GENERALIZATION OF THE CORRECTED MID-POINT RULE AND ERROR BOUNDS

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Abstract — A generalization of the corrected mid-point rule is derived. Various error bounds for this generalization are established.

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1. Introduction

In recent years a number of authors have considered error inequalities for some known and some new quadrature rules. Sometimes they have considered generalizations of these rules. For example, the well-known mid-point and corrected mid-point quadrature rules are considered in [1, 2, 7–9] and some generalizations are given in [1] and [2]. In [2] we can find

$$\int_a^b f(t)dt = \sum_{k=0}^{n-1} [1 + (-1)^k] \frac{(b-a)^{k+1}}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right) + (-1)^n \int_a^b K_n(t) f^{(n)}(t)dt,$$

where

$$K_n(t) = \begin{cases} \frac{(t-a)^n}{n!}, & t \in [a, \frac{a+b}{2}] , \\ \frac{(t-b)^n}{n!}, & t \in (\frac{a+b}{2}, b] . \end{cases}$$

For $n = 1$ we get the mid-point rule

$$\int_a^b f(t)dt = (b-a)f\left(\frac{a+b}{2}\right) - \int_a^b K_1(t)f'(t)dt.$$

In this paper we consider the generalization of the corrected mid-point rule and give corresponding error inequalities. Similar error inequalities are established in [9] and [10]. We also give a numerical example which shows that this generalization can be very effective.

2. Main results

Theorem 2.1. *Let $f : [a, b] \rightarrow R$ be a function such that $f^{(n-1)}$ is absolutely continuous. Then*

$$\begin{aligned} \int_a^b f(x)dx &= f\left(\frac{a+b}{2}\right)(b-a) + \frac{f'(b) - f'(a)}{24}(b-a)^2 \\ &\quad - \frac{1}{6} \sum_{i=2}^m \frac{(b-a)^{2i+1}(2i-2)(2i+3)}{2^{2i}(2i+1)!} f^{(2i)}\left(\frac{a+b}{2}\right) + R(f), \end{aligned} \quad (1)$$

where $m = \left[\frac{n-1}{2}\right]$, the integer part of $(n-1)/2$,

$$R(f) = (-1)^n \int_a^b S_n(t) f^{(n)}(t) dt \quad (2)$$

and

$$S_n(t) = \begin{cases} \frac{(t-a)^{n-2}}{n!} \left[(t-a)^2 - \frac{n(n-1)}{24}(b-a)^2 \right], & t \in \left[a, \frac{a+b}{2} \right], \\ \frac{(t-b)^{n-2}}{n!} \left[(t-b)^2 - \frac{n(n-1)}{24}(b-a)^2 \right], & t \in \left(\frac{a+b}{2}, b \right]. \end{cases} \quad (3)$$

Proof. We prove (1) by induction. First we note that

$$\begin{aligned} S_2(t) &= \begin{cases} \frac{1}{2}(t-a)^2 - \frac{1}{24}(b-a)^2, & t \in \left[a, \frac{a+b}{2} \right], \\ \frac{1}{2}(t-b)^2 - \frac{1}{24}(b-a)^2, & t \in \left(\frac{a+b}{2}, b \right], \end{cases} \\ S_3(t) &= \begin{cases} \frac{(t-a)}{3!} \left[(t-a)^2 - \frac{1}{4}(b-a)^2 \right], & t \in \left[a, \frac{a+b}{2} \right], \\ \frac{(t-b)}{3!} \left[(t-b)^2 - \frac{1}{4}(b-a)^2 \right], & t \in \left(\frac{a+b}{2}, b \right], \end{cases} \\ S_4(t) &= \begin{cases} \frac{(t-a)^2}{4!} \left[(t-a)^2 - \frac{1}{2}(b-a)^2 \right], & t \in \left[a, \frac{a+b}{2} \right], \\ \frac{(t-b)^2}{4!} \left[(t-b)^2 - \frac{1}{2}(b-a)^2 \right], & t \in \left(\frac{a+b}{2}, b \right] \end{cases} \end{aligned}$$

are the Peano kernels for the corrected mid-point quadrature rule, that is, we have

$$\begin{aligned} \int_a^b S_4(t) f^{(4)}(t) dt &= - \int_a^b S_3(t) f'''(t) dt = \int_a^b S_2(t) f''(t) dt \\ &= - \frac{f'(b) - f'(a)}{24} (b-a)^2 - f\left(\frac{a+b}{2}\right)(b-a) + \int_a^b f(t) dt. \end{aligned}$$

We easily show that (1) holds for $n = 5$. Now suppose that (1) holds for an arbitrary n . We have to prove that (1) holds for $n \rightarrow n+1$. To simplify the proof, we introduce the notations

$$P_n(t) = \frac{(t-a)^{n-2}}{n!} \left[(t-a)^2 - \frac{n(n-1)}{24}(b-a)^2 \right], \quad (4)$$

$$Q_n(t) = \frac{(t-b)^{n-2}}{n!} \left[(t-b)^2 - \frac{n(n-1)}{24}(b-a)^2 \right]. \quad (5)$$

We see that P_n and Q_n form Appell sequences of polynomials, that is

$$P'_n(t) = P_{n-1}(t), \quad Q'_n(t) = Q_{n-1}(t).$$

Then we have

$$\begin{aligned}
& (-1)^{n+1} \int_a^b S_{n+1}(t) f^{(n+1)}(t) dt \\
&= (-1)^{n+1} \int_a^{\frac{a+b}{2}} P_{n+1}(t) f^{(n+1)}(t) dt + (-1)^{n+1} \int_{\frac{a+b}{2}}^b Q_{n+1}(t) f^{(n+1)}(t) dt \\
&= (-1)^{n+1} \left[P_{n+1} \left(\frac{a+b}{2} \right) f^{(n)} \left(\frac{a+b}{2} \right) - P_{n+1}(a) f^{(n)}(a) \right] + (-1)^n \int_a^{\frac{a+b}{2}} P_n(t) f^{(n)}(t) dt \\
&\quad + (-1)^{n+1} \left[Q_{n+1}(b) f^{(n)}(b) - Q_{n+1} \left(\frac{a+b}{2} \right) f^{(n)} \left(\frac{a+b}{2} \right) \right] + (-1)^n \int_{\frac{a+b}{2}}^b Q_n(t) f^{(n)}(t) dt \\
&= (-1)^n \int_a^b S_n(t) f^{(n)}(t) dt + (-1)^{n+1} \left[P_{n+1} \left(\frac{a+b}{2} \right) - Q_{n+1} \left(\frac{a+b}{2} \right) \right] f^{(n)} \left(\frac{a+b}{2} \right) \\
&= -f \left(\frac{a+b}{2} \right) (b-a) - \frac{f'(b) - f'(a)}{24} (b-a)^2 + \int_a^b f(t) dt \\
&\quad + \frac{1}{6} \sum_{i=2}^m \frac{(b-a)^{2i+1} (2i-2)(2i+3)}{2^{2i} (2i+1)!} f^{(2i)} \left(\frac{a+b}{2} \right) \\
&\quad + (-1)^{n+1} \left[P_{n+1} \left(\frac{a+b}{2} \right) - Q_{n+1} \left(\frac{a+b}{2} \right) \right] f^{(n)} \left(\frac{a+b}{2} \right) \\
&= -f \left(\frac{a+b}{2} \right) (b-a) - \frac{f'(b) - f'(a)}{24} (b-a)^2 + \int_a^b f(t) dt \\
&\quad + \frac{1}{6} \sum_{i=2}^{m_1} \frac{(b-a)^{2i+1} (2i-2)(2i+3)}{2^{2i} (2i+1)!} f^{(2i)} \left(\frac{a+b}{2} \right),
\end{aligned}$$

where $m_1 = \left\lfloor \frac{n}{2} \right\rfloor$, since

$$\begin{aligned}
& (-1)^{n+1} \left[P_{n+1} \left(\frac{a+b}{2} \right) - Q_{n+1} \left(\frac{a+b}{2} \right) \right] f^{(n)} \left(\frac{a+b}{2} \right) \\
&= (-1)^n \frac{(b-a)^{n+1}}{2^{n+1} (n+1)!} [1 - (-1)^{n+1}] \frac{(n+3)(n-2)}{6} f^{(n)} \left(\frac{a+b}{2} \right).
\end{aligned}$$

This completes the proof. $\square$

Lemma 2.1. *The Peano kernels $S_n(t)$, $n > 3$, satisfy*

$$\int_a^b S_n(t) dt = 0, \quad \text{if } n \text{ is odd}, \quad (6)$$

$$\int_a^b |S_n(t)| dt = \frac{(n+3)(n-2)}{3} \frac{(b-a)^{n+1}}{2^{n+1}(n+1)!}, \quad (7)$$

$$\max_{t \in [a,b]} |S_n(t)| = \frac{(n-3)(n+2)}{3} \frac{(b-a)^n}{2^{n+1}n!}. \quad (8)$$

Proof. A simple calculation gives

$$\int_a^b S_n(t) dt = \frac{(b-a)^{n+1}}{2^{n+1}(n+1)!} [1 - (-1)^{n+1}] \left[1 - \frac{n(n+1)}{6} \right].$$

From the above relation we see that (6) holds, since $1 - (-1)^{n+1} = 0$ if n is odd.

We now consider some properties of the Appell sequences of polynomials $P_n(t)$ and $Q_n(t)$, given by (4) and (5), respectively. We have that $(t-a)^{n-2} \geq 0$ for each $n > 3$ and $t \in [a, \frac{a+b}{2}]$. We also have

$$(t-a)^2 - \frac{n(n-1)}{24}(b-a)^2 \leq 0, \quad t \in \left[a, \frac{a+b}{2} \right], \quad n > 3,$$

such that $P_n(t) \leq 0$ for $n > 3$ and $t \in [a, \frac{a+b}{2}]$. Since $P'_n(t) = P_{n-1}(t)$, we conclude that $P_n(t)$ are decreasing functions.

We have that $(t-b)^{n-2} \leq 0$ if $n > 3$, n is odd and $(t-b)^{n-2} \geq 0$ if $n > 3$, n is even. Since

$$(t-b)^2 - \frac{n(n-1)}{24}(b-a)^2 \leq 0, \quad t \in \left[\frac{a+b}{2}, b \right], \quad n > 3,$$

we have that $Q_n(t) \leq 0$ if n is even and $Q_n(t) \geq 0$ if n is odd. As we know, $Q'_n(t) = Q_{n-1}(t)$ such that $Q_n(t)$ are decreasing functions if n is odd and $Q_n(t)$ are increasing functions if n is even. We use these properties to prove (7) and (8).

We have

$$\begin{aligned} \int_a^b |S_n(t)| dt &= \int_a^{\frac{a+b}{2}} |P_n(t)| dt + \int_{\frac{a+b}{2}}^b |Q_n(t)| dt \\ &= \left| \int_a^{\frac{a+b}{2}} P_n(t) dt \right| + \left| \int_{\frac{a+b}{2}}^b Q_n(t) dt \right| \\ &= \frac{1}{3} \frac{(n^2 + n - 6)(b-a)^{n+1}}{2^{n+1}(n+1)!}. \end{aligned}$$

Finally, we have

$$\begin{aligned}\max_{t \in [a, b]} |S_n(t)| &= \max \left\{ \max_{t \in [a, \frac{a+b}{2}]} |P_n(t)|, \max_{t \in [\frac{a+b}{2}, b]} |Q_n(t)| \right\} \\ &= \max \left\{ \left| P_n \left(\frac{a+b}{2} \right) \right|, \left| Q_n \left(\frac{a+b}{2} \right) \right| \right\} \\ &= \frac{1}{3} \frac{(n^2 - n - 6)(b - a)^n}{2^{n+1}n!}.\end{aligned}$$

□

We introduce the notations

$$\begin{aligned}I &= \int_a^b f(t) dt, \\ F &= f \left(\frac{a+b}{2} \right) (b-a) + \frac{f'(b) - f'(a)}{24} (b-a)^2 \\ &\quad - \frac{1}{6} \sum_{i=2}^m \frac{(b-a)^{2i+1} (2i-2)(2i+3)}{2^{2i} (2i+1)!} f^{(2i)} \left(\frac{a+b}{2} \right).\end{aligned}$$

Theorem 2.2. Let $f : [a, b] \rightarrow \mathbb{R}$ be a function such that $f^{(n-1)}$, $n > 3$, is absolutely continuous and there exist real numbers γ_n, Γ_n such that $\gamma_n \leq f^{(n)}(t) \leq \Gamma_n$, $t \in [a, b]$. Then

$$|I - F| \leq \frac{1}{3} \frac{\Gamma_n - \gamma_n}{(n+1)!} \frac{(n+3)(n-2)}{2^{n+2}} (b-a)^{n+1} \quad \text{if } n \text{ is odd} \quad (9)$$

and

$$|I - F| \leq \frac{1}{3} \frac{(n+3)(n-2)(b-a)^{n+1}n}{2^{n+1}(n+1)!} \|f^{(n)}\|_{\infty} \quad \text{if } n \text{ is even.} \quad (10)$$

Proof. Let n be odd. From (2) and (6) we get

$$R(f) = (-1)^n \int_a^b S_n(t) f^{(n)}(t) dt = (-1)^n \int_a^b S_n(t) \left[f^{(n)}(t) - \frac{\gamma_n + \Gamma_n}{2} \right] dt$$

such that we have

$$|R(f)| = |I - F| \leq \max_{t \in [a, b]} \left| f^{(n)}(t) - \frac{\gamma_n + \Gamma_n}{2} \right| \int_a^b |S_n(t)| dt. \quad (11)$$

We also have

$$\max_{t \in [a, b]} \left| f^{(n)}(t) - \frac{\gamma_n + \Gamma_n}{2} \right| \leq \frac{\Gamma_n - \gamma_n}{2}. \quad (12)$$

From (11), (12) and (7) we get

$$|I - F| \leq \frac{1}{3} \frac{\Gamma_n - \gamma_n}{(n+1)!} \frac{n^2 + n - 6}{2^{n+2}} (b-a)^{n+1}.$$

Let n be even. Then we have

$$|R(f)| = |I - F| \leq \int_a^b |S_n(t)| dt \|f^{(n)}\|_\infty = \frac{1}{3} \frac{(n^2 + n - 6)(b - a)^{n+1}n}{2^{n+1}(n+1)!} \|f^{(n)}\|_\infty.$$

□

Theorem 2.3. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a function such that $f^{(n-1)}$, $n > 3$, is absolutely continuous and let n be odd. If there exists a real number γ_n such that $\gamma_n \leq f^{(n)}(t)$, $t \in [a, b]$, then*

$$|I - F| \leq \frac{(n-3)(n+2)}{3} (T_n - \gamma_n) \frac{(b-a)^{n+1}}{2^{n+1}(n+1)!}, \quad (13)$$

where

$$T_n = \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b - a}.$$

If there exists a real number Γ_n such that $f^{(n)}(t) \leq \Gamma_n$, $t \in [a, b]$, then

$$|I - F| \leq \frac{(n-3)(n+2)}{3} (\Gamma_n - T_n) \frac{(b-a)^{n+1}}{2^{n+1}(n+1)!}. \quad (14)$$

Proof. We have

$$|R(f)| = |I - F| = \left| \int_a^b (f^{(n)}(t) - \gamma_n) S_n(t) dt \right|,$$

since (6) holds. Then we have

$$\begin{aligned} \left| \int_a^b (f^{(n)}(t) - \gamma_n) S_n(t) dt \right| &\leq \max_{t \in [a, b]} |S_n(t)| \int_a^b (f^{(n)}(t) - \gamma_n) dt \\ &= \frac{n^2 - n - 6}{3} \frac{(b-a)^n}{2^{n+1}(n+1)!} [f^{(n-1)}(b) - f^{(n-1)}(a) - \gamma_n(b-a)] \\ &= \frac{n^2 - n - 6}{3} \frac{(b-a)^{n+1}}{2^{n+1}(n+1)!} (T_n - \gamma_n). \end{aligned}$$

In a similar way we can prove that (14) holds. □

Remark 2.1. Note that we can apply estimations (9) and (10) only if $f^{(n)}$ is bounded. On the other hand, we can apply estimation (13) if $f^{(n)}$ is unbounded above and we can apply estimation (14) if $f^{(n)}$ is unbounded below.

3. A numerical example

Here we consider the integral (special function) $Ei(x) = \int_0^x \frac{\exp(t)-1}{t} dt$ and apply the summation formula (1) to this integral. We get the summation formula $Ei(x) = F(x) + R(x)$,

where

$$F(x) = \frac{2}{x}(\exp(\frac{x}{2}) - 1) + \frac{\exp(x)(x-1) + 1 - \frac{x^2}{2}}{24} - \frac{1}{6} \sum_{i=2}^m \frac{(2i-2)(2i+3)x^{2i+1}}{2^{2i}(2i+1)!} f^{(2i)}(\frac{x}{2}) \quad (15)$$

and $f(t) = \frac{\exp(t)-1}{t}$, $f'(0) = \frac{1}{2}$. We calculate the derivatives $f^{(j)}(t)$ as follows. We have

$$(g(t)h(t))^{(j)} = \sum_{k=0}^j \binom{j}{k} g^{(k)}(t)h^{(j-k)}(t).$$

If we choose $g(t) = \exp(t) - 1$ and $h(t) = 1/t$, then we get

$$f^{(j)}(\frac{x}{2}) = \sum_{k=0}^j \binom{j}{k} (-1)^{j-k} e_k(\frac{x}{2}) \frac{(j-k)!2^{j-k+1}}{x^{j-k+1}},$$

where $e_0(t) = e^t - 1$ and $e_k(t) = e^t$, $k \geq 1$.

We now compare the summation formula (15) with the known compound formula (for the corrected mid-point rule)

$$\int_0^x f(t)dt = \frac{f'(x) - f'(0)}{24}h^2 + h \sum_{i=0}^{n-1} f(\frac{x_i + x_{i+1}}{2}) + R(x), \quad (16)$$

where $x_i = ih$, $h = x/n$, $f(t) = (\exp t - 1)/t$.

Let us choose $x = 1$. The "exact" value is $Ei(1) = 1.3179021514544$. If we use (15) with $m = 5$, then we get $Ei(1) \approx 1.31790215145452$. The calculations take $2.8E - 06$ seconds. If we use (16) with $n = 200$, then we get $Ei(1) \approx 1.3179021514546$. The calculations take $4.51E - 05$ seconds. All calculations are done in double precision arithmetic. As we see, the first approximate result is obtained 16 times faster than the second one. This is a consequence of the fact that we have to calculate the function $\exp(t)$ many times when we apply the compound formula and we have only to calculate $\exp(x)$ when we apply the summation formula. (In fact, it is possible to find examples where the summation formula is a few dozen times faster than the compound formula.)

Similar summation formulas can be obtained for the integrals (special functions): $\int_0^x [(\sin t)/t] dt$, $\int_0^x [(\cos t - 1)/t] dt$, $\int_0^x \exp(-t^2)dt$, etc.

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