APPLICATION OF JACOBI POLYNOMIALS TO THE APPROXIMATE SOLUTION OF A SINGULAR INTEGRAL EQUATION WITH CAUCHY KERNEL ON THE REAL HALF-LINE

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Abstract — In this paper, exact solution of the characteristic equation with Cauchy kernel on the real half-line is presented. Next, Jacobi polynomials are used to derive approximate solutions of this equation. Moreover, estimations of errors of the approximated solutions are presented and proved.

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1. Introduction

We consider the characteristic equation

$$a\varphi(x) + \frac{1}{\pi i} \int_0^{+\infty} \frac{b\varphi(\sigma)}{\sigma - x} d\sigma = f(x), \quad x \in (0, +\infty), \quad (1.1)$$

where a, b are given complex numbers satisfying the conditions $a^2 - b^2 \neq 0$ and $b \neq 0$, $f(x)$ is the given complex-valued Hölder continuous function and $\varphi(x)$ is the unknown function.

We say, that a function $f(x)$, $x \in [0, +\infty)$, belongs to the Hölder class if it satisfies the inequality

$$|f(x_1) - f(x_2)| \leq A_1 |x_1 - x_2|^\gamma, \quad 0 < \gamma \leq 1, \quad \forall x_1, x_2 \in [0, c], \quad A_1 = \text{const},$$

on each interval $[0, c]$, $c > 0$, and the condition

$$|f(x) - f(\infty)| \leq A_2 / |x|^\gamma, \quad A_2 = \text{const},$$

is satisfied as $x \rightarrow \infty$.

If the integration contour is a smooth closed or open curve of finite length, the theory of equations of the form (1.1) has been developed comprehensively in [1, 2, 6, 11]. Moreover, the equation

$$\frac{1}{\pi i} \int_0^{+\infty} \frac{\varphi(\sigma)}{\sigma - x} d\sigma = f(x), \quad x > 0, \quad (1.2)$$

which is a special case of (1.1), was considered in [1, 8, 10].

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In the present paper, we find explicit formulas for the solution of (1.1). We also find conditions for the unique solvability of (1.1). Moreover, we present numerical schemes for (1.1) based on Jacobi polynomials and estimate the order of accuracy of the approximate solution.

2. The characteristic singular integral equation with a Cauchy kernel on the real half-line

We apply the transformation [9] of the form

$$\frac{1}{\sigma - x} = \frac{x + 1}{\sigma + 1} \frac{1}{\sigma - x} + \frac{1}{\sigma + 1}$$

to the kernel of (1.1) and rewrite the equation in the form

$$a\varphi(x) + \frac{1}{\pi i} \int_0^{+\infty} \frac{x+1}{\sigma+1} \frac{b}{\sigma-x} \varphi(\sigma) d\sigma + \frac{1}{\pi i} \int_0^{+\infty} \frac{b}{\sigma+1} \varphi(\sigma) d\sigma = f(x), \quad x \in (0, +\infty), \quad a^2 - b^2 \neq 0. \quad (2.1)$$

By setting $x = (1+t)/(1-t)$, $\sigma = (1+\tau)/(1-\tau)$ and introducing the notation

$$\psi(t) = \varphi\left(\frac{1+t}{1-t}\right), \quad g(t) = f\left(\frac{1+t}{1-t}\right),$$

we reduce (2.1) to the form

$$a\psi(t) + \frac{1}{\pi i} \int_{-1}^1 b \frac{\psi(\tau)}{\tau - t} d\tau - \frac{1}{\pi i} \int_{-1}^1 b \frac{\psi(\tau)}{\tau - 1} d\tau = g(t), \quad t \in (-1, 1),$$

$$\lim_{t \rightarrow 1-0} \psi(t) = 0, \quad \lim_{t \rightarrow 1-0} g(t) = 0. \quad (2.2)$$

Let $X(z)$ be a canonical function for the linear conjugation problem

$$X^+(t) = \frac{a-b}{a+b} X^-(t), \quad -1 < t < 1, \quad (2.3)$$

and

$$Z(t) = (a+b) X^+(t) = (a-b) X^-(t), \quad -1 < t < 1.$$

If the solution $\psi(t)$ belongs to $h(1)$ (i.e., the class of Hölder continuous functions on $(-1, 1)$, bounded in the neighborhood of the point $z = 1$ and admitting an integrable singularity at the point $z = -1$ [6]), then, for $0 < \theta < \pi$, we have

$$X(z) = \left(\frac{z-1}{z+1}\right)^{\omega_1 + i\omega_2}, \quad z \notin (-1, 1), \quad (2.4)$$

and for $-\pi < \theta < 0$

$$X(z) = \left(\frac{z-1}{z+1}\right)^{\omega_1 + i\omega_2} \frac{z-1}{z+1}, \quad z \notin (-1, 1), \quad (2.5)$$

where $G = (a-b)/(a+b)$, $\theta = \arg G$, $\omega_1 = (2\pi)^{-1}\theta$ and $\omega_2 = -(2\pi)^{-1} \ln |G|$.

Hence, in the first case we have

$$Z(t) = \sqrt{a^2 - b^2} p(t), \quad (2.6)$$

where

$$p(t) = (1-t)^\alpha (1+t)^\beta, \quad \alpha = \omega_1 + i\omega_2, \quad \beta = -\omega_1 - i\omega_2, \quad (2.7)$$

and in the second case

$$Z(t) = (-1) \sqrt{a^2 - b^2} p(t), \quad (2.8)$$

where

$$p(t) = (1-t)^\alpha (1+t)^\beta, \quad \alpha = 1 + \omega_1 + i\omega_2, \quad \beta = -1 - \omega_1 - i\omega_2. \quad (2.9)$$

The index of the characteristic operator in the class $h(1)$ is equal to 0 ($\kappa = -(\alpha + \beta) = 0$).

If the solution $\psi(t)$ is sought in the class $h(-1, 1)$ (i.e., the class of Hölder continuous functions on $(-1, 1)$, bounded in the neighbourhoods of points $z = \pm 1$ [6]), then the fundamental solution $X(z)$ has the form

$$X(z) = \left(\frac{z-1}{z+1} \right)^{\omega_1 + i\omega_2} (z+1), \quad 0 < \theta < \pi, \quad (2.10)$$

and

$$X(z) = \left(\frac{z-1}{z+1} \right)^{\omega_1 + i\omega_2} (z-1), \quad -\pi < \theta < 0 \quad (2.11)$$

respectively. Thus, for $0 < \theta < \pi$, we have

$$Z(t) = \sqrt{a^2 - b^2} (1-t)^\alpha (1+t)^\beta, \quad \alpha = \omega_1 + i\omega_2, \quad \beta = 1 - \omega_1 - i\omega_2, \quad (2.12)$$

and, for $-\pi < \theta < 0$, we have

$$Z(t) = -\sqrt{a^2 - b^2} (1-t)^\alpha (1+t)^\beta, \quad \alpha = 1 + \omega_1 + i\omega_2, \quad \beta = -\omega_1 - i\omega_2. \quad (2.13)$$

In this case, the index κ is equal to -1 ($\kappa = -(\alpha + \beta) = -1$).

Passing in (2.2) to the new unknown function $u(t)$ by the rule

$$\psi(t) = \frac{Z(t)}{a^2 - b^2} u(t), \quad (2.14)$$

and introducing the notation $A = a/(a^2 - b^2)$, $B = b/(a^2 - b^2)$, we reduce (2.2) to the form

$$AZ(t)u(t) + \frac{1}{\pi i} \int_{-1}^1 BZ(\tau) \frac{u(\tau)}{\tau - t} d\tau - \frac{1}{\pi i} \int_{-1}^1 BZ(\tau) \frac{u(\tau)}{\tau - 1} d\tau = g(t), \quad t \in (-1, 1), \quad g(1) = 0. \quad (2.15)$$

Setting

$$\frac{1}{\pi i} \int_{-1}^1 BZ(\tau) \frac{u(\tau)}{\tau - 1} d\tau = \gamma_0, \quad (2.16)$$

and solving (2.15) in the class $h(1)$, we obtain

$$u(t) = aZ^{-1}(t)g(t) - \frac{1}{\pi i} \int_{-1}^1 bZ^{-1}(\tau) \frac{g(\tau)}{\tau - t} d\tau + \gamma_0. \quad (2.17)$$

Hence by substituting (2.17) into (2.15) we can show that the function (2.17), where γ_0 is an arbitrary constant, is the general solution of (2.15) [9].

The constant γ_0 on the right-hand side in expression (2.17) is uniquely determined if (2.15) is supplemented by the condition

$$\frac{1}{\pi i} \int_{-1}^1 BZ(\tau) \frac{u(\tau)}{\tau-1} d\tau = A_0^*, \quad (2.18)$$

equivalent to (2.16). Substituting function (2.17) for $u(t)$ into this relation, we obtain $\gamma_0 = A_0^*$.

Let us introduce function classes that will be used during the solution of (1.1). We write $\varphi(x) \in h(\infty)$, $x > 0$, if $\varphi(x)$ belongs to the class of functions that are Hölder continuous on $[\epsilon, +\infty)$, $\epsilon > 0$, vanish at infinity ($\lim_{x \rightarrow +\infty} \varphi(x) = 0$), and can have an integrable singularity in the neighbourhood of $x = 0$. Further, we write $\varphi(x) \in h(0, \infty)$, $x > 0$, if it belongs to the class of bounded Hölder functions on $(0, +\infty)$ such that $\lim_{x \rightarrow +\infty} \varphi(x) = 0$.

We have thereby proved the following theorem.

Theorem 2.1. *Let a complex valued function $f(x) \in h(0, \infty)$ be given on $[0, +\infty)$. Then the general solution of Eq. (1.1) in the class $h(\infty)$ has the form*

$$\begin{aligned} \varphi(x) = & \frac{a}{a^2 - b^2} f(x) - \frac{x^\beta}{a^2 - b^2} \frac{1}{\pi i} \int_0^{+\infty} \frac{bf(\sigma)}{\sigma^\beta (\sigma - x)} d\sigma + \\ & \frac{x^\beta}{a^2 - b^2} \frac{1}{\pi i} \int_0^{+\infty} \frac{bf(\sigma)}{\sigma^\beta (\sigma + 1)} d\sigma + \frac{\gamma_0}{\sqrt{a^2 - b^2} x^{-\beta}}, \quad x \in (0, +\infty), \end{aligned} \quad (2.19)$$

where $\operatorname{Re} \beta < 0$, $0 < \theta < \pi$, and γ_0 is an arbitrary constant. Additionally, if the solution $\varphi(x)$ satisfies the condition

$$\frac{1}{\pi i} \int_0^{+\infty} \frac{b\varphi(\sigma)}{\sigma + 1} d\sigma = A_0^*, \quad (2.20)$$

where A_0^* is an arbitrary number, then the unique solution of problem (1.1) is given by formula (2.19) with $\gamma_0 = A_0^*$.

Now we will look for the solution $u(t)$ of (2.15) in $h(-1, 1)$. From [2, 6, 9] it follows that (2.15) is solvable if and only if $(\pi i)^{-1} \int_{-1}^1 bZ^{-1}(t) [g(t) + A_0^*] dt = 0$, where A_0^* is given by (2.18). Since it follows from [11] that $(\pi i)^{-1} \int_{-1}^1 bZ^{-1}(t) dt = \operatorname{Res}_{z=\infty} (X(z))^{-1} = -1$, we have

$$A_0^* = \frac{1}{\pi i} \int_{-1}^1 BZ(\tau) \frac{u(\tau)}{\tau-1} d\tau = \frac{1}{\pi i} \int_{-1}^1 bZ^{-1}(\tau) g(\tau) d\tau. \quad (2.21)$$

Therefore, (2.15) can be rewritten in the form

$$AZ(t) u(t) + \frac{1}{\pi i} \int_{-1}^1 \frac{BZ(\tau) u(\tau)}{\tau - t} d\tau = g(t) + \frac{1}{\pi i} \int_{-1}^1 bZ^{-1}(\tau) g(\tau) d\tau, \quad t \in (-1, 1). \quad (2.22)$$

If the necessary and sufficient condition (2.21) is satisfied, then in view of [2, 6, 9] the solution of this equation is given by the formula

$$\begin{aligned} u(t) &= aZ^{-1}(t)(g(t) + A_0^*) - \frac{1}{\pi i} \int_{-1}^1 bZ^{-1}(\tau) \frac{(g(\tau) + A_0^*)}{\tau - t} d\tau = \\ &= aZ^{-1}(t)g(t) - \frac{1}{\pi i} \int_{-1}^1 bZ^{-1}(\tau) \frac{g(\tau)}{\tau - t} d\tau, \quad t \in (-1, 1). \end{aligned} \quad (2.23)$$

This finishes the proof of the following theorem.

Theorem 2.2. *Let a complex valued function $f(x) \in h(0, \infty)$ be given on $[0, +\infty)$. Then the solution $\varphi(x)$ of (1.1) in the class $h(0, \infty)$ satisfying the condition*

$$\frac{1}{\sqrt{a^2 - b^2}} \frac{1}{\pi i} \int_0^{+\infty} \frac{1}{\sigma^\beta} \frac{f(\sigma)}{\sigma + 1} d\sigma = \frac{1}{\pi i} \int_0^{+\infty} \frac{\varphi(\sigma)}{\sigma + 1} d\sigma, \quad (2.24)$$

is given by the formula

$$\varphi(x) = \frac{af(x)}{a^2 - b^2} - \frac{x^\beta}{\pi i} \int_0^{+\infty} \frac{1}{\sigma^\beta} \frac{b}{\sigma - x} f(\sigma) d\sigma, \quad x \in (0, +\infty),$$

where $\operatorname{Re} \alpha > 0$, $\operatorname{Re} \beta > 0$, $0 < \theta < \pi$, $\lim_{x \rightarrow +\infty} f(x) = 0$.

Now we introduce one more formula, which will be used later to obtain an approximate solution of (2.15).

Let $P_k^{(\alpha, \beta)}(x)$ be a Jacobi polynomial of degree k with a weight function $p(x) = (1 - x)^\alpha \times (1 + x)^\beta$, $\alpha, \beta > -1$. Then it follows from [3] that

$$\begin{aligned} AZ(x)P_k^{(\alpha, \beta)}(x) + \frac{1}{\pi i} \int_{-1}^1 BZ(t) \frac{P_k^{(\alpha, \beta)}(t)}{t - x} dt - \frac{1}{\pi i} \int_{-1}^1 BZ(t) \frac{P_k^{(\alpha, \beta)}(t)}{t - 1} dt = \\ \begin{cases} P_k^{(-\alpha, -\beta)}(x) - P_k^{(-\alpha, -\beta)}(1), & \alpha + \beta = 0, \quad \kappa = 0, \quad k = 0, 1, 2, \dots, \\ 2[P_{k+1}^{(-\alpha, -\beta)}(x) - P_{k+1}^{(-\alpha, -\beta)}(1)], & \alpha + \beta = 1, \quad \kappa = -1, \quad k = 0, 1, 2, \dots, \\ [P_{k-1}^{(-\alpha, -\beta)}(x) - P_{k-1}^{(-\alpha, -\beta)}(1)]/2, & \alpha + \beta = -1, \quad \kappa = 1, \quad k = 1, 2, 3, \dots \end{cases} \end{aligned} \quad (2.25)$$

3. Approximate solution

In this section, we will derive an approximate solution of (2.15). For this purpose, we will interpolate the function $g(t)$ at Chebyshev nodes

$$t_k = \cos \frac{(2k - 1)\pi}{2(n + 1)}, \quad k = 1, 2, \dots, n + 1, \quad (3.1)$$

by a polynomial $g_n(t)$ of degree n . We will use the following approximation formula (cf. [7])

$$g_n(t) = \frac{2}{n + 1} \sum_{j=0}^n \left(\sum_{k=1}^{n+1} T_j(t_k) g(t_k) \right) T_j(t), \quad (3.2)$$

where $T_j(t) = \cos(j \arccos t)$ are Chebyshev polynomials of the first kind. Expressing Chebyshev polynomials $T_j(t)$ in terms of Jacobi polynomials, we obtain

$$T_j(t) = \sum_{l=0}^j \rho_{jl} P_l^{(-\alpha, -\beta)}(t), \quad (3.3)$$

where

$$\begin{aligned} \rho_{jl} &= \frac{1}{h_l^{(-\alpha, -\beta)}} \frac{1}{\pi} \int_{-1}^1 q(t) T_j(t) P_l^{(-\alpha, -\beta)}(t) dt = \\ &= (-1) \frac{1}{h_l^{(-\alpha, -\beta)}} \frac{1}{\sin \pi \alpha} \operatorname{Res}_{z=\infty} \{(z-1)^{-\alpha} (z+1)^{-\beta} T_j(z) P_l^{(-\alpha, -\beta)}(z)\}, \\ q(t) &= (1-t)^{-\alpha} (1+t)^{-\beta}, \quad \alpha + \beta = 0, \\ h_l^{(-\alpha, -\beta)} &= \frac{1}{\pi} \int_{-1}^1 q(t) [P_l^{(-\alpha, -\beta)}(t)]^2 dt = \frac{2}{(2l+1)\pi} \frac{\Gamma(l-\alpha+1) \Gamma(l-\beta+1)}{l! \Gamma(l+1)}. \end{aligned}$$

Using (3.3), the interpolation polynomial (3.2) takes the form

$$g_n(t) = \sum_{k=0}^n g_k P_k^{(-\alpha, -\beta)}(t), \quad (3.4)$$

where

$$g_0 = \frac{2}{n+1} \sum_{j=0}^n \left(\sum_{i=1}^{n+1} T_j(t_i) g(t_i) \right) \rho_{j0}, \quad g_k = \frac{2}{n+1} \sum_{j=k}^n \left(\sum_{i=1}^{n+1} T_j(t_i) g(t_i) \right) \rho_{jk}, \quad k=1, 2, \dots, n. \quad (3.5)$$

Following [11], an approximate solution $u_n(x)$ of problem (2.15), (2.18) is defined as a solution of the following problem:

$$AZ(t) u_n(t) + \frac{1}{\pi i} \int_{-1}^1 BZ(\tau) \frac{u_n(\tau)}{\tau - t} d\tau - \frac{1}{\pi i} \int_{-1}^1 BZ(\tau) \frac{u_n(\tau)}{\tau - 1} d\tau = g_n(t) - g_n(1), \quad t \in (-1, 1), \quad (3.6)$$

$$\frac{1}{\pi i} \int_{-1}^1 BZ(\tau) \frac{u_n(\tau)}{\tau - 1} d\tau = A_0^*, \quad (3.7)$$

where

$$u_n(t) = \sum_{k=0}^n c_k P_k^{(\alpha, \beta)}(t), \quad (3.8)$$

with unknown coefficients c_k .

Substituting (3.8) into (3.6), (3.7) and using formula (2.25) for the computation of the singular integral we obtain

$$\sum_{k=0}^n c_k [P_k^{(-\alpha, -\beta)}(t) - P_k^{(-\alpha, -\beta)}(1)] = \sum_{k=0}^n g_k [P_k^{(-\alpha, -\beta)}(t) - P_k^{(-\alpha, -\beta)}(1)],$$

$$\sum_{k=0}^n c_k \frac{1}{\pi i} \int_{-1}^1 BZ(\tau) \frac{P_k^{(\alpha, \beta)}(\tau)}{\tau - 1} d\tau = \sum_{k=0}^n c_k P_k^{(-\alpha, -\beta)}(1) = A_0^*.$$

Hence, we get

$$c_n = g_n, \dots, c_1 = g_1, \quad c_0 = A_0^* - c_1 P_1^{(-\alpha, -\beta)}(1) - \dots - c_n P_n^{(-\alpha, -\beta)}(1).$$

If $\kappa = -1$, then the approximate solution of (2.15)

$$u_{n-1}(t) = \sum_{k=0}^{n-1} c_k P_k^{(\alpha, \beta)}(t), \quad (3.9)$$

where c_k are the coefficients to be determined, can be found as a solution of the following problem:

$$AZ(t)u_{n-1}(t) + \frac{1}{\pi i} \int_{-1}^1 BZ(\tau) \frac{u_{n-1}(\tau)}{\tau - t} d\tau - \frac{1}{\pi i} \int_{-1}^1 BZ(\tau) \frac{u_{n-1}(\tau)}{\tau - 1} d\tau = g_n(t) - g_n(1), \quad t \in (-1, 1), \quad (3.10)$$

with $g_n(t) = \sum_{k=0}^n g_k P_k^{(-\alpha, -\beta)}(t)$, where

$$g_0 = \frac{2}{n+1} \sum_{j=0}^n \left(\sum_{i=1}^{n+1} T_j(t_i) g(t_i) \right) \rho_{j0}, \quad g_k = \frac{2}{n+1} \sum_{j=k}^n \left(\sum_{i=1}^{n+1} T_j(t_i) g(t_i) \right) \rho_{jk}, \quad k=1, 2, \dots, n,$$

and

$$\begin{aligned} \rho_{jk} &= \frac{1}{h_k^{(-\alpha, -\beta)}} \frac{1}{\pi} \int_{-1}^1 q(t) T_j(t) P_k^{(-\alpha, -\beta)}(t) dt = \\ &= (-1) \frac{1}{h_k^{(-\alpha, -\beta)}} \frac{1}{\sin \pi \alpha} \operatorname{Res}_{z=\infty} \{ (z-1)^{-\alpha} (z+1)^{-\beta} T_j(z) P_k^{(-\alpha, -\beta)}(z) \}, \\ q(t) &= (1-t)^{-\alpha} (1+t)^{-\beta}, \quad \alpha + \beta = 1, \\ h_k^{(-\alpha, -\beta)} &= \frac{1}{\pi} \int_{-1}^1 q(t) [P_k^{(-\alpha, -\beta)}(t)]^2 dt = \frac{2}{(k+1)\pi} \frac{\Gamma(k-\alpha+1)\Gamma(k-\beta+1)}{k! \Gamma(k+2)}. \end{aligned}$$

Using (2.22) formula (3.10) can be rewritten in the form

$$\begin{aligned} AZ(t)u_{n-1}(t) + \frac{1}{\pi i} \int_{-1}^1 BZ(\tau) \frac{u_{n-1}(\tau)}{\tau - t} d\tau &= g_n(t) + \frac{1}{\pi i} \int_{-1}^1 bZ^{-1}(\tau) g_n(\tau) d\tau = \\ g_n(t) - g_0 &= \sum_{k=1}^n g_k P_k^{(-\alpha, -\beta)}(t), \quad t \in (-1, 1). \end{aligned} \quad (3.11)$$

Substituting (3.9) into (3.11) and using formula (2.25), we obtain

$$2 \sum_{k=0}^{n-1} c_k P_{k+1}^{(-\alpha, -\beta)}(t) = \sum_{k=1}^n g_k P_k^{(-\alpha, -\beta)}(t).$$

Hence, we finally obtain

$$c_{n-1} = \frac{1}{2} g_n, \quad c_{n-2} = \frac{1}{2} g_{n-1}, \quad \dots, \quad c_0 = \frac{1}{2} g_1.$$

4. Estimation of errors

In this section, we will estimate the orders of errors of approximate solutions.

Theorem 4.1. *Let the right-hand side $g(t)$ of (2.15) satisfy the Hölder condition*

$$|g(t') - g(t'')| \leq K |t' - t''|^\mu, \quad -1 < t', t'' < 1, \quad 0 < \mu \leq 1. \quad (4.1)$$

Moreover, let $g(t)$ be approximated by the interpolation polynomial (3.4) with respect to the Chebyshev nodes of the first kind (3.1), and let $u(t)$ and $u_n(t)$ denote the exact and approximate solutions of the problem given by (2.15), (2.18) and (3.6), (3.10). Then the following estimation holds:

$$\|(1-t)^\alpha (u(t) - u_n(t))\|_\infty \leq M n^{-\mu} \ln^2 n, \quad (4.2)$$

where α is given by (2.7) and (2.9) and M is a constant not depending on n .

Proof. Since

$$u_n(t) = aZ^{-1}(t)g_n(t) - \frac{1}{\pi i} \int_{-1}^1 bZ^{-1}(\tau) \frac{g_n(\tau)}{\tau - t} d\tau + \gamma_0, \quad -1 < t < 1,$$

we have

$$u(t) - u_n(t) = aZ^{-1}(t)[g(t) - g_n(t)] - \frac{1}{\pi i} \int_{-1}^1 bZ^{-1}(\tau) \frac{g(\tau) - g_n(\tau)}{\tau - t} d\tau, \quad -1 < t < 1. \quad (4.3)$$

In this case, the integral on the right-hand side of (4.3) can have an integrable singularity at the point $t = 1$. Therefore, we have to estimate the product

$$(1-t)^\alpha \frac{1}{\pi i} \int_{-1}^1 bZ^{-1}(\tau) \frac{g(\tau) - g_n(\tau)}{\tau - t} d\tau, \quad -1 < t < 1,$$

for which we have the following estimation [11]:

$$\left\| (1-t)^\alpha \frac{1}{\pi i} \int_{-1}^1 bZ^{-1}(\tau) \frac{g(\tau) - g_n(\tau)}{\tau - t} d\tau \right\| \leq M_1 \frac{\ln^2 n}{n^\mu},$$

where M_1 is a constant not depending on n . Taking into account inequality [7]

$$\|g(t) - g_n(t)\|_\infty \leq M_2 n^{-\mu} \ln n,$$

where M_2 is a constant not depending on n , we obtain result (4.2). $\square$

Theorem 4.2. *Suppose that the conditions of the Theorem 4.1 are satisfied (i.e., the right-hand side function $g(t)$ and its approximation $g_n(t)$ are the same as in Theorem 4.1), and let $u(t)$, $u_{n-1}(t)$ denote the exact and approximate solutions of the problem defined by (2.15), (3.10) respectively. Then the following estimate holds:*

$$\|Z(t)[u(t) - u_{n-1}(t)]\|_\infty \leq M n^{-\mu} \ln^2 n, \quad (4.4)$$

where M is a constant not depending on n .

In the last case, the integral on the right-hand side of (4.3) can have an integrable singularity at the endpoints of the interval $[-1, 1]$. Therefore, we have to estimate the product

$$Z(t) \frac{1}{\pi i} \int_{-1}^1 bZ^{-1}(\tau) \frac{g(\tau) - g_n(\tau)}{\tau - t} d\tau, \quad -1 < t < 1,$$

for which we have the following estimation [11]:

$$\left\| Z(t) \frac{1}{\pi i} \int_{-1}^1 bZ^{-1}(\tau) \frac{g(\tau) - g_n(\tau)}{\tau - t} d\tau \right\|_{\infty} \leq M_1 \frac{\ln^2 n}{n^{\mu}},$$

where M_1 is a constant not depending on n .

Remark 4.1. If $g(t)$, $t \in (-1, 1)$, being the right-hand side of (2.15), satisfies the Hölder condition (4.1), then the function $f(x)$, $x > 0$, being the right-hand side of (1.1) also satisfies the Hölder inequality.

5. Numerical example

In this section we will present a numerical example which illustrates usefulness of the approximation algorithms presented in the previous sections.

Let us consider the characteristic equation (1.1) in the class $h(0, \infty)$ ($\kappa = -1$), where

$$a = 1 + \frac{i}{2}, \quad b = -\frac{i}{2}, \quad f(x) = \frac{1}{2x + 3}.$$

Then

$$g(t) = f\left(\frac{1+t}{1-t}\right) = \frac{t-1}{t-5}.$$

It is easy to check that the exact solution of (2.15) in the class $h(-1, 1)$ ($\kappa = -1$) is equal to

$$u(t) = \frac{4}{X(5)(t-5)}.$$

The values of the exact and approximated solutions $u_{n-1}(x)$ of (2.15) for $n = 10$ are shown in Table.

Comparison of the values of the exact and approximated solutions of (2.15)

x	t	$u(t)$	$u_n(t)$
0,11	-0,8	-0,1208880939359 + 0,002704114228534i	-0,1208880939315 + 0,002704114229119i
0,25	-0,6	-0,1252055258622 + 0,002800689736696i	-0,1252055258366 + 0,002800689749419i
0,538	-0,3	-0,1322926310997 + 0,002959219344433i	-0,1322926310882 + 0,002959219344380i
1	0	-0,1402301889657 + 0,003136772505099i	-0,1402301889676 + 0,003136772510523i
1,5	0,2	-0,1460731135059 + 0,003267471359479i	-0,1460731135102 + 0,003267471359345i
2,333	0,4	-0,1524241184410 + 0,003409535331630i	-0,1524241184220 + 0,00340953533329i
3	0,5	-0,1558113210730 + 0,00348530278344i	-0,1558113210675 + 0,003485302787656i
4	0,6	-0,1593524874610 + 0,003564514210340i	-0,1593524874720 + 0,003564514214022i
5,666	0,7	-0,1630583592624 + 0,003647409889651i	-0,1630583592671 + 0,00364740988958i
9	0,8	-0,1669407011496 + 0,003734252982261i	-0,1669407011326 + 0,003734252980918i
19	0,9	-0,171012425567 + 0,003825332323292i	-0,1710124255673 + 0,003825332327957i

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