discontinuous elements*, SIAM J. Numer. Anal., **19** (1982), pp. 742–760.
- [2] S. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer, 1994.
- [3] M. Dauge, *Elliptic Boundary Value Problems on Corner Domains*, Springer-Verlag, 1988.
- [4] C. Lasser and A. Toselli, *Convergence of some two-level overlapping domain decomposition preconditioner with smoothed aggregation coarse space*, in *Recent Developments in Domain Decomposition Methods* (L. F. Pavarino and A. Toselli eds.), Springer, 2002, pp. 95–117.
- [5] R. D. Lazarov, S. Z. Tomov and P. S. Vassilevski, *Interior penalty discontinuous approximations of elliptic problems*, Comput. Meth. Appl. Math., **1** (2001), No. 4, pp. 367–382.
- [6] M. Petzoldt, *Regularity results for interface problems in 2D*, Weierstrass Institute for Applied Mathematics and Stochastic (TR), No. 565, Berlin, 2000.
- [7] B. Riviere, M. F. Wheeler and V. Girault, *A priori error estimates for finite element methods based on discontinuous approximation spaces for elliptic problems*, SIAM J. Numer. Anal., **39** (2001), pp. 902–931.
- [8] B. Smith, P. Bjorstad and W. Gropp, *Domain Decomposition, Parallel Multilevel Methods for Elliptic Partial Differential Equations*, Cambridge University Press, 1996.