APPROXIMATION TO THE SOLUTION OF THE SYSTEM OF NONLINEAR STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract — In this paper we consider the system of nonlinear stochastic differential equations. The solution of these equations is approximated by finite series of definite integrals where every term of the series can be evaluated. To construct the approximation, the linearization technique and the technique for noncommuting matrices are used.

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1. Introduction

Reaction-diffusion systems have been the subject of many experimental and theoretical studies. Such systems appear in many biological, chemical and physical processes [5, 11, 12]. Reaction-diffusion systems are described by mean-field type nonlinear equations [3, 9], which do not contain the stochastic nature, and by nonlinear stochastic differential equations [8]

$$\begin{aligned}dx_1(t) &= k_1x_1(t)dt + a_1x_1(t)x_2(t)dt + \sigma_1x_1(t)d\omega(t) \\dx_2(t) &= k_2x_2(t)dt + a_2x_1(t)x_2(t)dt + \sigma_2x_2(t)d\omega(t) \\x_1(0) &= x_{10}, x_2(0) = x_{20}.\end{aligned}\tag{1}$$

where $\omega(t)$ is the Wiener process, $x_1(t), x_2(t)$ are concentrations of reactants; the coefficients of the equation $k_i, a_i, \sigma_i, i = 1, 2$ are some real or complex numbers characterizing the reaction.

There is a method for evaluating the transition probability density for the process which is the solution of one stochastic differential equation [4, 6]. But we cannot use this method to find the solution of the system of nonlinear stochastic differential equations. It is highly nontrivial task to get an analytic solution of equation (1). Deterministic and stochastic diffusion type systems are considered by M. Arato [1]. The approach proposed in [1] gives the solution of a nonlinear stochastic differential equation as an exponential function, but the exponent contains the desired solution. Such a form of the solution gives only asymptotic behavior of the solution as $t \rightarrow \infty$. Therefore we propose an approximation for solving the nonlinear stochastic differential equations (1). The approximation is represented as a finite series of increasing multiplicity definite integrals, where every term of series can be

evaluated. The represented approximate solution makes it possible to analyze the solution for $0 \leq t < \infty$.

2. The approximate formulas to the solution of the system

Our approximations $\tilde{x}_1(t), \tilde{x}_2(t)$ to the solution of equations (1) are as follows:

$$\begin{aligned} \tilde{x}_1 &= \sum_{k=0}^{n-1} \int_0^t \dots \int_0^{s_2} \frac{\partial^k}{\partial b_1 \dots \partial b_k} \exp \left(\int_0^t \psi_k(\tau) (f(\bar{s}^{(k)}, \tau) + 1) \right) \\ &\quad \times \sum_{j=0}^n \int_0^t \dots \int_0^{\tau_2} \prod_{l=1}^j \psi_{qk}(\tau_l) \exp \left(\int_0^{\tau_l} \psi_{dk}(u) \right) \Big|_{b_1=\dots=b_k=0} ds_1 \dots ds_k x_{10}^{k+1} x_{20}^j a_2^k, \\ \tilde{x}_2 &= \sum_{k=0}^n \int_0^t \dots \int_0^{s_2} \frac{\partial^k}{\partial b_1 \dots \partial b_k} \exp \left(\int_0^t \psi_k(\tau) f(\bar{s}^{(k)}, \tau) \right) \exp \left(\int_0^t \psi_{dk}(\tau) \right) \\ &\quad \times \sum_{j=0}^{n-1} \int_0^t \dots \int_0^{\tau_2} \prod_{l=1}^j (\psi_{qk}(\tau_l) - a_1 d\tau_l) \times \exp \left(\int_0^{\tau_l} \psi_{dk}(u) \right) \Big|_{b_1=\dots=b_k=0} d\bar{s}^k x_{10}^k x_{20}^{j+1} a_2^k, \end{aligned} \quad (2)$$

where $f(\bar{s}^{(k)}, \tau) = \sum_{j=1}^{k+1} (k+1-j) \chi_{[s_{j-1}, s_j]}(\tau)$, $s_0 = 0, s_{k+1} = t$, $\psi_{dk}(\tau) = (\phi(\bar{s}^{(k)}) + k_2 - \frac{1}{2}\sigma_2^2)d\tau + \sigma_2 d\omega(\tau)$, $\psi_{qk}(\tau) = a_1(f(\bar{s}^{(k)}) + 1)d\tau$, $\psi_k(\tau) = (k_1 - \frac{1}{2}\sigma_1^2)d\tau + \sigma_1 d\omega(\tau)$, $k \geq 0$, $\phi(\bar{s}^{(k)}, \tau) = \sum_{j=1}^k b_k \delta(\tau - s_j)$.

The construction of the approximation or proof of equalities (2) will be divided into several steps. In first step we reduce the nonlinear equation to a linear one and introduce the matrices D, Q, d, q ; in the second step, we remove the matrix D ; in the third step, we remove the matrix d ; the fourth step is to remove the matrices Q, q .

2.1. Reduction of the nonlinear equation to a linear one

Using the linearization technique of [7], in order to find solutions x_1, x_2 we introduce $y_{ij}(t) = x_1^i(t)x_2^j(t)$, $i, j = 0, 1, 2, \dots$. Then using the Ito formula [4] we get the expression for $dy_{ij}(t)$:

$$\begin{aligned} dy_{ij}(t) &= (ik_1 + jk_2)y_{ij}(t)dt + (ia_1y_{ij+1}(t) + ja_2y_{i+1j}(t))dt \\ &\quad + (i\sigma_1 + j\sigma_2)y_{ij}(t)d\omega(t) + ij\sigma_1\sigma_2y_{ij}(t)dt \\ &\quad + \frac{1}{2}(i(i-1)\sigma_1^2 + j(j-1)\sigma_2^2)y_{ij}(t)dt \end{aligned} \quad (3)$$

If $y_{ij}(t)$, $i, j = 0, 1, 2, \dots$ are the solutions of system (3), then $x_1(t) = y_{10}(t)$, $x_2(t) = y_{01}(t)$ are the solutions of system (1).

Let us consider $\tilde{x}_1(t) = y_{10}(t)$, $\tilde{x}_2(t) = y_{01}(t)$ – the solutions of the system of equation (3), where $i, j \leq n$, as approximation of $x_1(t), x_2(t)$ from system (1). The convergence $\tilde{x}_1(t), \tilde{x}_2(t)$ to $x_1(t), x_2(t)$ will be discussed in 2.5.

Let $\vec{y}$ be $(\vec{y}_0, \vec{y}_1, \dots, \vec{y}_n)$, $\vec{y}_i = (y_{i0}, y_{i1}, \dots, y_{in})$, $i = 0, 1, \dots, n$, $\vec{y}(0) = (x_{20}, x_{10}x_{20}, \dots, x_{10}^n x_{20})$, $x_{20} = (1, x_{20}, x_{20}^2, \dots, x_{20}^n)$. To write equation (3) in matrix form, let us introduce

the matrices

$$d = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & n \end{pmatrix}, \quad q = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

which have dimensions $(n+1) \times (n+1)$ and satisfy the commutative rules $qd - dq = q$, we also introduce the matrices

$$D = \begin{pmatrix} 0e & 0 & 0 & \dots & 0 & 0 \\ 0 & 1e & 0 & \dots & 0 & 0 \\ 0 & 0 & 2e & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & ne \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & e & 0 & \dots & 0 & 0 \\ 0 & 0 & e & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & e \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

which have dimensions $(n+1)^2 \times (n+1)^2$ and satisfy the commutative rules $QD - DQ = Q$, e is the $(n+1) \times (n+1)$ identical matrix, E is the $(n+1)^2 \times (n+1)^2$ identical matrix.

Now we can write equation (3) as a matrix linear stochastic differential equation

$$\begin{aligned} d\vec{y}(t) = & (D\sigma_1 + dE\sigma_2)\vec{y}(t)d\omega(t) + k_1D + k_2dE + a_1Dq + a_2dQ \\ & + dD\sigma_1\sigma_2 + \frac{1}{2}D(D-E)\sigma_1^2 + \frac{1}{2}dE(d-e)\sigma_2^2\vec{y}(t)dt. \end{aligned} \quad (4)$$

Using the definition of the chronological exponential or $T \exp$ [10], we get

$$\vec{y}(t) = T \exp \left\{ \int_0^t (H(D, d) + a_1Dq + a_2dQ) \right\} \vec{y}(0), \quad (5)$$

where $H(D, d) = (k_1D + k_2dE - \frac{1}{2}D\sigma_1^2 - \frac{1}{2}dE\sigma_2^2)ds + (D\sigma_1 + dE\sigma_2)d\omega(s)$.

2.2. Removal of matrix D

Using the equality

$$\begin{aligned} & T \exp \left\{ \int_0^t (H(D, d) + a_1Dq + a_2dQ) \right\} \\ &= T \exp \left\{ \int_0^t (H(D, d) + a_1Dq) \right\} + \int_0^t T \exp \left\{ \int_{s_1}^t (H(D, d) + a_1Dq) \right\} a_2dQ \\ & \quad \times T \exp \left\{ \int_0^{s_1} (H(D, d) + a_1Dq + a_2dQ) \right\} ds_1 \end{aligned}$$

we can write $T \exp$ as the series

$$\begin{aligned}
 & T \exp \left\{ \int_0^t (H(D, d) + a_1 Dq + a_2 dQ) \right\} \\
 &= T \exp \left\{ \int_0^t (H(D, d) + a_1 Dq) \right\} + \int_0^t T \exp \left\{ \int_{s_1}^t (H(D, d) + a_1 Dq) \right\} a_2 dQ \\
 &\quad \times T \exp \left\{ \int_0^{s_1} (H(D, d) + a_1 Dq) \right\} ds_1 + \dots \\
 &\quad + \int_0^t \dots \int_0^{s_2} T \exp \left\{ \int_{s_k}^t (H(D, d) + a_1 Dq) \right\} a_2 dQ \\
 &\quad \times T \exp \left\{ \int_{s_{k-1}}^{s_k} (H(D, d) + a_1 Dq) \right\} a_2 dQ \times \dots \times T \exp \left\{ \int_{s_1}^{s_2} (H(D, d) + a_1 Dq) \right\} a_2 dQ \\
 &\quad \times T \exp \left\{ \int_0^{s_1} (H(D, d) + a_1 Dq) \right\} ds_1 \dots ds_k + \dots
 \end{aligned} \tag{6}$$

We consider separately the evaluation of $\tilde{x}_1$ and $\tilde{x}_2$. For that we need to remove the matrices D, d, Q, q . Let us start with the removal of the matrix D and use a technique analogous to the quantum mechanics technique to evaluate the vacuum mean of the operator [2]. Here the creating operators are relocated to the left, the annihilator operators are relocated to the right, and the action of the annihilator operator on vacuum is zero.

To evaluate $\tilde{x}_1$, we need to know $(n+2)$ -th component of the vector $T \exp \left\{ \int_0^t (H(D, d) + a_1 Dq + a_2 dQ) \right\} \vec{y}(0)$. Since after the action of the matrix $D - E$ on any vector the $(n+2)$ -th component of the resulting vector is zero, we relocate the matrices D in the expression for $T \exp$ to the left and relocate the matrices Q to the right. After relocating the matrices D and Q and taking into account that $Qf(D) = f(D+E)Q$ and for any vector $\vec{z}$ the $(n+2)$ -th component of $((D-E)\vec{z})_{n+2}$ is zero, we get the following expression for $T \exp$:

$$\begin{aligned}
 & T \exp \left\{ \int_0^t (H(D, d) + a_1 Dq + a_2 dQ) \right\} \\
 &= T \exp \left\{ \int_0^t (H(1, d) + a_1 qE) \right\} + \sum_{k=1}^{\infty} \int_0^t \dots \int_0^{s_2} T \exp \left\{ \int_{s_k}^t (H(1, d) + a_1 qE) \right\} a_2 Ed \\
 &\quad \times T \exp \left\{ \int_{s_{k-1}}^{s_k} (H(2, d) + 2a_1 qE) \right\} a_2 Ed \times \dots \times T \exp \left\{ \int_{s_1}^{s_2} (H(k, d) + ka_1 qE) \right\} \\
 &\quad \times a_2 Ed \times T \exp \left\{ \int_0^{s_1} (H(k+1, d) + (k+1)a_1 qE) \right\} ds_1 \dots ds_k Q^k.
 \end{aligned} \tag{7}$$

This expression does not contain the matrix D , and we have numbers instead of D . Thus, the process of removal of the matrix D from the expression for $T \exp$ is finished.

To evaluate $\tilde{x}_2$, we need 2-th component of the vector $\vec{y}$. Since after the action of the matrix D on any vector the 2-th component of the resulting vector is zero, we relocate the matrices D in expression (6) for $T \exp$ to the left and relocate the matrices Q to the right. After relocating the matrices D and Q and taking into account that $Qf(D) = f(D+E)Q$ and for any vector $\vec{z}$ the 2-th component of $(D\vec{z})_2$ is zero, we get the following expression for $T \exp$:

$$\begin{aligned}
& T \exp \left\{ \int_0^t (H(D, d) + a_1 D q + a_2 d Q) \right\} \\
&= T \exp \left\{ \int_0^t (H(0, d) + 0 a_1 q E) \right\} + \sum_{k=1}^{\infty} \int_0^t \dots \int_0^{s_2} T \exp \left\{ \int_{s_k}^t (H(0, d) + 0 a_1 q E) \right\} a_2 E d \\
&\quad \times T \exp \left\{ \int_{s_{k-1}}^{s_k} (H(1, d) + 1 a_1 q E) \right\} a_2 E d \times \dots \\
&\quad \times T \exp \left\{ \int_{s_1}^{s_2} (H(k-1, d) + (k-1) a_1 q E) \right\} \\
&\quad \times a_2 E d \times T \exp \left\{ \int_0^{s_1} (H(k, d) + k a_1 q E) \right\} ds_1 \dots ds_k Q^k.
\end{aligned} \tag{7'}$$

2.3. Removal of matrix d

Represent the product of $T \exp$ under the integral sign in (7) as single $T \exp$. In addition, in the exponent of $T \exp$ we group together the terms containing the matrix d and the terms containing the matrix q . Let the coefficient before the matrix d be ψ_d , the coefficient before the matrix q be ψ_q , and the term without matrices d and q be ψ . Then

$$\begin{aligned}
& T \exp \left\{ \int_{s_k}^t (H(1, d) + a_1 q E) \right\} E d \times T \exp \left\{ \int_{s_{k-1}}^{s_k} (H(2, d) + 2 a_1 q E) \right\} E d \times \dots \\
&\quad \times T \exp \left\{ \int_{s_1}^{s_2} (H(k, d) + k a_1 q E) \right\} E d \times T \exp \left\{ \int_0^{s_1} (H(k+1, d) + (k+1) a_1 q E) \right\} \\
&= \frac{\partial^k}{\partial b_1 \dots \partial b_k} T \exp \left\{ \int_0^t (\psi_{dk} d + \psi_{qk} q + (f(\vec{s}^{(k)}) + 1) \psi_k) \right\} \Big|_{b_1 = \dots = b_k = 0}
\end{aligned} \tag{8}$$

To represent $T \exp \left\{ \int_0^t (\psi_{dk}(\tau) d + \psi_{qk}(\tau) q + \psi_k(\tau) (f(\vec{s}^{(k)}) + 1)) \right\}$ in the form of exponen-

tial without the chronological ordered sign, write $T \exp$ in the following form:

$$\begin{aligned}
 T \exp \left\{ \int_0^t (\psi_{dk}(\tau)d + \psi_{qk}(\tau)q + \psi_k(\tau)(f(\bar{s}^{(k)}) + 1)) \right\} &= \exp \left(\int_0^t \psi_k(\tau)(f(\bar{s}^{(k)}) + 1) \right) \\
 &\times \left(\exp \left\{ \int_0^t d\psi_{dk}(\tau) \right\} + \sum_{j=1}^{\infty} \int_0^t \dots \int_0^{s_2} \exp \left(\int_{s_j}^t d\psi_{dk}(\tau) \right) q\psi_{qk}(s_j) \right. \\
 &\times \exp \left\{ \int_{s_{j-1}}^{s_j} d\psi_{dk}(\tau) \right\} q\psi_{qk}(s_{j-1}) \times \dots \\
 &\times \exp \left\{ \int_{s_1}^{s_2} d\psi_{dk}(\tau) \right\} q\psi_{qk}(s_1) \times \exp \left\{ \int_0^{s_1} d\psi_{dk}(\tau) \right\} ds_1 \dots ds_j \Big).
 \end{aligned}$$

Because after the action of the matrix d on any vector the $(n+2)$ -th component of the resulting vector is zero, we relocate the matrices d to the left and the matrices q to the right in the expression for $T \exp$. Then taking into account that $qf(d) = f(d+e)q$ and for any vector $\vec{z}$ the $(n+2)$ -th component of $(d\vec{z})_{n+2}$ is zero, we obtain

$$\begin{aligned}
 T \exp \left\{ \int_0^t (\psi_{dk}(\tau)d + \psi_{qk}(\tau)q + \psi_k(\tau)(f(\bar{s}^{(k)}) + 1)) \right\} &= \exp \left(\int_0^t \psi_k(\tau)(f(\bar{s}^{(k)}) + 1) \right) \\
 &\times \left(1 + \sum_{j=1}^{\infty} \int_0^t \dots \int_0^{s_2} \exp \left(\int_{s_j}^t 0\psi_{dk}(\tau) \right) \psi_{qk}(s_j) \times \exp \left\{ \int_{s_{j-1}}^{s_j} 1\psi_{dk}(\tau) \right\} \psi_{qk}(s_{j-1}) \times \dots \right. \\
 &\times \exp \left\{ \int_{s_1}^{s_2} (j-1)d\psi_{dk}(\tau) \right\} \psi_{qk}(s_1) \times \exp \left\{ \int_0^{s_1} j\psi_{dk}(\tau) \right\} ds_1 \dots ds_j q^j \Big).
 \end{aligned}$$

Let $P_k(t)$ be $T \exp \left\{ \int_0^t (\psi_{dk}(\tau)d + \psi_{qk}(\tau)q) \right\}$ and evaluate the differential of the function $P_k(t)$.

$$dP_k(t) = \psi_{qk}(t)q \exp \left(\int_0^t \psi_{dk}(\tau) \right) P_k(t), P_k(0) = 1.$$

Solving this equation, one gets the expression for $P_k(t)$ and for $T \exp$

$$\begin{aligned}
 &T \exp \left\{ \int_0^t (\psi_{dk}(\tau)d + \psi_{qk}(\tau)q + \psi_k(\tau)(f(\bar{s}^{(k)}) + 1)) \right\} \\
 &= \exp \left(\int_0^t \psi_k(\tau)(f(\bar{s}^{(k)}) + 1) \right) \exp \left(q \int_0^t \psi_{qk}(\tau) \exp \left(\int_0^\tau \psi_{dk}(u) \right) \right). \tag{9}
 \end{aligned}$$

From equalities (5), (7)–(9) one infers that the solution $\tilde{x}_1$ is the $(n+2)$ -th component of the vector

$$\begin{aligned} & \left(\exp \left(\int_0^t \psi_0(\tau) \right) \exp \left(q \int_0^t \psi_{q0}(\tau) \exp \left(\int_0^\tau \psi_{d0}(s) \right) \right) \right) \\ & + \sum_{k=1}^{\infty} \int_0^t \dots \int_0^{s_2} \frac{\partial^k}{\partial b_1 \dots \partial b_k} \exp \left(\int_0^t \psi_k(f(\bar{s}^{(k)}) + 1) d\tau \right) \\ & \times \exp \left(q \int_0^t \psi_{qk}(\tau) \exp \left(\int_0^\tau \psi_{dk}(u) \right) \right) \Big|_{b_1=\dots=b_k=0} ds_1 \dots ds_k a_2^k Q^k \Big) \vec{y}(0). \end{aligned}$$

Let us write the product of $T \exp$ under the integral sign in expression (7') as single $T \exp$. In addition, in the exponent of $T \exp$ we group together the terms containing the matrix d and the terms containing the matrix q . Then

$$\begin{aligned} & T \exp \left\{ \int_{s_k}^t (H(0, d) + 0a_1qE) \right\} Ed \times T \exp \left\{ \int_{s_{k-1}}^{s_k} (H(1, d) + 1a_1qE) \right\} \times Ed \times \dots \\ & \times T \exp \left\{ \int_{s_1}^{s_2} (H((k-1), d) + (k-1)a_1qE) \right\} Ed \times T \exp \left\{ \int_0^{s_1} (H(k, d) + ka_1qE) \right\} (8') \\ & = \frac{\partial^k}{\partial b_1 \dots \partial b_k} T \exp \left\{ \int_0^t (\psi_{dk}(\tau)d + (\psi_{qk}(\tau) - a_1d\tau)q + \psi_k(\tau)f(\bar{s}^{(k)})) \right\} \Big|_{b_1=\dots=b_k=0}. \end{aligned}$$

Because after the action of the matrix $d-1$ on any vector the 2-th component of the resulting vector is zero, we will have

$$\begin{aligned} & T \exp \left\{ \int_0^t (\psi_{dk}(\tau)d + (\psi_{qk}(\tau) - a_1d\tau)q + \psi_k(\tau)f(\bar{s}^{(k)})) \right\} \\ & = \exp \left(\int_0^t \psi_k(\tau)f(\bar{s}^{(k)}) \right) \left(\exp \left(\int_0^t \psi_{dk}(\tau) \right) \right. \\ & + \sum_{j=1}^{\infty} \int_0^t \dots \int_0^{s_2} \exp \left(\int_{s_j}^t 1\psi_{dk}(\tau) \right) (\psi_{qk}(s_j) - a_1) \\ & \times \exp \left\{ \int_{s_{j-1}}^{s_j} 2\psi_{dk}(\tau) \right\} (\psi_{qk}(s_{j-1}) - a_1) \times \dots \times \exp \left\{ \int_{s_1}^{s_2} j\psi_{dk}(\tau) \right\} \\ & \left. \times (\psi_{qk}(s_1) - a_1) \times \exp \left\{ \int_0^{s_1} (j+1)\psi_{dk}(\tau) \right\} ds_1 \dots ds_j Q^j \right). \end{aligned}$$

Let $R_k(t)$ be

$$T \exp \left\{ \int_0^t (\psi_{dk}(\tau) d + (\psi_{qk}(\tau) - a_1 d\tau) q) \right\} \times \exp \left\{ - \int_0^t \psi_{dk}(\tau) \right\}$$

and evaluate the differential of the function $R_k(t)$.

$$dR_k(t) = \psi_{qk}(t) q \exp \left(\int_0^t \psi_{dk}(\tau) \right) R_k(t), R_k(0) = 1.$$

Solving this equation, we obtain the expression for $R_k(t)$ and for $T \exp$

$$\begin{aligned} & T \exp \left\{ \int_0^t (\psi_{dk}(\tau) d + (\psi_{qk}(\tau) - a_1 d\tau) q + \psi_k(\tau) f(\bar{s}^{(k)})) \right\} \\ &= \exp \left(\int_0^t \psi_k(\tau) f(\bar{s}^{(k)}) \right) \exp \left(\int_0^t \psi_{dk}(\tau) \right) \\ & \quad \times \exp \left(q \int_0^t (\psi_{qk}(\tau) - a_1 d\tau) \exp \left(\int_0^\tau \psi_{dk}(u) \right) \right). \end{aligned} \quad (9')$$

From (5), (7') – (9') one infers that the solution $\tilde{x}_2$ is 2-th component of the vector

$$\begin{aligned} & \left(\exp \left(\int_0^t \psi_{d0}(\tau) \right) + \sum_{k=1}^{\infty} \int_0^t \dots \int_0^{s_2} \frac{\partial^k}{\partial b_1 \dots \partial b_k} \exp \left(\int_0^t \psi_k(\tau) f(\bar{s}^{(k)}) d\tau \right) \right. \\ & \quad \times \exp \left(\int_0^t \psi_{dk}(\tau) \right) \exp \left[q \int_0^t (\psi_{qk}(\tau) - a_1 d\tau) \right. \\ & \quad \left. \left. \times \exp \left(\int_0^\tau \psi_{dk}(u) \right) \right] \right|_{b_1=\dots=b_k=0} ds_1 \dots ds_k a_2^k Q^k \Big) \vec{y}(0). \end{aligned}$$

2.4. Removal of matrices Q, q

This is the last stage in the construction of the approximation. At this stage the matrices Q, q are removed. Thus, the removal of all introduced matrices is finished. Expanding the exponential containing the matrix q , with respect to the degree of q and using the equality

$$(q^j Q^k \vec{y}(0))_{n+2} = x_{10}^{k+1} x_{20}^j, k \leq n-1, j \leq n,$$

one can write

$$\begin{aligned} \tilde{x}_1 &= \sum_{k=0}^{n-1} \int_0^t \dots \int_0^{s_2} \frac{\partial^k}{\partial b_1 \dots \partial b_k} \exp \left(\int_0^t \psi_k(\tau) (f(\bar{s}^{(k)}) + 1) \right) \\ & \quad \times \sum_{j=0}^n \int_0^t \dots \int_0^{\tau_2} \prod_{l=1}^j \psi_{qk}(\tau_l) \exp \left(\int_0^{\tau_l} \psi_{dk}(u) \right) \Big|_{b_1=\dots=b_k=0} ds_1 \dots ds_k x_{10}^{k+1} x_{20}^j a_2^k. \end{aligned}$$

Returning from $\tilde{x}_1$ to $\tilde{x}_2$, let us expand the exponential containing the matrix q with respect to the degree of q and using the equality

$$(q^j Q^k \vec{y}(0))_2 = x_{10}^k x_{20}^{j+1}, k \leq n, j \leq n-1,$$

one gets

$$\begin{aligned} \tilde{x}_2 = & \sum_{k=0}^n \int_0^t \dots \int_0^{s_2} \frac{\partial^k}{\partial b_1 \dots \partial b_k} \exp \left(\int_0^t \psi_k(\tau) f(\bar{s}^{(k)}) \right) \\ & \times \exp \left(\int_0^t \psi_{dk}(\tau) \right) \sum_{j=0}^{n-1} \int_0^t \dots \int_0^{\tau_2} \prod_{l=1}^j (\psi_{qk}(\tau_l) - a_1 d\tau_l) \\ & \times \exp \left(\int_0^{\tau_l} \psi_{dk}(u) \right) \Big|_{b_1=\dots=b_k=0} d\bar{s}^k x_{10}^k x_{20}^{j+1} a_2^k. \end{aligned}$$

Thus, equalities (2) are proved and the construction of the approximation is finished.

2.5. Convergence of the approximation

Now let us discuss the convergence of solutions of the truncated system $\tilde{x}_1(t), \tilde{x}_2(t)$ as the dimension of the truncated system n goes to infinity. To investigate the convergence of $\tilde{x}_1(t)$, consider the k -th term of $\tilde{x}_1(t)$ in the form

$$\begin{aligned} \tilde{x}_{1,k} = & \int_0^t \dots \int_0^{s_2} \frac{\partial^k}{\partial b_1 \dots \partial b_k} \exp \left(\int_0^t \psi_k(\tau) (f(\bar{s}^{(k)}, \tau) + 1) \right) \\ & \times \exp \left[x_{20} \int_0^t \psi_{qk}(\tau) \exp \left(\int_0^{\tau} \psi_{dk}(u) \right) \right] \Big|_{b_1=\dots=b_k=0} ds_1 \dots ds_k x_{10}^{k+1} a_2^k. \end{aligned}$$

Evaluating the derivatives, we get

$$\begin{aligned} |\tilde{x}_{1,k}| \leq & \int_0^t \dots \int_0^{s_2} \exp \left(\int_0^t \psi_k(\tau) (f(\bar{s}^{(k)}, \tau) + 1) \right) \\ & \times \left(\left| x_{20} \int_0^t \psi_{qk}(\tau) \exp \left(\int_0^{\tau} (\psi_{dk}(u) - \phi(\bar{s}^{(k)}, u)) \right) \right| \right)_k \\ & \times \exp \left[x_{20} \int_0^t \psi_{qk}(\tau) \exp \left(\int_0^{\tau} (\psi_{dk}(u) - \phi(\bar{s}^{(k)}, u)) \right) \right] ds_1 \dots ds_k |x_{10}^{k+1} a_2^k|, \end{aligned}$$

where $(c)_k = c \cdot (c+1) \cdot \dots \cdot (c+k-1)$.

Estimating the expression $(|x_{20} \int_0^t \psi_{qk}(\tau) \exp(\int_0^\tau (\psi_{dk}(u) - \phi(\bar{s}^{(k)}, u)))|)_k$, one gets

$$|\tilde{x}_{1,k}| \leq \int_0^t \dots \int_0^{s_2} \exp \left(\int_0^t \psi_k(\tau) (f(\bar{s}^{(k)}, \tau) + 1) \right) 2^{k-1} k! \\ \times \exp \left((1 - \ln(2)) \left[x_{20} \int_0^t \psi_{qk}(\tau) \exp \left(\int_0^\tau (\psi_{dk}(u) - \phi(\bar{s}^{(k)}, u)) \right) \right] \right) ds_1 \dots ds_k |x_{10}^{k+1} a_2^k|.$$

If σ_1, σ_2 are imaginary and $x_{20}a_1 < 0$, using the estimation for $\psi_k(\tau)$ and $\psi_{dk}(u)$, we have

$$|\tilde{x}_{1,k}| \leq |x_{10}^{k+1} a_2^k| \int_0^t \dots \int_0^{s_2} \exp((k_1 - \frac{\sigma_1^2}{2})(s_1 + \dots + s_k + t)) 2^{k-1} k! \\ \times \exp((1 - \ln(2))x_{20}a_1(s_1 + \dots + s_k + t) \min \left\{ 1, \exp((k_2 - \frac{\sigma_2^2}{2})t) \right\}) ds_1 \dots ds_k.$$

For $C = \min \left\{ 1, \exp((k_2 - \frac{\sigma_2^2}{2})t) \right\} (1 - \ln 2)x_{20}a_1 + k_1 - \frac{\sigma_1^2}{2} < 0$, evaluating the integrals, we get

$$|\tilde{x}_{1,k}| \leq |x_{10}^{k+1} a_2^k| 2^{k-1} k! \frac{\exp(Ct)}{k! |C|^k}.$$

If $\delta = \left| \frac{2x_{10}a_2}{C} \right| < 1$ $\tilde{x}_1(t)$ has a limiting value as the dimension of the truncated system n tends to infinity and we suppose that this limiting value is the solution of infinite system. The estimate for the error of the transition from the infinite to the truncated system is $\frac{|x_{10}|}{2} \exp(Ct) \sum_{j=n}^{\infty} \delta^j \leq \frac{|x_{10}|}{2} \exp(Ct) \frac{\delta^n}{1-\delta}$.

Analyzing the convergence of $\tilde{x}_2(t)$, consider the k -th term of $\tilde{x}_2(t)$ in the form

$$\tilde{x}_{2k} = \int_0^t \dots \int_0^{s_2} \frac{\partial^k}{\partial b_1 \dots \partial b_k} \exp \left(\int_0^t \psi_k(\tau) f(\bar{s}^{(k)}, \tau) \right) \\ \times \exp \left(\int_0^t \psi_{dk}(\tau) \right) \exp \left(x_{20} \int_0^t (\psi_{qk}(\tau) - a_1 d\tau) \right) \\ \times \exp \left(\int_0^\tau \psi_{dk}(u) \right) \Big|_{b_1=\dots=b_k=0} d\bar{s}^k x_{10}^k x_{20} a_2^k.$$

After evaluating the derivatives we will have

$$|\tilde{x}_{2k}| \leq \int_0^t \dots \int_0^{s_2} \exp \left(\int_0^t \psi_k(\tau) f(\bar{s}^{(k)}, \tau) \right) \\ \times \left(\left| x_{20} \int_0^t (\psi_{qk}(\tau) - a_1 d\tau) \times \exp \left(\int_0^\tau (\psi_{dk}(u) - \phi(\bar{s}^{(k)}, u)) \right) \right| \right)_{k+1} \\ \times \exp \left(x_{20} \int_0^t (\psi_{qk}(\tau) - a_1 d\tau) \times \exp \left(\int_0^\tau (\psi_{dk}(u) - \phi(\bar{s}^{(k)}, u)) \right) \right) d\bar{s}^k |x_{10}^k x_{20} a_2^k|.$$

Substituting the estimate of $(|x_{20} \int_0^t (\psi_{qk}(\tau) - a_1 d\tau) \times \exp(\int_0^\tau (\psi_{dk}(u) - \phi(\bar{s}^{(k)}, u)))|)_{k+1}$ we get

$$\begin{aligned} |\tilde{x}_{2k}| &\leq \int_0^t \cdots \int_0^{s_2} \exp \left(\int_0^t \psi_k(\tau) f(\bar{s}^{(k)}, \tau) \right) \\ &\quad \times 2^k (k+1)! \exp \left(x_{20} (1 - \ln(2)) \int_0^t (\psi_{qk}(\tau) - a_1 d\tau) \right) \\ &\quad \times \exp \left(\int_0^\tau (\psi_{dk}(u) - \phi(\bar{s}^{(k)}, u)) \right) d\bar{s}^k |x_{10}^k x_{20} a_2^k|. \end{aligned}$$

For σ_1, σ_2 are imaginary and $x_{20} a_1 < 0$, write the last inequality in the form

$$\begin{aligned} |\tilde{x}_{2k}| &\leq |x_{10}^k x_{20} a_2^k| \int_0^t \cdots \int_0^{s_2} \exp \left(\left(k_1 - \frac{\sigma_1^2}{2} \right) (s_1 + \dots + s_k) \right) 2^k (k+1)! \\ &\quad \times \exp(x_{20} a_1 (1 - \ln(2)) (s_1 + \dots + s_k) \min \left\{ 1, \exp((k_2 - \frac{\sigma_2^2}{2})t) \right\}) d\bar{s}^k. \end{aligned}$$

For $C = \min \left\{ 1, \exp((k_2 - \frac{\sigma_2^2}{2})t) \right\} (1 - \ln(2)) x_{20} a_1 + k_1 - \frac{\sigma_1^2}{2} < 0$, after evaluating the integrals we get

$$|\tilde{x}_{2k}| \leq |x_{10}^k x_{20} a_2^k| 2^k (k+1)! \frac{1}{k! |C|^k}.$$

If $\delta = |\frac{2x_{10}a_2}{C}| < 1$ $\tilde{x}_2(t)$ has a limiting value as the dimension of the truncated system n tends to infinity and we suppose that this limiting value is the solution of infinite system. The estimate for the error of the transition from the infinite to the truncated system is $|x_{20}| \sum_{j=n}^\infty (j+1) \delta^j \leq 2|x_{20}| n \frac{\delta^n}{1-\delta}$.

3. Example

Let us evaluate the mathematical expectation from the solutions $\tilde{x}_1, \tilde{x}_2$ for small values of a_1, a_2 . In this case, we can use only the first terms in the expansion with respect to the degrees of a_1, a_2 . For example, taking into account the zero and first degrees of a_1, a_2 , one infers that

$$\begin{aligned} M\tilde{x}_1 &\approx x_{10} \exp(k_1 t) \left(1 + x_{20} a_1 \frac{\exp((k_2 + \sigma_1 \sigma_2)t) - 1}{k_2 + \sigma_1 \sigma_2} \right), \\ M\tilde{x}_2 &\approx x_{20} \exp(k_2 t) \left(1 + x_{10} a_2 \frac{\exp((k_1 + \sigma_1 \sigma_2)t) - 1}{k_1 + \sigma_1 \sigma_2} \right). \end{aligned}$$

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