ON ITERATIVE REGULARIZATION METHODS FOR VARIATIONAL INEQUALITIES OF THE SECOND KIND WITH PSEUDOMONOTONE OPERATORS ¹

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Abstract — Variational inequality of the second kind

$$\langle A_0 u, \eta - u \rangle + F_1(\eta) - F_1(u) \geq \langle f, \eta - u \rangle$$

with a pseudomonotone operator A_0 and a convex non-differentiable functional F_1 in the Banach or Hilbert space is considered. A "semi-implicit" iterative method with explicit approximation of the operator A_0 and regularized functional F_ε for its solution is studied.

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1. Introduction

In this paper, we study variational inequalities of the second kind with a pseudomonotone operator and convex non-differentiable functional in Banach spaces that occur, in particular, in describing stationary seepage processes and problems of defining soft shell equilibrium. To solve a variational inequality, a two-level method of iterative regularization is suggested. It allows to reduce the solution of the original variational inequality and the regularized functional to the solution of the variational inequality with a duality operator. The duality operator has better properties compared to the original operator. Iterative processes of this type have already been considered earlier (see, e.g., [1, 2, 4, 5, 8, 11, 14, 16]). However, these investigations considered only cases with strongly monotone operators or maximal monotone operators.

We investigate the convergence of an iterative process. It is proved that the iterative sequence is bounded and any of its weak limit points are a solutions of the original inequality. In the case of a Hilbert space under an additional limitation, a weak convergence of the whole iterative sequence is proved.

Note that in [6] we investigated an iterative method for solving the same variational inequalities without regularization. At each step of this method it is necessary to solve a

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convex non-differentiable problem, which causes an additional difficulty in implementation of the algorithm. The method suggested in this paper leads to a much easier problem (in the sense of its implementation) if the regularized functional becomes differentiable.

The general results obtained are applied to a stationary seepage problem for an incompressible liquid that follows a discontinuous seepage law with the limit gradient (see [3, 12, 16]). Some results regarding properties of the solution of seepage problems are also obtained.

The proposed algorithm can be applied to the solution of a wide class of problems with non-monotone operators. One of the most important examples is the problem of the soft shells theory (see, e.g., [15]) where operator is the sum of the nonlinear monotone operator and the linear self-adjoint indefinite one.

2. The case of the Banach space

Let V be a reflexive Banach space with a uniformly convex conjugate space V^* , $\langle \cdot, \cdot \rangle$ is the duality relation between V and V^* , M is the convex closed set in V , $A_0 : V \rightarrow V^*$ is the pseudomonotone, coercive operator that satisfies the conditions (see [3, 9])

$$\int_0^1 (\langle A_0(t(u+v)), u+v \rangle - \langle A_0(tu), u \rangle) dt = \int_0^1 \langle A_0(u+tv), v \rangle dt \quad \forall u, v \in V, \quad (1)$$

$$\|A_0u - A_0v\|_{V^*} \leq \mu(R)\Phi(\|u - v\|_V) \quad \forall u, v \in V, \quad (2)$$

where $R = \max\{\|u\|_V, \|v\|_V\}$, μ is the nondecreasing function on $[0, +\infty)$, Φ is the strictly increasing continuous function on $[0, +\infty)$ such that $\Phi(0) = 0$, $\Phi(\xi) \rightarrow +\infty$ as $\xi \rightarrow +\infty$.

It will be recalled (see [13]) that the operator A_0 is pseudomonotone if A_0 is a bounded operator and from the weak convergence of the sequence $\{u_n\}_{n=1}^\infty \in V$ to u in V and inequality

$$\limsup_{n \rightarrow \infty} \langle A_0u_n, u_n - u \rangle \leq 0 \quad (3)$$

it follows that

$$\liminf_{n \rightarrow \infty} \langle A_0u_n, u_n - v \rangle \geq \langle A_0u, u - v \rangle \quad \forall v \in V. \quad (4)$$

By the operator coerciveness is meant the realization of the following inequality:

$$\langle A_0u, u \rangle \geq \rho(\|u\|_V)\|u\|_V; \quad \lim_{\xi \rightarrow +\infty} \rho(\xi) = +\infty. \quad (5)$$

Let later $F_1 : V \rightarrow R^1$ be a convex (in general, non-differentiable) weak lower semi-continuous functional.

We define the functional $F : V \rightarrow R^1$ by the following relation:

$$F(u) = F_0(u) + F_1(u) - \langle f, u \rangle, \quad F_0(u) = \int_0^1 \langle A_0(tu), u \rangle dt, \quad f \in V^*. \quad (6)$$

Here from (1), (6) it follows that for any $u, v \in V$ the following equality is valid:

$$F(u) - F(v) = \int_0^1 \langle A_0(v + t(u-v)), u - v \rangle dt + F_1(u) - F_1(v) - \langle f, u - v \rangle. \quad (7)$$

We observe that by virtue of the convexity of the functional F_1 and condition (5) the functional F is coercive (see, e.g., [7]).

Now we consider the problem of searching the element $u \in M$, which is a solution of the following variational inequality of the second kind:

$$\langle A_0 u, \eta - u \rangle + F_1(\eta) - F_1(u) \geq \langle f, \eta - u \rangle \quad \forall \eta \in M. \quad (8)$$

We consider for $\varepsilon > 0$ a convex functional F_ε satisfying the conditions

$$|F_\varepsilon(\eta) - F_1(\eta)| \leq c(\varepsilon), \quad \forall \eta \in V, \quad \lim_{\varepsilon \rightarrow 0} c(\varepsilon) = 0. \quad (9)$$

$$|F_\varepsilon(v) - F_\varepsilon(u)| \leq \gamma^* \|v - u\|_V \quad \forall u, v \in V, \quad \gamma^* > 0. \quad (10)$$

From (9), (10) it follows that the functional F_1 is Lipschitz continuous with the constant γ^* .

In order to solve problem (8), we consider the following iterative process which permits reducing it to a variational inequality of the second kind with a duality operator and a regularized functional whose solution can be obtained by the known methods (see, e.g., [8, 10, 14, 19]). It should be noted that the approach under consideration is very useful when the regularized functional F_ε becomes differentiable.

Let u_0 be an arbitrary element of M . For $n = 0, 1, 2, \dots$ we define $u_{n+1} \in M$ as a solution of the variational inequality

$$\langle J(u_{n+1} - u_n), v - u_{n+1} \rangle + \tau(F_{\varepsilon_n}(v) - F_{\varepsilon_n}(u_{n+1})) \geq \tau \langle f - A_0 u_n, v - u_{n+1} \rangle \quad \forall v \in M, \quad (11)$$

$J : V \rightarrow V^*$ is the duality operator generated by the function Φ which satisfies the conditions (see [13])

$$\langle Jv, v \rangle = \|Jv\|_{V^*} \|v\|_V = \Phi(\|v\|_V) \|v\|_V \quad \forall v \in V, \quad (12)$$

the sequence $\{\varepsilon_n\}_{n=1}^\infty$ satisfies the condition

$$\sum_{n=1}^{+\infty} c(\varepsilon_n) = \sigma < +\infty, \quad (13)$$

$\tau > 0$ is the iterative parameter that satisfies the condition

$$0 < \tau < \min\{1, 1/\mu_0\}, \quad \mu_0 = \mu(R_0 + \Phi^{-1}(R_1 + \gamma^*)), \quad (14)$$

where

$$R_0 = \sup_{u \in S_0} \|u\|_V, \quad R_1 = \sup_{u \in S_0} \|A_0 u - f\|_{V^*}, \quad S_0 = \{u \in M : F(u) \leq F(u_0) + 2\sigma\}.$$

We observe that from the coerciveness of the functional F it follows that $R_0 < +\infty$, while from the boundedness of the operator A_0 we have that $R_1 < +\infty$. Thus, $\mu_0 < +\infty$, i.e., the iterative process (11) is well defined.

The solvability of the variational inequalities (8), (11) follows from the results of [13, p. 265], [7, p. 52].

The following result is valid.

Theorem 2.1. *Let condition (13) be satisfied. Then the iterative sequence $\{u_n\}_{n=1}^\infty$ constructed according to (11) is bounded in V and all of its weakly limited points are solutions of problem (8).*

Proof. Let us first prove the boundedness of the iterative sequence and check that the following condition is satisfied:

$$\{u_n\}_{n=1}^{\infty} \subset S_0, \quad (15)$$

and hence $\|u_n\| \leq R_0$, $n = 0, 1, 2, \dots$

By definition, $u_0 \in S_0$. Let $u_n \in S_0$, then we prove that $u_{n+1} \in S_0$. By setting in (11) $v = u_n$ and using (10), (12) and $\tau \leq 1$, we obtain that

$$\begin{aligned} \Phi(\|u_n - u_{n+1}\|_V) \|u_n - u_{n+1}\|_V &= \langle J(u_{n+1} - u_n), u_{n+1} - u_n \rangle \\ &\leq \tau [\langle f - A_0 u_n, u_{n+1} - u_n \rangle + F_{\varepsilon_n}(u_n) - F_{\varepsilon_n}(u_{n+1})] \\ &\leq [\|A_0 u_n - f\|_{V^*} + \gamma^*] \|u_n - u_{n+1}\|_V, \end{aligned}$$

and using a strict increase of Φ we have

$$\|u_n - u_{n+1}\|_V \leq \Phi^{-1}(\|A_0 u_n - f\|_{V^*} + \gamma^*). \quad (16)$$

Furthermore, according to (2), for $t \in [0, 1]$ we have

$$\begin{aligned} |\langle A_0(u_{n+1} + t(u_n - u_{n+1})) - A_0 u_n, (u_n - u_{n+1}) \rangle| \\ \leq \mu(R_*) \Phi(\|(t-1)(u_n - u_{n+1})\|_V) \|u_n - u_{n+1}\|_V \\ \leq \mu(R_*) \Phi(\|u_n - u_{n+1}\|_V) \|u_n - u_{n+1}\|_V, \end{aligned} \quad (17)$$

where $R_* = \max\{\|u_{n+1} + t(u_n - u_{n+1})\|_V, \|u_n\|_V\}$.

On the other hand, by virtue of (16)

$$\begin{aligned} \|u_{n+1} + t(u_n - u_{n+1})\|_V - \|u_n\|_V &\leq \|(1-t)(u_n - u_{n+1})\|_V \leq \|u_n - u_{n+1}\|_V \\ &\leq \Phi^{-1}(\|A_0 u_n - f\|_{V^*} + \gamma^*). \end{aligned}$$

Therefore

$$R_* \leq \|u_n\|_V + \Phi^{-1}(\|A_0 u_n - f\|_{V^*} + \gamma^*) \leq R_0 + \Phi^{-1}(R_1 + \gamma^*).$$

Since μ is a nondecreasing function, $\mu(R_*) \leq \mu_0$. Consequently, from (17) it follows that

$$-|\langle A_0(u_{n+1} + t(u_n - u_{n+1})) - A_0 u_n, (u_n - u_{n+1}) \rangle| \geq -\mu_0 \Phi(\|u_n - u_{n+1}\|_V) \|u_n - u_{n+1}\|_V. \quad (18)$$

Furthermore, similarly to [2, 9] from (7) we have

$$\begin{aligned} F(u_n) - F(u_{n+1}) &= \int_0^1 \langle A_0(u_{n+1} + t(u_n - u_{n+1})), u_n - u_{n+1} \rangle dt - \langle f, u_n - u_{n+1} \rangle + F_1(u_n) - F_1(u_{n+1}) \\ &= \int_0^1 \langle A_0(u_{n+1} + t(u_n - u_{n+1})) - A_0 u_n, u_n - u_{n+1} \rangle dt - \langle f - A_0 u_n, u_n - u_{n+1} \rangle \\ &\quad + F_1(u_n) - F_1(u_{n+1}). \end{aligned}$$

Hence, using (11) with $v = u_n$ and (9), (18), we obtain

$$\begin{aligned}
 F(u_n) - F(u_{n+1}) &\geq - \int_0^1 |\langle A_0(u_{n+1} + t(u_n - u_{n+1})) - A_0 u_n, (u_n - u_{n+1}) \rangle| dt \\
 &\quad + \tau^{-1} [\tau \langle f - A_0 u_n, u_{n+1} - u_n \rangle + \tau (F_{\varepsilon_n}(u_n) - F_{\varepsilon_n}(u_{n+1}))] \\
 &\quad + (F_1(u_n) - F_{\varepsilon_n}(u_n)) + (F_{\varepsilon_n}(u_{n+1}) - F_1(u_{n+1})) \\
 &\geq -\mu_0 \Phi(\|u_n - u_{n+1}\|_V) \|u_n - u_{n+1}\|_V \\
 &\quad + \tau^{-1} \langle J(u_{n+1} - u_n), u_{n+1} - u_n \rangle - 2c(\varepsilon_n) \\
 &\geq \lambda \Phi(\|u_n - u_{n+1}\|_V) \|u_n - u_{n+1}\|_V - 2c(\varepsilon_n), \quad \lambda = \frac{1}{\tau} - \mu_0 > 0.
 \end{aligned} \tag{19}$$

Thus, we have

$$F(u_{n+1}) \leq F(u_n) + 2c(\varepsilon_n). \tag{20}$$

Adding these inequalities at $n = 0, 1, \dots, N-1$, we obtain

$$F(u_{n+1}) \leq F(u_0) + 2 \sum_{n=0}^N c(\varepsilon_n) \leq F(u_0) + 2\sigma,$$

i.e., $u_{n+1} \in S_0$. Statement (15) has been proved.

We need the following result (see [21, p. 93]):

Lemma 2.1. *Let the numerical sequence $\{a_k\}_{k=1}^\infty$ satisfy the conditions*

$$a_{k+1} \leq a_k + \delta_k, \delta_k \geq 0, \quad k = 1, 2, \dots, \quad \sum_{k=1}^{+\infty} \delta_k < +\infty.$$

Then there exists

$$\lim_{k \rightarrow +\infty} a_k < +\infty.$$

If, in addition, this sequence $\{a_k\}_{k=1}^\infty$ is lower-bounded, then $\lim_{k \rightarrow +\infty} a_k$ is finite.

The conditions of Lemma 2.1 for the sequence $\{F(u_n)\}$ are true because of (13), (20).

Indeed, from the definition of the functional F given in (6) we have

$$F(u) \geq -\mu(\|u\|_V) \Phi(\|u\|_V) \|u\|_V - \gamma^* \|u\|_V - \|f\|_{V^*} \|u\|_V + F_1(0) - \|A_0(0)\|_{V^*} \|u\|_V.$$

From this inequality, the boundedness of the operator A_0 and the boundedness of the iterative sequence it follows that the sequence $\{F(u_n)\}_{n=1}^\infty$ is bounded. It means that the numerical sequence $\{F(u_n)\}_{n=1}^\infty$ has a finite limit. Therefore, from (19) we have

$$\lim_{n \rightarrow +\infty} \lambda \Phi(\|u_n - u_{n+1}\|_V) \|u_n - u_{n+1}\|_V = 0.$$

From this equality, by virtue of the continuity and the strict increase of the function Φ , it follows that

$$\lim_{n \rightarrow +\infty} \|u_n - u_{n+1}\|_V = 0. \tag{21}$$

Furthermore, for an arbitrary function $v \in M$ from (11) we obtain

$$\begin{aligned}
\langle A_0 u_n, u_n - v \rangle &\leq \langle A_0 u_n, u_n - u_{n+1} \rangle + \frac{1}{\tau} \langle J(u_{n+1} - u_n), v - u_{n+1} \rangle + (F_{\varepsilon_n}(v) - F_{\varepsilon_n}(u_n)) \\
&\quad + (F_{\varepsilon_n}(u_n) - F_{\varepsilon_n}(u_{n+1})) + \langle f, u_n - v \rangle + \langle f, u_{n+1} - u_n \rangle \\
&\leq (\|A_0 u_n\|_{V^*} + \|f\|_{V^*} + \gamma^*) \|u_{n+1} - u_n\|_V \\
&\quad + \frac{1}{\tau} \Phi(\|u_{n+1} - u_n\|_V) \|v - u_{n+1}\|_V + (F_{\varepsilon_n}(v) - F_{\varepsilon_n}(u_n)) + \langle f, u_n - v \rangle \\
&\leq C_v (\Phi(\|u_{n+1} - u_n\|_V) + \|u_{n+1} - u_n\|_V) + (F_1(v) - F_1(u_n)) \\
&\quad + \langle f, u_n - v \rangle + 2c(\varepsilon_n),
\end{aligned} \tag{22}$$

where C_v is a positive constant that depends on v .

From the boundedness of the iterative sequence and reflexivity of the V space there follows the existence of the subsequence $\{u_{n_k}\}_{k=1}^{\infty}$, which converges weakly to u^* in V as $k \rightarrow +\infty$. Let us see that for this subsequence inequality (3) is satisfied. From (22) with $v = u^*$, taking into account (21) and the weak lower semi-continuity of the functional F_1 , we obtain

$$\begin{aligned}
\limsup_{k \rightarrow \infty} \langle A_0 u_{n_k}, u_{n_k} - u^* \rangle &\leq \limsup_{k \rightarrow \infty} C_{u^*} (\Phi(\|u_{n_k+1} - u_{n_k}\|_V) + \|u_{n_k+1} - u_{n_k}\|_V) \\
&\quad + \limsup_{k \rightarrow \infty} (F_1(u^*) - F_1(u_{n_k})) + \limsup_{k \rightarrow \infty} \langle f, u_{n_k} - u^* \rangle + 2 \limsup_{k \rightarrow \infty} c(\varepsilon_{n_k}) \leq 0.
\end{aligned}$$

We show that u^* is a solution of problem (8). From (22) for an arbitrary function $v \in M$ we have

$$\begin{aligned}
C_v [\Phi(\|u_{n_k+1} - u_{n_k}\|_V) + \|u_{n_k+1} - u_{n_k}\|_V] \\
\geq \langle A_0 u_{n_k}, u_{n_k} - v \rangle + (F_1(u_{n_k}) - F_1(v)) + \langle f, u_{n_k} - v \rangle - 2c(\varepsilon_{n_k}),
\end{aligned}$$

consequently, by virtue of (21), the weak lower semi-continuity of the functional F_1 , and the pseudomonotonicity of the operator A_0 (inequality (4)) we have

$$\begin{aligned}
0 &= \liminf_{k \rightarrow \infty} C_v [\Phi(\|u_{n_k+1} - u_{n_k}\|_V) + \|u_{n_k+1} - u_{n_k}\|_V] + 2c(\varepsilon_{n_k}) \\
&\geq \liminf_{k \rightarrow \infty} \langle A_0 u_{n_k}, u_{n_k} - v \rangle + \liminf_{k \rightarrow \infty} (F_1(u_{n_k}) - F_1(v)) + \liminf_{k \rightarrow \infty} \langle f, v - u_{n_k} \rangle \\
&\geq \langle A_0 u^*, u^* - v \rangle + F_1(u^*) - F_1(v) + \langle f, v - u^* \rangle,
\end{aligned}$$

i.e., u^* is the solution of the variational inequality (8). $\square$

Remark 2.1. If the subsequence $\{u_{n_k}\}_{k=0}^{\infty}$ of the iterative sequence $\{u_n\}_{n=0}^{\infty}$ converges weakly to u^* in V as $k \rightarrow +\infty$, then the following equality is true:

$$\lim_{k \rightarrow +\infty} \langle A_0 u_{n_k} - A_0 u^*, u_{n_k} - u^* \rangle = 0.$$

3. The case of a Hilbert space

We consider another case where the space V is a Hilbert space. We denote by $(\cdot, \cdot)_V$ the scalar product in it and identify V^* with V . Let, further, the operator A_0 instead of (2) satisfy the condition

$$\|A_0 u - A_0 v\|_V^2 \leq d_0 (A_0 u - A_0 v, u - v)_V, \quad d_0 > 0 \quad \forall u, v \in V. \tag{23}$$

We observe that inequality (2) is a sequence of inequality (23) with $\mu(\xi) = d_0$, $\Phi(\xi) = \xi$. We shall prove the following.

Theorem 3.1. Let u_0 be an arbitrary element of M . For $n = 0, 1, 2, \dots$ we define $u_{n+1} \in M$ as a solution of the variational inequality

$$(u_{n+1} - u_n, v - u_{n+1})_V + \tau(F_{\varepsilon_n}(v) - F_{\varepsilon_n}(u_{n+1})) \geq \tau(f - A_0 u_n, v - u_{n+1})_V \quad \forall v \in M, \quad (24)$$

where $0 < \tau < \tau_0 = 2/d_0$. Then the iterative sequence $\{u_n\}_{n=0}^\infty$ converges weakly in V for the solution of problem (8).

Proof. Following the proof of Theorem 2.1 with explicit changes, it is easy to check that the sequence $\{u_n\}_{n=0}^\infty$ is bounded and any of its weak limit points is the solution of problem (8).

Let u^* be an arbitrary solution of problem (8), i.e.,

$$\tau(F_1(v) - F_1(u^*)) \geq \tau(f - A_0 u^*, v - u^*)_V \quad \forall v \in M.$$

We substitute into the last inequality $v = u_{n+1}$, $v = u^*$ into (24) and add the obtained inequalities. We have

$$\begin{aligned} (w_{n+1}, w_{n+1})_V &\leq (w_n, w_{n+1})_V - \tau(A_0 u_n - A_0 u^*, w_{n+1})_V \\ &\quad + \tau(F_{\varepsilon_n}(u^*) - F_{\varepsilon_n}(u_{n+1})) + \tau(F_1(u_{n+1}) - F_1(u^*)), \end{aligned}$$

where $w_n = u_n - u^*$, $n = 0, 1, \dots$. Hence, using (10), we obtain

$$\|w_{n+1}\|_V \leq \|w_n - \tau(A_0 u_n - A_0 u^*)\|_V + 2\tau c(\varepsilon_n).$$

Then

$$\|w_{n+1}\|_V^2 \leq \frac{1}{2} \|w_n - \tau(A_0 u_n - A_0 u^*)\|_V^2 + \frac{1}{2} \|w_{n+1}\|_V^2 + 2\tau c(\varepsilon_n),$$

and finally

$$\|w_{n+1}\|_V^2 \leq \|w_n - \tau(A_0 u_n - A_0 u^*)\|_V^2 + 4\tau c(\varepsilon_n). \quad (25)$$

From inequality (25) we obtain

$$\|w_{n+1}\|_V^2 \leq \|w_n\|_V^2 - 2\tau(A_0 u_n - A_0 u^*, u_n - u^*)_V + \tau^2 \|A_0 u_n - A_0 u^*\|_V^2 + 4\tau c(\varepsilon_n). \quad (26)$$

Using inequality (23), from (26) we have

$$\|w_{n+1}\|_V^2 \leq \|w_n\|_V^2 - \tau(2 - d_0 \tau)(A_0 u_n - A_0 u^*, u_n - u^*)_V + 4\tau c(\varepsilon_n).$$

Thus, when $0 < \tau \leq 2/d_0$ the numerical sequence $\{\|u_n - u^*\|_V\}_{n=1}^\infty$ satisfies the conditions of Lemma 2.1 and, consequently, there is a finite limit of this numerical sequence as $n \rightarrow +\infty$

$$\lim_{n \rightarrow +\infty} \|u_n - u^*\|_V = \lambda_{u^*}. \quad (27)$$

We shall prove that all weak limit points of the sequence $\{u_n\}_{n=0}^\infty$ coincide. From this there will follow a weak convergence of the whole iterative sequence to a solution of problem (8).

Let v, w be two weak limit points of the sequence $\{u_n\}_{n=0}^\infty$, i.e., there exist subsequences $\{u_{n_k}\}_{k=0}^\infty, \{u_{m_k}\}_{k=0}^\infty$ that converge weakly in V to v and w respectively as $k \rightarrow +\infty$. By virtue of Theorem 2.1 the elements v and w are solutions of problem (8), consequently, for the whole sequence $\{u_n\}_{n=0}^\infty$ relations (27) with $u^* = v$ and $u^* = w$ hold, i.e.,

$$\lim_{n \rightarrow +\infty} \|u_n - v\|_V = \lambda_v, \quad \lim_{n \rightarrow +\infty} \|u_n - w\|_V = \lambda_w.$$

Obviously, these relations also hold for any subsequences of the sequence $\{u_n\}_{n=0}^\infty$, in particular for the subsequences $\{u_{n_k}\}_{k=0}^\infty$, $\{u_{m_k}\}_{k=0}^\infty$

$$\lim_{k \rightarrow +\infty} \|u_{n_k} - v\|_V = \lambda_v, \quad \lim_{k \rightarrow +\infty} \|u_{m_k} - w\|_V = \lambda_w. \quad (28)$$

According [17, 18], we consider the numerical sequence $\{e_k\}_{k=0}^\infty$ defined by the formula

$$e_k = \|u_{n_k} - v\|_V^2 - \|u_{n_k} - w\|_V^2 - \|u_{m_k} - v\|_V^2 + \|u_{m_k} - w\|_V^2, \quad k = 0, 1, 2, \dots$$

From (28) we have

$$\lim_{k \rightarrow +\infty} e_k = 0.$$

On the other hand, by the direct calculations it is easy to verify that

$$e_k = -2(u_{n_k} - u_{m_k}, v - w)_V,$$

hence, by the weak convergence of the subsequences $\{u_{n_k}\}_{k=0}^\infty$, $\{u_{m_k}\}_{k=0}^\infty$ in V to v and w , respectively, as $k \rightarrow +\infty$ we obtain that

$$\lim_{k \rightarrow +\infty} e_k = -2(v - w, v - w)_V = -2\|v - w\|_V^2.$$

Thus, we have $\|v - w\|_V = 0$, i.e., $v = w$. □

4. Application to seepage problems with a limiting gradient in a domain with a semipermeable boundary

Let Ω be a bounded domain in R^m , $m \geq 1$, with a Lipschitz continuous boundary Γ , $\xi \rightarrow g(\xi^2)\xi$ is a function defining the filtration law. We assume that $g(\xi^2)\xi = g_0(\xi^2)\xi + g_1(\xi^2)\xi$ and the following conditions are satisfied:

$$g_i(\xi^2)\xi \geq 0 \quad \text{for} \quad \xi \geq 0, \quad g_i(\xi^2)\xi = 0 \quad \text{for} \quad \xi \leq \beta, \quad i = 0, 1, \quad (29)$$

$$g_0(\xi^2)\xi \quad \text{is continuous,} \quad (g_0(\xi^2)\xi)'_\xi > 0 \quad \text{for} \quad \xi \geq \beta, \quad (30)$$

$\beta \geq 0$ is a limiting gradient,

there exist $c_0, c_1, c_2 > 0$, $p > 1$ such that

$$c_1(\xi - \beta)^{p-1} \leq g_0(\xi^2)\xi \leq c_2(\xi - \beta)^{p-1} \quad \text{for} \quad \xi \geq \beta, \quad (31)$$

$$(g_0(\xi^2)\xi - g_0(\eta^2)\eta)/(\xi - \eta) \leq c_0(1 + \xi + \eta)^{p-2} \quad \forall \xi, \eta \in R^1 > 0, \quad (32)$$

$$g_1(\xi^2)\xi = \vartheta \quad \text{for} \quad \xi > \beta. \quad (33)$$

Furthermore let $\Gamma = \Gamma_0 \cup \Gamma_1$, $\text{mes } \Gamma_0 > 0$, $\text{mes } \Gamma_1 > 0$, $V = \{v \in W_p^{(1)}(\Omega) : v(x) = 0, x \in \Gamma_0\}$, $M = \{v \in V : v(x) \geq 0, x \in \Gamma_1\}$ (Γ_1 is a semipermeable part of the boundary), $A_0 : V \rightarrow V^*$ is an operator generated by the form

$$\langle A_0 u, v \rangle = \int_{\Omega} g_0(|\nabla u|^2)(\nabla u, \nabla v) dx, \quad u, v \in V, \quad (34)$$

where $(\cdot, \cdot)$ and $|\cdot|$ are the inner production and norm in R^m respectively.

We also define the functional $F_1 : V \rightarrow R^1$ by the relation

$$F_1(v) = \int_{\Omega} \int_0^{|\nabla v|} g_1(\xi^2) \xi d\xi dx = \vartheta \int_{\Omega} (|\nabla v| - \beta)^+ dx, \quad a^+ = (|a| + a)/2.$$

We call the function $u \in M$ the solution of the seepage problem of an incompressible liquid which is the solution of the variational inequality (8) (see [3, 12]).

We investigate some solution properties of problem (8). We need the following lemma (see [5]).

Lemma 4.1. *Let the function $\psi : R^1 \rightarrow R^1$ be such that $\psi(x^2)x$ monotonely increases when $x \geq 0$. Then the function $H : R^m \times R^m \rightarrow R^1$,*

$$H(\xi, \zeta) \equiv (\psi(|\xi|^2)\xi - \psi(|\zeta|^2)\zeta, \xi - \zeta)$$

is non-negative. Moreover, if $\psi(x^2) = 0$ when $0 \leq x \leq \alpha$, $\alpha > 0$, $\psi(x^2)x$ is strictly increasing when $x > \alpha$ then for the case where $H(\xi, \zeta) = 0$:

$$1) \xi = \zeta \text{ for } |\xi| > \alpha, \quad 2) |\zeta| \leq \alpha \text{ for } |\xi| \leq \alpha.$$

Lemma 4.2. *Let for $\eta_1, \eta_2 \in V$ the equality*

$$\langle A_0\eta_1 - A_0\eta_2, \eta_1 - \eta_2 \rangle = 0$$

hold. Then with an accuracy to the set of measure zero the following equalities are satisfied:

$$\Omega_{\eta_1}^+ = \Omega_{\eta_2}^+ = \Omega^*, \quad \Omega_{\eta}^+ = \{x \in \Omega : |\nabla \eta(x)| > \beta\}, \quad (35)$$

besides,

$$\nabla \eta_1 = \nabla \eta_2 \quad \text{almost everywhere on } \Omega^*, \quad (36)$$

$$g(|\nabla \eta_1|^2) \nabla \eta_1 = g(|\nabla \eta_2|^2) \nabla \eta_2 \quad \text{almost everywhere on } \Omega. \quad (37)$$

Proof. By virtue of Lemma 4.1 with $\psi = g$ we have

$$0 = \langle A_0\eta_1 - A_0\eta_2, \eta_1 - \eta_2 \rangle = \int_{\Omega} H(\nabla \eta_1, \nabla \eta_2) dx \geq 0,$$

i.e.,

$$\int_{\Omega} H(\nabla \eta_1, \nabla \eta_2) dx = 0.$$

The subintegral function is non-negative, hence $H(\nabla \eta_1, \nabla \eta_2) = 0$ almost everywhere on Ω . But then from statement 2) of Lemma 4.1 it follows that

$$\Omega \setminus \Omega_{\eta_1}^+ = \Omega \setminus \Omega_{\eta_2}^+,$$

from where (35) follows. From statement 1) of Lemma 4.1 we obtain that equality (36) holds.

From (35), (36) follow the equalities

$$g(|\nabla \eta_1|^2) \nabla \eta_1 = g(|\nabla \eta_2|^2) \nabla \eta_2,$$

almost everywhere on Ω^* .

On the set $\Omega \setminus \Omega^*$ we have

$$g(|\nabla \eta_1|^2) \nabla \eta_1 = g(|\nabla \eta_2|^2) \nabla \eta_2 = 0,$$

i.e., equality (37) holds. $\square$

Theorem 4.1. *The seepage velocity v , constructed by the solution u of problem (8) in accordance with the formula*

$$v = -g(|\nabla u|^2) \nabla u,$$

is defined uniquely; for any solutions u_1, u_2 of problem (8) with an accuracy to the set of measure zero the equality

$$\Omega_{u_1}^+ = \Omega_{u_2}^+ = \Omega^*$$

holds.

Proof. Let u_1, u_2 be two solutions of problem (8)

$$\langle A_0 u_1, \eta - u_1 \rangle + F_1(\eta) - F_1(u_1) \geq \langle f, \eta - u_1 \rangle \quad \forall \eta \in M, \quad (38)$$

$$\langle A_0 u_2, \eta - u_2 \rangle + F_1(\eta) - F_1(u_2) \geq \langle f, \eta - u_2 \rangle \quad \forall \eta \in M. \quad (39)$$

Putting $\eta = u_2$ in (38), $\eta = u_1$ in (39) and adding the obtained inequalities, taking into account the monotonicity of the operator A_0 that follows from Lemma 4.1, we have

$$\langle A_0 u_1 - A_0 u_2, u_1 - u_2 \rangle = 0.$$

The statement of the theorem follows now from Lemma 4.2. $\square$

By virtue of conditions (29) - (33) the operator A_0 is monotone, coercive, potential (thus, equality (1) holds) (see [3, 16]) and satisfies condition (2) (see [2]) with the functions Φ, μ , that are defined by the formulas

$$\Phi(\xi) = \begin{cases} \xi^{p-1}, & 1 < p < 2, \\ \xi, & p \geq 2, \end{cases} \quad \mu(\xi) = \begin{cases} c_4, & c_4 > 0, \quad 1 < p < 2, \\ c_5(1 + 2\xi)^{p-2}, & c_5 > 0, \quad p \geq 2. \end{cases}$$

From the definition of the functional F_1 , the second of the conditions (29), and equality (33) the next lemma follows (see [4]).

Lemma 4.3. *The functional F_1 must satisfy the condition*

$$|F_1(v) - F_1(u)| \leq \vartheta \|\nabla(v - u)\|_{L_1}.$$

From Lemma 4.3 it follows that the functional F_1 is Lipschitz continuous.

We now define the regularized function

$$g_{1\varepsilon}(\xi^2)\xi = \begin{cases} 0, & \xi \leq \beta - \varepsilon, \\ \vartheta(\xi - \beta + \varepsilon)/\varepsilon, & \beta - \varepsilon \leq \xi \leq \beta, \\ \vartheta, & \xi \geq \beta. \end{cases} \quad (40)$$

This function generates the corresponding regularized functional $F_{1\varepsilon}$. We shall prove the following.

Lemma 4.4. *Let the function $g_{1\varepsilon}(\xi^2)\xi$ be defined by (40). Then the functional $F_{1\varepsilon}$ satisfies condition (9) with $c(\varepsilon) = 2\varepsilon \text{mes } \Omega$.*

Proof. We denote $\Omega_{1\varepsilon\eta} = \{x \in \Omega : \beta - \varepsilon \leq |\nabla\eta| \leq \beta\}$, $\Omega_{1\eta} = \{x \in \Omega : |\nabla\eta| > \beta\}$. On $\Omega_{1\varepsilon\eta}$ we have $0 \leq \beta - |\nabla\eta| \leq -\varepsilon$, hence,

$$\begin{aligned} |F_{1\varepsilon}(\eta) - F_1(\eta)| &\leq \left| \int_{\Omega_{1\varepsilon\eta}} \int_{\beta-\varepsilon}^{|\nabla\eta|} g_{1\varepsilon}(\xi^2)\xi d\xi dx \right| \\ &\quad + \left| \int_{\Omega_{1\eta}} \left[\int_{\beta-\varepsilon}^{\beta} g_{1\varepsilon}(\xi^2)\xi d\xi + \int_{\beta}^{|\nabla\eta|} (g_{1\varepsilon}(\xi^2)\xi - g_1(\xi^2)\xi) d\xi \right] dx \right| \\ &= \left| \frac{\vartheta}{2\varepsilon} \int_{\Omega_{1\varepsilon\eta}} (|\nabla\eta| - \beta + \varepsilon)^2 dx \right| + \frac{\vartheta\varepsilon}{2} \text{mes } \Omega_{1\eta} \\ &\leq \frac{\vartheta}{2\varepsilon} \int_{\Omega_{1\varepsilon\eta}} [(|\nabla\eta| - \beta)^2 + 2\varepsilon(|\nabla\eta| - \beta)] dx + \frac{\vartheta\varepsilon}{2} \int_{\Omega_{1\varepsilon\eta}} dx \\ &\quad + \frac{\vartheta}{2\varepsilon} 3\varepsilon^2 \text{mes } \Omega_{1\eta\varepsilon} + \frac{\vartheta\varepsilon}{2} \text{mes } \Omega \leq 2\varepsilon \text{mes } \Omega. \end{aligned}$$

□

When $p = 2$ the space V is a Hilbert space, in [16] inequality (23) is determined for the operator A_0 defined in (34).

Thus, for the seepage problems under consideration the conditions of the previous sections of this paper are satisfied and, consequently, Theorems 2.1 and 3.1 hold.

Additional properties of the iterative sequence can be obtained with the help of the property of the kind of $(S)_+$ (the property $(S)_+$ is defined in [20]). Namely, in [5] it has been proved that if the sequence v_n converges weakly to v in V and

$$\lim_{n \rightarrow +\infty} \langle A_0 v_n, v_n - v \rangle = 0,$$

then the sequence ∇v_n converges to ∇v in $[L_p(\Omega_v^+)]^m$, the sequence $G_0(\nabla v_n)$ converges to $G_0(\nabla v)$ in $[L_q(\Omega)]^m$ as $n \rightarrow +\infty$, where

$$G_0(a) = g_0(|a|^2)a/|a|, \quad a \in R^m.$$

From Theorem 2.1, according to [5], it is easy to see that the following theorem holds.

Theorem 4.2. *Let u be any solution of the seepage problem (8) and $\{u_n\}_{n=1}^\infty$ is the iterative sequence constructed according to the conditions of Theorem 2.1. Then the sequence ∇u_n converges to ∇u in $[L_p(\Omega(u))]^m$ and the sequence $G_0(\nabla u_i)$ converges to $G_0(\nabla u)$ in $[L_q(\Omega)]^m$ as $n \rightarrow +\infty$, $q = p/(p-1)$.*

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