Complex Manifolds

Research Article

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A dual of the Chow transformation

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Abstract: We define a dual of the Chow transformation of currents on the complex projective space. This transformation factorizes a left inverse of the Chow transformation and its composition with the Chow transformation is a right inverse of a linear differential operator. In such a way we complete the general scheme of integral geometry for the Chow transformation. On another hand we prove the existence of a well defined closed positive conormal current associated to every closed positive current on the projective space. This is a consequence of the existence of a dual current, defined on the dual projective space. This allows us to extend to the case of a closed positive current the known inversion formula for the conormal of the Chow divisor of an effective algebraic cycle.

Keywords: Complex projective space, Homogeneous space, Chow divisor, Cycle space, Radon transform, John differential equations, Closed positive current, Slicing, Dual subvariety

MSC: 14C05, 32C30, 32J25, 53C65, 58A25

1 Introduction

For T a (q, q) -current on $\mathbb{P}_N$, we denote by $\mathcal{C}(T)$ the Chow transform of T , which is a $(1, 1)$ -current on the Grassmannian $\mathbb{G}_{q-1, N}$ of $(q-1)$ -dimensional projective subspaces of $\mathbb{P}_N$. We define $\mathcal{C}(T)$ by integrating T on the projective subspaces of $\mathbb{P}_N$ of dimension $q-1$ i.e. $\mathcal{C}(T)$ is obtained from T by means of the double fibration

$$\begin{array}{ccc} & \Gamma & \\ \swarrow & & \searrow \\ \mathbb{P}_N & & \mathbb{G}_{q-1, N} \end{array}$$

with $\Gamma \subset \mathbb{P}_N \times \mathbb{G}_{q-1, N}$ the incidence manifold. We prove the existence of a dual integral transform $\mathcal{C}^*$, defined for $(1, 1)$ -currents on $\mathbb{G}_{q-1, N}$, with values in

$$\begin{cases} \{(q, q)\text{-currents on } \mathbb{P}_N\} & \text{if } 2q \leq N+1, \\ \{(N+1-q, N+1-q)\text{-currents on } \mathbb{P}_N\} & \text{if } 2q \geq N+1 \end{cases}$$

and satisfying the following property (see Theorem 2).

Theorem. *There is a linear differential operator $\mathcal{P}$ of order $2q-2$ with smooth coefficients, defined on*

$$\begin{cases} \{(q, q)\text{-currents on } \mathbb{P}_N\} & \text{if } 2q \leq N+1, \\ \{(N+1-q, N+1-q)\text{-currents on } \mathbb{P}_N\} & \text{if } 2q \geq N+1 \end{cases}$$

with values in $\{(q, q)\text{-currents on } \mathbb{P}_N\}$ such that $T = (\mathcal{P}\mathcal{C}^\mathcal{C})(T)$ for all (q, q) -currents T on $\mathbb{P}_N$.*

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In other words, $\mathcal{C}$ is injective and a left inverse for $\mathcal{C}$ is $\mathcal{P}\mathcal{C}^*$. In this way we complete for the Chow transformation the general scheme of integral geometry (see [18]).

We know by [24] that the Chow transformation of currents defined on a complex projective manifold is injective and we recall here the proof by slicing of this property (see Proposition 2 and Lemma 8). An application to the problem of the approximation by algebraic cycles is also given (see subsection 6.1).

When we take for T the integration current $[Z]$ associated to an algebraic cycle Z in $\mathbb{P}_N$ of codimension q , the injectivity of $\mathcal{C}^*\mathcal{C}$ is a consequence of the following result (see Proposition 10).

Proposition. *The singular support of the current $(\mathcal{C}^*\mathcal{C})([Z])$ is $\text{sing supp } ((\mathcal{C}^*\mathcal{C})([Z])) = \text{supp } Z$.*

The injectivity of $Z \rightarrow (\mathcal{C}^*\mathcal{C})([Z])$ is also obtained by using the Schubert calculus in the Grassmannian.

When T is closed positive on $\mathbb{P}_N$, by replacing T by its conormal current $\text{con}(T)$, we give a more explicit inversion formula.

In effect, $\text{con}(T)$ is a well defined closed positive current on $T^*\mathbb{P}_N$ (see Proposition 9 and Theorem 4). Note that the existence of a tangent current to T at a point is studied in [2, 3, 22].

By denoting by $\Phi : N^*\Gamma \rightarrow T^*\mathbb{P}_N$ and $\Psi : N^*\Gamma \rightarrow T^*\mathbb{G}_{q-1,N}$ the restrictions to the conormal $N^*\Gamma \subset T^*\mathbb{P}_N \times T^*\mathbb{G}_{q-1,N}$ of the canonical projections, we can transform $\text{con}(T)$ by means of the double fibration

$$\begin{array}{ccc} & N^*\Gamma & \\ \swarrow & & \searrow \\ T^*\mathbb{P}_N & & T^*\mathbb{G}_{q-1,N} \end{array}$$

obtained by considering the cotangent bundles. Then, on the conormal, the transposed map of $\Psi_*\Phi^*$ is also a left inverse, in other words, we have the following result (see Theorem 4).

Theorem. *For all closed positive (q, q) -currents T on $\mathbb{P}_N$, we have $\text{con}(T) = \Phi_*\Psi^*\Psi_*\Phi^*(\text{con}(T))$.*

By using the Schubert calculus, we also obtain the Skoda potential of T from the Chow transform $\mathcal{C}(T)$ by means of a similar dual integral transform (see Theorem 5).

In the rest of the paper, we give new proofs of the characterization by PDE of closed $(1, 1)$ -currents Θ on $\mathbb{G}_{q-1,N}$ which are in the image of $\mathcal{C}$ i.e. can be written $\Theta = \mathcal{C}(T)$ with some (q, q) -current T on $\mathbb{P}_N$ necessarily closed. We use homogeneous coordinates

$$\tau : (\mathbb{C}^{N+1})^q \rightarrow \mathbb{G}_{q-1,N}$$

defined by $\tau(z^0, \dots, z^{q-1}) = \text{vect}(z^0, \dots, z^{q-1})$ and we write the inverse image $\tau^*\Theta = \text{dd}^c U$ with some distribution $U(z^0, \dots, z^{q-1})$. We denote by z_m^l the coordinates of $z^l \in \mathbb{C}^{N+1}$ and we consider the differential operator

$$\partial_I = \det \left(\frac{\partial}{\partial z_j^k} \right)_{\substack{0 \leq k \leq q-1 \\ j \in I}}$$

for $I = (i_1, \dots, i_q)$ with $0 \leq i_1 < \dots < i_q \leq N$.

Then Θ is a Chow transform if and only if U satisfies the Gelfand-Gindikin-Graev equations

$$0 = \left(\frac{\partial^2}{\partial z_m^l \partial z_{m'}^{l'}} - \frac{\partial^2}{\partial z_m^l \partial z_{m'}^l} \right) \partial_I \bar{\partial}_J U \text{ and } 0 = \left(\frac{\partial^2}{\partial \bar{z}_m^l \partial \bar{z}_{m'}^{l'}} - \frac{\partial^2}{\partial \bar{z}_m^l \partial \bar{z}_{m'}^l} \right) \partial_I \bar{\partial}_J U$$

for all $0 \leq l, l' \leq q-1$, all $0 \leq m, m' \leq N$ and all $|I| = |J| = q$.

We give a proof of this characterization by slicing (see subsection 2.2 and Proposition 4) and another proof by using the kinematic formula (see subsection 6.2). We also give a proof by using the point of view of homogeneous spaces (see section 4). Precisely we denote by $U(N+1)$ the unitary group of $\mathbb{C}^{N+1}$ and we write the double fibration in the form

$$\begin{array}{ccc} & U(N+1)/(K \cap L) & \\ \swarrow & & \searrow \\ U(N+1)/K & & U(N+1)/L \end{array}$$

where $K = U(1) \times U(N)$ and $L = U(q) \times U(N+1-q)$. In such a manner, we easily express the Chow transform of a smooth differential $(q-1, q-1)$ -form on $\mathbb{P}_N$. By complexifying, we retrieve a distribution on the linear group $GL(N+1, \mathbb{C})$, which satisfies an equivalent system of PDE (see Theorem 3).

2 Chow transformation and slicing

We denote by $P(V)$ the projective space of lines in a complex vector space V of dimension $N+1$ and $[x]$ the point of $P(V)$ associated to a nonzero vector x in V . For $1 \leq q \leq N$, let $G(q, V) = \mathbb{G}_{q-1, N}$ be the Grassmannian of vector subspaces of V of dimension q and let $P(s)$ be the projective subspace of $P(V)$ associated to $s \in G(q, V)$.

Let Γ be the incidence manifold i.e. the submanifold in $P(V) \times G(q, V)$ of $([x], s)$ satisfying $s \ni x$. Then $\varphi: \Gamma \rightarrow P(V)$ and $\psi: \Gamma \rightarrow G(q, V)$ are the restrictions to Γ of the canonical projections.

Given T a current on $P(V)$ of bidimension (p, p) with $p = N - q$, we set

$$\mathcal{C}(T) = \psi_* \varphi^* T = \text{pr}_{2*}([\Gamma] \wedge \text{pr}_1^* T)$$

as being the Chow transform of T . Then $\mathcal{C}(T)$ is a current of bidegree $(1, 1)$ on $G(q, V)$.

2.1 Injectivity of $\mathcal{C}$

We first recall the proof (see [24]) by slicing of the injectivity of the Chow transformation in the set of currents of bidegree (q, q) on the projective space.

Proposition 1. *The transformation $\mathcal{C}$ is injective in the set of currents of bidegree (q, q) on $P(V)$.*

Proof. This is a consequence of the injectivity of $\mathcal{C}$ in the set of C^∞ differential forms of bidegree (q, q) on $P(V)$, by regularizing T and $\mathcal{C}(T)$ by means of mollifier operators J_ε on $P(V)$ and $\tilde{J}_\varepsilon$ on $G(q, V)$ such that

$$(\tilde{J}_\varepsilon \circ \mathcal{C})(T) = \mathcal{C}(J_\varepsilon(T))$$

and obtained from C^∞ approximations χ_ε of the Dirac mass at the origin of the unitary group $U(V) = U(N+1)$, for a Hermitian scalar product on V . We set

$$J_\varepsilon(T) = \int_{g \in U(V)} \chi_\varepsilon(g) g^* T d\nu(g) \quad (1)$$

with ν the Haar measure on $U(V)$, which is of total mass equal to 1, and

$$T_n = J_{\frac{1}{n}}(T).$$

This sequence of C^∞ differential forms on $P(V)$ weakly converges to T and satisfies

$$\mathcal{C}(T_n) = \int_{g \in U(V)} \chi_{\frac{1}{n}}(g) g^* \mathcal{C}(T) d\nu(g) = \tilde{J}_{\frac{1}{n}}(\mathcal{C}(T))$$

thus $\mathcal{C}(T) = 0 \Rightarrow \mathcal{C}(T_n) = 0$.

Now for the injectivity of $\mathcal{C}$ in the set of C^∞ differential forms of bidegree (q, q) on $P(V)$, we use a slicing, to reduce to the case when the dimension of the differential form is 0. For T of class C^∞ , we have

$$\mathcal{C}(T)|_{P(W^*)} = \mathcal{C}_W(T|_{P(W)}) \quad (2)$$

with $P(W^*) = G(q, W)$ for all $W \in G(q+1, V)$. Then if $T|_{P(W)} = 0$ for all W , then $T = 0$. In effect, with $\pi: V \rightarrow P(V)$ the canonical map, we have

$$(\pi^* T)_x(v_1, \dots, v_q, \bar{v}_1, \dots, \bar{v}_q) = (\pi^* T)|_W(x; v_1, \dots, v_q, \bar{v}_1, \dots, \bar{v}_q)$$

$$= (\pi|_W)^*(T|_{P(W)})(x; v_1, \dots, v_q, \bar{v}_1, \dots, \bar{v}_q)$$

when $W = \text{vect}(x, v_1, \dots, v_q)$, thus $\pi^*T = 0$ since we can assume T real.

In the case of the dimension 0 i.e. $q = N$, we have

$$\mathcal{C}(g\omega^N) \wedge (\omega^*)^{N-1} = \mathcal{R}_{N-1}(g)(\omega^*)^N$$

with ω resp. ω^* the Fubini-Study form on $P(V)$ resp. $P(V^*)$ and with $\mathcal{R}_{N-1}$ the projective hyperplane Radon transformation, defined by

$$\mathcal{R}_{N-1}(g)(s) = \int_{P(s)} g\omega^{N-1}$$

for $g \in \mathcal{C}^\infty(P(V))$ and for $s \in P(V^*) = G(N, V)$.

The injectivity of $\mathcal{R}_{N-1}$ is proved in [20, 21]. Precisely there is a polynomial $\mathcal{P}$ of degree $N - 1$ whose coefficients depend only on N , such that for all $g \in \mathcal{C}^\infty(P(V))$, we have the relation

$$g = \mathcal{P}(\Delta)\mathcal{R}_{N-1}^*\mathcal{R}_{N-1}g \quad (3)$$

with Δ the Laplacian on $P(V)$ associated to ω and with $\mathcal{R}_{N-1}^*$ the dual Radon transformation. $\square$

Note that formula (2) is still valid when T is a current, thanks to the following result.

Proposition 2. For T a current of bidegree (q, q) on $P(V)$,

- (i) a restriction current $T|_{P(W)}$ can be defined on $P(W)$, for all $W \in G(q+1, V)$, from a strict transform of T ,
- (ii) there is a sequence of smooth differential forms T_n of bidegree (q, q) on $P(V)$, weakly converging to T , such that $T_n|_{P(W)}$ weakly converges to a current which is equal to the restriction $T|_{P(W)}$.

Proof. Let $\mu : M \rightarrow P(V)$ be the blowing up of $P(V)$ with center $P(W)$ and let $H \subset M$ be the exceptional divisor. According to Poly (see [26]), there is a current μ^*T on M such that $\mu_*\mu^*T = T$. Another inverse image by μ of T is of the form $\mu^*T + i_*r$ where $i : H \rightarrow M$ is the canonical injection and r is a current on H such that $\nu_*r = 0$, with $\nu : H \rightarrow P(W)$ the fibration obtained as restriction of μ to H .

To define a restriction $T|_{P(W)}$, we are reduced to define $(\mu^*T)|_H$ thanks to the formula

$$T|_{P(W)} = \nu_*((\mu^*T)|_H \wedge \gamma)$$

with γ a closed smooth differential $(N - q - 1, N - q - 1)$ -form on H such that $\nu_*\gamma = 1$.

But when $\lambda = \lambda(X_1, \dots, X_N)$ is any distribution in $(X_1, \dots, X_N) \in \mathbb{C}^N$, a product

$$\begin{aligned} \lambda \delta(X_1) \frac{i}{2} dX_1 \wedge d\bar{X}_1 &= -\frac{i}{2\pi} \lambda \bar{\partial} \left(\frac{dX_1}{X_1} \right) \\ &= -\frac{i}{2\pi} \left\{ \bar{\partial} \left(\lambda \frac{dX_1}{X_1} \right) - (\bar{\partial} \lambda) \wedge \frac{dX_1}{X_1} \right\} \end{aligned}$$

is defined, by the Hörmander-Lojasiewicz division theorem. With $S = \mu^*T$, an intersection $S \wedge [H]$ is therefore defined, by assuming that $X_1 = 0$ is a local holomorphic equation of H .

Let $\sigma \in H^0(M, \mathcal{O}(H))$ be such that $\sigma^{-1}(0) = H$. Since $\|\sigma\|(S \wedge [H]) = 0$, we can write $S \wedge [H] = i_*(S|_H)$ with $S|_H$ some current on H . In such a way, a restriction $T|_{P(W)}$ has been defined.

Then we write this current $T|_{P(W)}$ as a weak limit of smooth differential (q, q) -forms R_n on $P(W)$. We can extend R_n to $P(V)$ by T_n smooth such that T_n weakly converges to T . When T is closed, we can take R_n closed, but T_n is not necessarily both closed and smooth. $\square$

When T is closed positive, the existence of almost every slice $T|_{P(W)}$ follows from [1, 7, 27].

Formula (3) is the projective version of the inversion formula for the hyperplane Radon transform. Let us now recall the inversion formula in the affine case, which is stated in [8, 9].

Let $u \in \mathcal{C}^\infty(V - \{0\})$ be the function defined by

$$g([x]) = \|x\|^{2N} u(x)$$

when $x \in V - \{0\}$ and let $\mathcal{I}_{N-1}u \in \mathcal{C}^\infty(V^* - \{0\})$ be the function defined by

$$(\mathcal{R}_{N-1}g)([\xi]) = \|\xi\|^2(\mathcal{I}_{N-1}u)(\xi)$$

when $\xi \in V^* - \{0\}$. First with $[P(\text{Ker } \xi)]$ the current of integration on the projective hyperplane of $\mathbb{P}_N$ associated to ξ , we have

$$(\mathcal{R}_{N-1}g)([\xi]) = \int_{\mathbb{P}_N} g\omega^{N-1} \wedge [P(\text{Ker } \xi)]$$

and in $V = \mathbb{C}^{N+1}$, we write

$$\pi^*[P(\text{Ker } \xi)] = [\langle x, \xi \rangle = 0] = \frac{i}{2}\delta(\langle x, \xi \rangle)d\langle x, \xi \rangle \wedge d\overline{\langle x, \xi \rangle}.$$

We choose $(e_0, \dots, e_N)$ an orthonormal basis of V and take $x = e_0 + X_1e_1 + \dots + X_Ne_N$ with $X = (X_1, \dots, X_N) \in \mathbb{C}^N$. We use that

$$\begin{aligned} & g([1, X_1, \dots, X_N]) \left(\frac{1}{2} d\bar{c} \log(1 + \|X\|^2) \right)^{N-1} \wedge \frac{i}{2} \delta(\langle X, v \rangle + \xi_0) d\langle X, v \rangle \wedge d\overline{\langle X, v \rangle} \\ &= \frac{\pi}{N} (\|v\|^2 + |\langle X, v \rangle|^2) \frac{g([1, X_1, \dots, X_N])}{(1 + \|X\|^2)^N} \delta(\langle X, v \rangle + \xi_0) \left(\frac{i}{2\pi} \partial\bar{\partial}(\|X\|^2) \right)^N \\ &= \frac{\pi}{N} (\|v\|^2 + |\xi_0|^2) \frac{g([1, X_1, \dots, X_N])}{(1 + \|X\|^2)^N} \delta(\langle X, v \rangle + \xi_0) \left(\frac{i}{2\pi} \partial\bar{\partial}(\|X\|^2) \right)^N \end{aligned}$$

where $v = (\xi_1, \dots, \xi_N)$ and $\langle X, v \rangle = \xi_1 X_1 + \dots + \xi_N X_N$. We obtain

$$(\mathcal{R}_{N-1}g)([\xi]) = \frac{\pi}{N} \|\xi\|^2 \int_{X \in \mathbb{C}^N} F(X) \delta(\langle X, v \rangle + \xi_0) \left(\frac{i}{2\pi} \partial\bar{\partial}(\|X\|^2) \right)^N$$

with $F(X) = F(X_1, \dots, X_N) = u(1, X_1, \dots, X_N)$ for $X \in \mathbb{C}^N$. The affine hyperplane Radon transform of F is then defined as

$$\hat{F}(v, s) = \int_{X \in \mathbb{C}^N} F(X) \delta(\langle X, v \rangle - s) \left(\frac{i}{2} \partial\bar{\partial}(\|X\|^2) \right)^N$$

for $v \in \mathbb{C}^N \setminus \{0\}$ and $s \in \mathbb{C}$, in such a way that

$$(\mathcal{I}_{N-1}u)(\xi_0, \xi_1, \dots, \xi_N) = \frac{1}{N\pi^{N-1}} \hat{F}(v, -\xi_0).$$

The classical inversion theorem states that there is a constant $c_N \neq 0$ such that

$$F(X) = c_N \int_{v \in K} \frac{\partial^{2(N-1)} \hat{F}}{\partial s^{N-1} \partial \bar{s}^{N-1}}(v, \langle X, v \rangle) \Phi(v) \quad (4)$$

where the differential form $\Phi(v) = \Phi(\xi_1, \dots, \xi_N)$ is given by

$$\begin{aligned} \Phi(v) = \frac{i}{4\pi} \sum_{1 \leq k \leq N} (-1)^{k-1} & (\xi_k d\xi_1 \wedge \dots \wedge \widehat{d\xi_k} \wedge \dots \wedge d\xi_N \wedge d\bar{\xi}_1 \wedge \dots \wedge d\bar{\xi}_N \\ & + (-1)^{N-k} \bar{\xi}_k d\xi_1 \wedge \dots \wedge d\xi_N \wedge d\bar{\xi}_1 \wedge \dots \wedge \widehat{d\bar{\xi}_k} \wedge \dots \wedge d\bar{\xi}_N) \end{aligned}$$

and where $K = \mathbb{S}^{2N-1} = \{v \in \mathbb{C}^N, \|v\| = 1\}$.

2.2 Determination of $\text{im } \mathcal{C} \cap \text{ker } d$

We obtain here, by slicing and by using the affine inversion formula, the Gelfand-Gindikin-Graev equations characterizing the images by the Chow transformation of closed (q, q) -currents on $\mathbb{P}_N$.

Let $\Theta = \mathcal{C}(T)$ be a real smooth differential $(1, 1)$ -form on $G(q, V)$, so which is in $\text{im } \mathcal{C}$. Then T is also real.

A first condition is that for all $W \in G(q+1, V)$, the restriction $\Theta|_{P(W^*)}$ of Θ to $P(W^*) = G(q, W)$ is closed i.e.

$$(d\Theta)|_{P(W^*)} = 0. \quad (5)$$

This follows from (2), since $T|_{P(W)}$ is of bidegree (q, q) on $P(W)$, thus closed.

On the other hand, by setting $T|_{P(W)} = g\omega_{|P(W)}^q$ with $g \in \mathcal{C}^\infty(P(W))$, we have

$$\Theta|_{P(W^*)} \wedge (\omega_W^*)^{q-1} = \mathcal{C}_W(T|_{P(W)}) \wedge (\omega_W^*)^{q-1} = \mathcal{R}_{q-1}(g)(\omega_W^*)^q$$

where ω_W^* is the Fubini-Study form on $P(W^*)$ and with the projective $(q-1)$ -dimensional Radon transform given by

$$\mathcal{R}_{q-1}(g)(s) = \int_{P(s)} g \omega^{q-1}$$

for $s \in P(W^*) = G(q, W)$. The inverse transform $\mathcal{C}_W^{-1}(\Theta|_{P(W^*)}) = g\omega_{|P(W)}^q$ can be calculated by applying formula (4) since $s \in P(W^*)$ is a hyperplane in W .

We fix $[x] \in P(V)$ and we take $W = \text{vect}(x, v_1, \dots, v_q)$ for $v_1, \dots, v_q \in V$. Then a second condition is that

$$\begin{aligned} & \mathcal{C}_W^{-1}(\Theta|_{P(W^*)})([x]; d\pi_x(v_1), \dots, d\pi_x(v_q), \overline{d\pi_x(v_1)}, \dots, \overline{d\pi_x(v_q)}) \\ &= T_{[x]}(d\pi_x(v_1), \dots, d\pi_x(v_q), \overline{d\pi_x(v_1)}, \dots, \overline{d\pi_x(v_q)}) \\ &= (\pi^*T)_x(v_1, \dots, v_q, \bar{v}_1, \dots, \bar{v}_q) \end{aligned}$$

is a Hermitian quadratic form of $v_1 \wedge \dots \wedge v_q \in \wedge^q \mathbb{C}^{N+1} = \wedge^q V$. We use the Plücker coordinates v_I defined by

$$v_1 \wedge \dots \wedge v_q = \sum_{|I|=q} v_I e_I$$

where $e_I = e_{i_1} \wedge \dots \wedge e_{i_q}$ for $I = (i_1, \dots, i_q)$ with $0 \leq i_1 < \dots < i_q \leq N$. This second condition is therefore that

$$\frac{\partial^2}{\partial v_I \partial \bar{v}_J} \{ \mathcal{C}_W^{-1}(\Theta|_{P(W^*)})([x]; d\pi_x(v_1), \dots, d\pi_x(v_q), \overline{d\pi_x(v_1)}, \dots, \overline{d\pi_x(v_q)}) \}$$

is independent of $v_1 \wedge \dots \wedge v_q$. Since $v_n \wedge v_{n'} = \sum_{m < m'} (v_m^n v_{m'}^{n'} - v_{m'}^n v_m^{n'}) e_m \wedge e_{m'}$, we use coordinates w of the type

$$w = w_{m, m'}^{n, n'} = v_m^n v_{m'}^{n'} - v_{m'}^n v_m^{n'}$$

for $1 \leq n, n' \leq q$ and $0 \leq m, m' \leq N$, where the v_m^n are the coordinates of v_n in the basis $e_0, \dots, e_N$. The above condition becomes

$$\begin{aligned} 0 &= \frac{\partial}{\partial w} \frac{\partial^2}{\partial v_I \partial \bar{v}_J} \{ \mathcal{C}_W^{-1}(\Theta|_{P(W^*)})([x]; d\pi_x(v_1), \dots, d\pi_x(v_q), \overline{d\pi_x(v_1)}, \dots, \overline{d\pi_x(v_q)}) \} \\ &= \frac{\partial}{\partial \bar{w}} \frac{\partial^2}{\partial v_I \partial \bar{v}_J} \{ \mathcal{C}_W^{-1}(\Theta|_{P(W^*)})([x]; d\pi_x(v_1), \dots, d\pi_x(v_q), \overline{d\pi_x(v_1)}, \dots, \overline{d\pi_x(v_q)}) \} \end{aligned}$$

the second equality being then automatic, since Θ is real. We use the parametrization λ of W given by

$$\lambda(t_0, t_1, \dots, t_q) = t_0 x + \sum_{1 \leq j \leq q} t_j v_j \in W$$

for $(t_0, t_1, \dots, t_q) \in \mathbb{C}^{q+1}$. This gives rise to an isomorphism induced by λ , defined by

$$\begin{aligned} \mathbb{P}_q^* &= G(q, \mathbb{C}^{q+1}) \xrightarrow{\lambda} G(q, W) = P(W^*) \\ \text{vect}(t^0, \dots, t^{q-1}) &\rightarrow \text{vect}(\sum_{0 \leq j \leq q} t_j^k v_j)_{0 \leq k \leq q-1} \end{aligned}$$

where $v_0 = x$. With $\mathcal{C}_{\mathbb{C}^{q+1}}$ the Chow transform on $\mathbb{P}_q = P(\mathbb{C}^{q+1})$, we have

$$\lambda^*(\mathcal{C}_W^{-1}(\Theta|_{P(W^*)})) = \mathcal{C}_{\mathbb{C}^{q+1}}^{-1}(\lambda^*(\Theta|_{P(W^*)}))$$

and with $b_0, \dots, b_q$ the canonical basis of $\mathbb{C}^{q+1}$, we have

$$\begin{aligned} & \mathcal{C}_W^{-1}(\Theta_{|P(W^*)})([X]; d\pi_X(v_1), \dots, d\pi_X(v_q), \overline{d\pi_X(v_1)}, \dots, \overline{d\pi_X(v_q)}) \\ &= \mathcal{C}_{\mathbb{C}^{q+1}}^{-1}(\lambda^*(\Theta_{|P(W^*)}))([b_0]; b_0^{-1} \otimes \tilde{b}_1, \dots, b_0^{-1} \otimes \tilde{b}_q, \overline{b_0^{-1} \otimes \tilde{b}_1}, \dots, \overline{b_0^{-1} \otimes \tilde{b}_q}) \end{aligned}$$

where $\tilde{b}_k = (b_k \bmod \mathbb{C}b_0) \in \mathbb{C}^{q+1}/\mathbb{C}b_0$. Therefore the second condition is

$$0 = \mathcal{C}_{\mathbb{C}^{q+1}}^{-1} \left(\frac{\partial}{\partial w} \frac{\partial^2}{\partial v_I \partial \bar{v}_J} \lambda^*(\Theta_{|P(W^*)})([b_0]; b_0^{-1} \otimes \tilde{b}_1, \dots, b_0^{-1} \otimes \tilde{b}_q, \overline{b_0^{-1} \otimes \tilde{b}_1}, \dots, \overline{b_0^{-1} \otimes \tilde{b}_q}) \right). \quad (6)$$

In effect, $\Theta_{|P(W^*)}$ is closed and we can write the closed smooth differential $(1, 1)$ -form

$$\frac{\partial}{\partial w} \frac{\partial^2}{\partial v_I \partial \bar{v}_J} \lambda^*(\Theta_{|P(W^*)}) = \mathcal{C}_{\mathbb{C}^{q+1}}(f\omega_{\mathbb{C}^{q+1}}^q)$$

with $f \in \mathcal{C}^\infty(\mathbb{P}_q)$ and with $\omega_{\mathbb{C}^{q+1}}$ the Fubini-Study form on $\mathbb{P}_q$. Then equation (6) reduces to

$$\begin{aligned} 0 &= (f\omega_{\mathbb{C}^{q+1}}^q)([b_0]; b_0^{-1} \otimes \tilde{b}_1, \dots, b_0^{-1} \otimes \tilde{b}_q, \overline{b_0^{-1} \otimes \tilde{b}_1}, \dots, \overline{b_0^{-1} \otimes \tilde{b}_q}) \\ &\Leftrightarrow 0 = f([b_0]). \end{aligned}$$

Assume now Θ closed in $G(q, V)$ i.e. $d\Theta = 0$, instead of (5). Denote by $\tau : V^q \rightarrow G(q, V)$ the map which associates to $z = (z^0, \dots, z^{q-1})$ in V^q the vector subspace $\tau(z) = \text{vect}(z^0, \dots, z^{q-1}) \in G(q, V)$. The inverse image by τ of Θ is then written as

$$\tau^*\Theta = dd^c U$$

with $U(z) = U(z^0, \dots, z^{q-1})$ some function on V^q . Since

$$\lambda^*(\Theta_{|P(W^*)}) = dd^c \lambda^*(U_{|W^q}) = dd^c \left(U \left(\sum_{0 \leq j \leq q} t_j^0 v_j, \dots, \sum_{0 \leq j \leq q} t_j^{q-1} v_j \right) \right),$$

the function f , which depends on $v_1, \dots, v_q$, satisfies the relation

$$dd^c \left\{ \frac{\partial}{\partial w} \frac{\partial^2}{\partial v_I \partial \bar{v}_J} \left(U \left(\sum_{0 \leq j \leq q} t_j^0 v_j, \dots, \sum_{0 \leq j \leq q} t_j^{q-1} v_j \right) \right) \right\} \wedge (\omega_{\mathbb{C}^{q+1}}^*)^{q-1} = \mathcal{R}_{q-1, \mathbb{C}^{q+1}}(f)(\omega_{\mathbb{C}^{q+1}}^*)^q \quad (7)$$

where $\omega_{\mathbb{C}^{q+1}}^*$ is the Fubini-Study form on $\mathbb{P}_q^* = G(q, \mathbb{C}^{q+1})$ and with $\mathcal{R}_{q-1, \mathbb{C}^{q+1}}$ the hyperplane Radon transform on $\mathbb{P}_q$.

But by using the coordinates v_m^n for $1 \leq n \leq q$ and $0 \leq m \leq N$ and by expressing the differential operators

$$\frac{\partial}{\partial w} = \frac{\partial}{\partial w_{m, m'}^{n, n'}} = \frac{\partial^2}{\partial v_m^n \partial v_{m'}^{n'}} - \frac{\partial^2}{\partial v_{m'}^n \partial v_m^{n'}}$$

and

$$\frac{\partial}{\partial v_I} = \det \left(\frac{\partial}{\partial v_j^k} \right)_{\substack{1 \leq k \leq q \\ j \in I}},$$

first we have the relation

$$\begin{aligned} & \frac{\partial}{\partial w} \frac{\partial^2}{\partial v_I \partial \bar{v}_J} \left(U(t_0^0 x + \sum_{1 \leq j \leq q} t_j^0 v_j, \dots, t_0^{q-1} x + \sum_{1 \leq j \leq q} t_j^{q-1} v_j) \right) = |\det(t_j^k)_{\substack{0 \leq k \leq q-1 \\ 1 \leq j \leq q}}|^2 \\ & \cdot \sum_{0 \leq l, l' \leq q-1} t_n^l t_{n'}^{l'} \left(\left(\frac{\partial^2}{\partial z_m^l \partial z_{m'}^{l'}} - \frac{\partial^2}{\partial z_{m'}^l \partial z_m^{l'}} \right) \partial_I \bar{\partial}_J U \right) (t_0^0 x + \sum_{1 \leq j \leq q} t_j^0 v_j, \dots, t_0^{q-1} x + \sum_{1 \leq j \leq q} t_j^{q-1} v_j) \end{aligned}$$

where

$$\partial_I = \det \left(\frac{\partial}{\partial z_j^k} \right)_{\substack{0 \leq k \leq q-1 \\ j \in I}}.$$

Then by calculating $f([b_0])$ by means of the formulas (4) and (7), we conclude that the condition (6) is equivalent to the Gelfand-Gindikin-Graev equations

$$0 = \left(\frac{\partial^2}{\partial z_m^l \partial \bar{z}_{m'}^{l'}} - \frac{\partial^2}{\partial z_{m'}^{l'} \partial \bar{z}_m^l} \right) \partial_I \bar{\partial}_J U \quad (8)$$

for all $0 \leq l, l' \leq q-1$, all $0 \leq m, m' \leq N$ and all $|I| = |J| = q$. Note that since Θ is assumed to be real, we also have the conjugate equations

$$0 = \left(\frac{\partial^2}{\partial \bar{z}_m^l \partial z_{m'}^{l'}} - \frac{\partial^2}{\partial \bar{z}_{m'}^{l'} \partial z_m^l} \right) \partial_I \bar{\partial}_J U \quad (9)$$

for all $0 \leq l, l' \leq q-1$, all $0 \leq m, m' \leq N$ and all $|I| = |J| = q$.

2.3 Closed current of bidegree $(p+1, p+1)$ on $P(V^*)$ dual of a closed current of bidegree (q, q) on $P(V)$

First for Z an algebraic cycle in $P(V) = \mathbb{P}_N$ of dimension $p = N - q$, we prove the existence of a closed dual $(p+1, p+1)$ -current $[Z]^*$ on $P(V^*)$ by using the dual variety $Y = Z^* \subset P(V^*)$ of Z . Let $\Sigma = \mathcal{C}(Z)$ be the divisor in $G(q, V)$, which is the Chow divisor of Z . When Z is irreducible,

$$\Sigma = \{s \in G(q, V), P(s) \cap Z \neq \emptyset\}. \quad (10)$$

We have an isomorphism $G(q, V) \simeq G(p+1, V^*)$ which associates to a vector subspace $s \subset V$ its orthogonal $s^\perp \subset V^*$. We now express the above condition with this dual point of view.

Since $Y = Z^* \subset P(V^*)$, we have $Z = Y^* \subset P(V)$ (see [10]). Let $I_Y \subset P(V^*) \times P(V)$ be the incidence variety associated to Y , defined as the Zariski closure of

$$\{([\xi], [x]) \in Y_{\text{reg}} \times P(V), (d\mu_\xi)^{-1}(T_{[\xi]}Y) \subset x^\perp\},$$

by denoting by $\mu: V^* \setminus \{0\} \rightarrow P(V^*)$ the canonical map. The dimension of I_Y is $N-1$ and its projection in $P(V)$ is equal to Z . Thus for $[x] \in Z$ generic, the Zariski closure of the fiber $\{[\xi] \in Y_{\text{reg}}, (d\mu_\xi)^{-1}(T_{[\xi]}Y) \subset x^\perp\} \subset P(x^\perp)$ has dimension $N-1 - \dim Z = q-1$ and so intersects $P(s^\perp)$ if $x \in s$. If $[\xi]$ is a point of intersection, then we have $\dim((d\mu_\xi)^{-1}(T_{[\xi]}Y) + s^\perp) \leq N$.

We write now $s = \{0 = \xi_0 = \dots = \xi_p\}$ with $\xi_0, \dots, \xi_p \in V^*$ and $H = P(x^\perp)$. Thus

$$\begin{aligned} P(s) \cap Z &\neq \emptyset \\ \Leftrightarrow \text{there is } H \supset P(\text{vect}(\xi_0, \dots, \xi_p)) \text{ s. t. } H \supset \mathbb{T}_{[\xi]}Y \text{ for some } [\xi] \in Y \cap P(\text{vect}(\xi_0, \dots, \xi_p)) \\ \Leftrightarrow \text{there is } [\xi] \in Y \cap P(\text{vect}(\xi_0, \dots, \xi_p)) \text{ s. t. } \dim(\mathbb{T}_{[\xi]}Y \cap P(\text{vect}(\xi_0, \dots, \xi_p))) &\geq p+1 - \text{codim } Y \end{aligned}$$

by setting $\mathbb{T}_{[\xi]}Y = P((d\mu_\xi)^{-1}(T_{[\xi]}Y)) \subset P(V^*)$.

Take $W' \in G(p+2, V^*)$ such that $P(W')$ cuts Y transversally in an algebraic variety of dimension $p+1 - \text{codim } Y$. Then $\Sigma \cap G(p+1, W')$ is the dual variety in $P(W'^*) = G(p+1, W')$ of $Y \cap P(W')$, in other words

$$\Sigma|_{P(W'^*)} = \Sigma|_{G(p+1, W')} = (Y \cap P(W'))^*.$$

In effect, for $\text{vect}(\xi_0, \dots, \xi_p) \subset W'$, if there is $[\xi] \in Y \cap P(\text{vect}(\xi_0, \dots, \xi_p))$ satisfying

$$\mathbb{T}_{[\xi]}Y \cap P(W') \subset P(\text{vect}(\xi_0, \dots, \xi_p)),$$

then

$$\mathbb{T}_{[\xi]}Y \cap P(\text{vect}(\xi_0, \dots, \xi_p)) = \mathbb{T}_{[\xi]}Y \cap P(W')$$

has dimension $\geq p+1 - \text{codim } Y$.

Now denote by C' the Chow transformation of currents of bidegree $(p+1, p+1)$ on $P(V^*)$ and by $C'_{W'}$ the Chow transformation of currents of bidegree $(p+1, p+1)$ on $P(W')$.

As a closed current of bidegree $(1, 1)$ on $P(W'^*)$, the current $[\Sigma]_{|P(W'^*)}$ can be written

$$[\Sigma]_{|P(W'^*)} = C'_{W'}(S_{W'})$$

for some current $S_{W'}$ of bidegree $(p+1, p+1)$ on $P(W')$ (see [25]). These currents satisfy the following property.

Proposition 3. (i) *There is a closed $(p+1, p+1)$ -current $[Z]^*$ on $P(V^*)$ satisfying $[Z]_{|P(W')}^* = S_{W'}$ for every $W' \in G(p+2, V^*)$. Thus $\mathcal{C}([Z]) = C'([Z]^*)$.*

(ii) *For every closed (q, q) -current T on $P(V)$, there is a closed $(p+1, p+1)$ -current T^* on $P(V^*)$ satisfying $\mathcal{C}(T) = C'(T^*)$ on $G(q, V) = G(p+1, V^*)$.*

Proof. Every closed (q, q) -current T on $P(V)$ can be written $T = \lim_{\nu} Z_{\nu}$ with Z_{ν} algebraic cycle with complex coefficients on $P(V)$. We set $T^* = \lim_{\nu} [Z_{\nu}]^*$ on $P(V^*)$. $\square$

Let us express the condition (5) for C' , by noting that

$$\{s \in G(q, V), s^{\perp} \subset W'\} = P(W'^*) = \{s \in G(q, V), s \supset W'^{\perp}\} = G(1, V/W'^{\perp}).$$

With $p_{W'} : P(V) \rightarrow P(V/W'^{\perp})$ the canonical projection, if $\Theta = \mathcal{C}(T)$ where T is some (q, q) -current on $P(V)$, we have

$$\Theta_{|P(W'^*)} = \mathcal{C}(T)_{|G(1, V/W'^{\perp})} = p_{W'*} T \quad (11)$$

since $p_{W'*} T$ is of bidegree $(1, 1)$ on $P(V/W'^{\perp})$ and the Chow transformation is the identity, when the bidegree is $(1, 1)$. Therefore, in that case, the condition (5) reduces to $p_{W'*} T$ closed, for every $W' \in G(p+2, V^*)$.

Lastly we can extend the method by slicing of subsection 2.2 to the case when Θ is not necessarily closed. In effect, for Q a function on $\wedge^q V = \wedge^q \mathbb{C}^{N+1}$, the condition

$$Q(\lambda v_1 \wedge \dots \wedge v_q) = |\lambda|^2 Q(v_1 \wedge \dots \wedge v_q)$$

for every $\lambda \in \mathbb{C}$ and every $v_1, \dots, v_q \in V$ amounts to

$$\tilde{Q}(\lambda v_n \wedge v_{n'}) = |\lambda|^2 \tilde{Q}(v_n \wedge v_{n'})$$

for all $1 \leq n, n' \leq q$ with $\tilde{Q}(v_n \wedge v_{n'}) = Q(v_1 \wedge \dots \wedge v_q)$ where the v_m for $m \notin \{n, n'\}$ are fixed. Therefore the elements in $\text{im } \mathcal{C}$ must satisfy the following property.

Proposition 4. *If a real smooth differential $(1, 1)$ -form Θ on $G(q, V)$ is the Chow transform of a real smooth differential (q, q) -form T on $P(V)$, then*

(i) *the condition (5) is satisfied i.e. $d\Theta_{|G(q, W)} = d\Theta_{|P(W^*)} = 0$ for all $W \in G(q+1, V)$,*

(ii) *the linear partial differential equation of order 6 in $(v_1, \dots, v_q)$*

$$0 = \frac{\partial}{\partial w_{m, m'}^{n, n'}} \frac{\partial^2}{\partial w_{i, i'}^{n, n'} \partial \bar{w}_{j, j'}^{n, n'}} \{C_W^{-1}(\Theta_{|P(W^*)})([x]; d\pi_x(v_1), \dots, d\pi_x(v_q), \overline{d\pi_x(v_1)}, \dots, \overline{d\pi_x(v_q)})\}$$

is satisfied with $W = \text{vect}(x, v_1, \dots, v_q)$ for all $x, v_1, \dots, v_q \in V$ and all $1 \leq n, n' \leq q$ and all $0 \leq m, m', i, i', j, j' \leq N$.

3 The dual of the Chow transformation

Let Ω be the fundamental $(1, 1)$ -form of the Hermitian metric induced in $G(q, V)$, which satisfies $\Omega = dd^c \log \|z^1 \wedge \dots \wedge z^q\|$ for $s = \text{vect}(z^1, \dots, z^q)$.

For $[x] \in P(V)$, the fiber $\varphi^{-1}([x])$ is identified with $G(q-1, V/\mathbb{C}x)$ and thus its dimension is $d = (q-1)(N+1-q)$.

For Θ a $(1, 1)$ -current on $G(q, V)$, we set

$$\mathcal{C}^*(\Theta) = \varphi_* \psi^* (\Omega^{d+q-1-j} \wedge \Theta)$$

where $j = \max(0, 2q - N - 1)$. In effect, we take $j \geq 0$ and we should have $d \leq d + q - j \leq \dim G(q, V)$ i.e. $2q - N - 1 \leq j \leq q$. Then $j \leq d + q - 1$.

$\mathcal{C}^*(\Theta)$ is a current of bidegree $(q - j, q - j)$ on $P(V)$ and $\mathcal{C}^*$ will be a dual of the Chow transformation $\mathcal{C}$.

Note that if $j = 2q - N - 1$, then $\binom{N+1}{q-j} = \binom{N+1}{q}$ in such a way that $\dim \wedge^{q-j} V = \dim \wedge^q V$.

3.1 Expression of $\mathcal{C}^*\mathcal{C}$ in coordinates

When u is a smooth differential (q, q) -form on $P(V)$, we now calculate

$$\mathcal{C}^*\mathcal{C}(u) = \varphi_* (\psi^* \Omega^{d+q-1-j} \wedge \psi^* \psi_* \varphi^* u). \quad (12)$$

We use that the direct image by a submersion of a smooth differential form can be calculated by integration along the fibers. With $v_1, \dots, v_{q-j} \in V$ and $\pi : V - \{0\} \rightarrow P(V)$ the canonical map, we obtain

$$\pi^* (\mathcal{C}^*\mathcal{C}(u))_x (v_1, \dots, v_{q-j}, \bar{v}_1, \dots, \bar{v}_{q-j}) = \int_{s \in \varphi^{-1}([x])} (\Omega^{d+q-1-j} \wedge \mathcal{C}(u))_s (\sigma_1, \dots, \sigma_{q-j}, \bar{\sigma}_1, \dots, \bar{\sigma}_{q-j}, \dots)$$

where $\sigma_k \in T_s G(q, V)$ is such that $(d\pi_x(v_k), \sigma_k) \in T_{([x], s)} \Gamma$. We now express the function $A(s)$ for $s \ni x$ such that

$$(\Omega_s^{d+q-1-j} \wedge \mathcal{C}(u)_s)(\sigma_1, \dots, \sigma_{q-j}, \bar{\sigma}_1, \dots, \bar{\sigma}_{q-j}, \dots)_{|T_s \varphi^{-1}([x])} = A(s) \Omega_s^d|_{T_s \varphi^{-1}([x])}.$$

Assume that $(e_0, \dots, e_N)$ is an orthonormal basis of V and that $[x] = [e_0]$ and $s = \text{vect}(e_l)_{l < q}$. Coordinates of $G(q, V)$ are the z_l^k for $k < q \leq l$, with the associated point of $G(q, V)$ equal to

$$\text{vect}(e_k + \sum_{q \leq l} z_l^k e_l)_{k < q}.$$

Then $\varphi^{-1}([e_0])$ is defined by $z_l^0 = 0$ for $q \leq l$. At this s , we can take

$$\begin{aligned} \sigma_k &= \frac{d}{dt} \text{vect}(e_0 + tv_k, e_1, \dots, e_{q-1})|_{t=0} \\ &= \frac{d}{dt} \text{vect}(e_0 + tv_{k,0}e_0 + t \sum_{q \leq l} v_{k,l}e_l, e_1, \dots, e_{q-1})|_{t=0} \\ &= \sum_{q \leq l} \frac{d}{dt} \left(\frac{tv_{k,l}}{1 + tv_{k,0}} \right) |_{t=0} \frac{\partial}{\partial z_l^0} = \sum_{q \leq l} v_{k,l} \frac{\partial}{\partial z_l^0}. \end{aligned}$$

At this s , we have $\Omega_s = \frac{i}{2\pi} \sum_{k < q \leq l} dz_l^k \wedge d\bar{z}_l^k$ and

$$\Omega_s|_{T_s \varphi^{-1}([e_0])} = \frac{i}{2\pi} \sum_{1 \leq k < q \leq l} dz_l^k \wedge d\bar{z}_l^k.$$

So there exists a constant $K = (-1)^{(q-j)(q-j-1)/2} (\frac{i}{2\pi})^d d!$ satisfying

$$KA(s) = (\Omega_s^{d+q-1-j} \wedge \mathcal{C}(u)_s)(\sigma_1, \bar{\sigma}_1, \dots, \sigma_{q-j}, \bar{\sigma}_{q-j}, \dots, \frac{\partial}{\partial z_l^k}, \frac{\partial}{\partial \bar{z}_l^k}, \dots) \quad (13)$$

where $1 \leq k < q$ and $q \leq l$. Now recall (see [25]) the expression of $\mathcal{C}(u)$ with the homogeneous coordinates.

The inverse image of u by the canonical map $\pi : V \rightarrow P(V)$ is written as

$$\pi^* u = \sum_{|I|=|J|=q} u_{I,J} dx_I \wedge d\bar{x}_J$$

with functions $u_{I,J}$ satisfying in particular the property of homogeneity $u_{I,J}(\lambda x) = |\lambda|^{-2q} u_{I,J}(x)$ for $\lambda \in \mathbb{C}^*$.

Denote by $\tau : V^q \rightarrow G(q, V)$ the map which associates to $z = (z^0, \dots, z^{q-1})$ in the product the vector subspace $\tau(z) = \text{vect}(z^0, \dots, z^{q-1})$. The inverse image by τ of the Chow transform $\mathcal{C}(u)$ is then written as

$$\tau^* \mathcal{C}(u) = \sum_{\substack{0 \leq k, m \leq q-1 \\ 0 \leq n, l \leq N}} C_{nl}^{km}(u) dz_n^k \wedge d\bar{z}_l^m.$$

The coefficients $C_{nl}^{km}(u)$ can be expressed by means of the $u_{I,J}$ in the following way :

$$C_{nl}^{km}(u)(z) = (-1)^{k+m} \sum_{|I|=|J|=q-1} \tilde{u}_{nI,lJ}(z) z_I^{\hat{k}} \bar{z}_J^{\hat{m}}$$

with $z_I^{\hat{k}} = \det(z_{i_\alpha}^{k'})_{\substack{1 \leq \alpha \leq q-1 \\ 0 \leq k' \leq q-1, k' \neq k}}$ and with the transform

$$\tilde{u}_{nI,lJ}(z) = \int_{t \in \mathbb{S}^{2q-1}} u_{nI,lJ}(t \cdot z) \Phi(t) \quad (14)$$

where $t \cdot z = t_1 z^0 + \dots + t_q z^{q-1}$ and

$$\begin{aligned} \Phi(t) = \frac{i}{4\pi} \sum_{1 \leq k \leq q} (-1)^{k-1} \left(t_k dt_1 \wedge \dots \wedge \widehat{dt_k} \wedge \dots \wedge dt_q \wedge d\bar{t}_1 \wedge \dots \wedge d\bar{t}_q \right. \\ \left. + (-1)^q \bar{t}_k dt_1 \wedge \dots \wedge dt_q \wedge d\bar{t}_1 \wedge \dots \wedge \widehat{d\bar{t}_k} \wedge \dots \wedge d\bar{t}_q \right). \end{aligned}$$

Set $f_{nI,lJ}([x]) = \|x\|^{2q} u_{nI,lJ}(x)$ and let $\mathcal{R}_{q-1}(f_{nI,lJ})$ be the Radon transform of $f_{nI,lJ}$ obtained by integration of that function in the projective subspaces of $P(V)$ of dimension $q-1$. Also

$$\mathcal{R}_{q-1}(f_{nI,lJ})_{\tau(z)} = \alpha_q \tilde{u}_{nI,lJ}(z) \Delta(z)$$

with $\alpha_q = \frac{(q-1)! i^{q^2-1}}{(2\pi)^{q-1}}$ and $\Delta(z)$ the Gram determinant of z .

But since we use the coordinates z_l^k for $k < q \leq l$, we can write

$$\mathcal{C}(u) = \sum_{\substack{0 \leq k, m \leq q-1 \\ q \leq n, l \leq N}} C_{nl}^{km}(u) dz_n^k \wedge d\bar{z}_l^m.$$

Moreover at $s = \text{vect}(e_0, e_1, \dots, e_{q-1})$ we have $z_I^{\hat{k}} = 0$ except for $I = \{0, \dots, k-1, k+1, \dots, q-1\}$ and in the same way $\bar{z}_J^{\hat{m}} = 0$ except for $J = \{0, \dots, m-1, m+1, \dots, q-1\}$. Thus at that z we have

$$C_{nl}^{km}(u)(z) = (-1)^{k+m} \tilde{u}_{n\hat{k},l\hat{m}}(z) = \tilde{u}_{\{0,\dots,k-1,n,k+1,\dots,q-1\},\{0,\dots,m-1,l,m+1,\dots,q-1\}}(z)$$

and therefore

$$\begin{aligned} & \Omega_s^{d+q-1-j} \wedge \mathcal{C}(u)_s \\ &= \left(\frac{i}{2\pi} \sum_{\substack{0 \leq k \leq q-1 \\ q \leq l \leq N}} dz_l^k \wedge d\bar{z}_l^k \right)^{d+q-1-j} \wedge \sum_{\substack{0 \leq k, m \leq q-1 \\ q \leq n, l \leq N}} (-1)^{k+m} \tilde{u}_{n\hat{k},l\hat{m}}(z) dz_n^k \wedge d\bar{z}_l^m \\ &= \left(\frac{i}{2\pi} \right)^{d+q-1-j} \sum dz_{l_1}^{k_1} \wedge d\bar{z}_{l_1}^{k_1} \wedge \dots \wedge dz_{l_{d+q-1-j}}^{k_{d+q-1-j}} \wedge d\bar{z}_{l_{d+q-1-j}}^{k_{d+q-1-j}} \wedge (-1)^{k+m} \tilde{u}_{n\hat{k},l\hat{m}}(z) dz_n^k \wedge d\bar{z}_l^m \end{aligned}$$

where $0 \leq k_1, \dots, k_{d+q-1-j}, k, m \leq q-1$ and $q \leq l_1, \dots, l_{d+q-1-j}, n, l \leq N$.

On the other hand, with $1 \leq k \leq q-1$ and $q \leq l \leq N$, we have

$$\begin{aligned} & \sigma_1 \wedge \bar{\sigma}_1 \wedge \dots \wedge \sigma_{q-j} \wedge \bar{\sigma}_{q-j} \wedge \dots \wedge \frac{\partial}{\partial z_l^k} \wedge \frac{\partial}{\partial \bar{z}_l^k} \wedge \dots \\ &= \left(\sum_{q \leq i_1} v_{1,i_1} \frac{\partial}{\partial z_{i_1}^0} \right) \wedge \left(\sum_{q \leq n_1} \bar{v}_{1,n_1} \frac{\partial}{\partial \bar{z}_{n_1}^0} \right) \wedge \dots \wedge \left(\sum_{q \leq i_1} v_{q-j,i_1} \frac{\partial}{\partial z_{i_1}^0} \right) \wedge \left(\sum_{q \leq n_1} \bar{v}_{q-j,n_1} \frac{\partial}{\partial \bar{z}_{n_1}^0} \right) \wedge \dots \end{aligned}$$

$$\wedge \frac{\partial}{\partial z_l^k} \wedge \frac{\partial}{\partial \bar{z}_l^k} \wedge \dots$$

$$= \sum v_{1,i_1} \bar{v}_{1,n_1} \dots v_{q-j,i_{q-j}} \bar{v}_{q-j,n_{q-j}} \frac{\partial}{\partial z_{i_1}^0} \wedge \frac{\partial}{\partial \bar{z}_{n_1}^0} \wedge \dots \wedge \frac{\partial}{\partial z_{i_{q-j}}^0} \wedge \frac{\partial}{\partial \bar{z}_{n_{q-j}}^0} \wedge \frac{\partial}{\partial z_{l'_1}^{k'_1}} \wedge \frac{\partial}{\partial \bar{z}_{l'_1}^{k'_1}} \wedge \dots$$

$$\wedge \frac{\partial}{\partial z_{l'_d}^{k'_d}} \wedge \frac{\partial}{\partial \bar{z}_{l'_d}^{k'_d}}$$

where $q \leq i_1, \dots, i_{q-j} \leq N$ and $q \leq n_1, \dots, n_{q-j} \leq N$ and where the vectors $\frac{\partial}{\partial z_{l'_1}^{k'_1}}, \dots, \frac{\partial}{\partial z_{l'_d}^{k'_d}}$ with $1 \leq k'_1, \dots, k'_d \leq q-1$ and $q \leq l'_1, \dots, l'_d \leq N$ constitute a basis of $T_s \varphi^{-1}([e_0])$.

As a consequence, when we calculate $A(s)$ by means of the formula (13), we must choose the indices such that the following equalities between sets are satisfied

$$\left\{ \frac{\partial}{\partial z_{i_1}^0}, \dots, \frac{\partial}{\partial z_{i_{q-j}}^0}, \frac{\partial}{\partial z_{l'_1}^{k'_1}}, \dots, \frac{\partial}{\partial z_{l'_d}^{k'_d}} \right\} = \left\{ \frac{\partial}{\partial z_{l_1}^{k_1}}, \dots, \frac{\partial}{\partial z_{l_{d+q-1-j}}^{k_{d+q-1-j}}}, \frac{\partial}{\partial z_n^k} \right\}$$

and

$$\left\{ \frac{\partial}{\partial \bar{z}_{n_1}^0}, \dots, \frac{\partial}{\partial \bar{z}_{n_{q-j}}^0}, \frac{\partial}{\partial \bar{z}_{l'_1}^{k'_1}}, \dots, \frac{\partial}{\partial \bar{z}_{l'_d}^{k'_d}} \right\} = \left\{ \frac{\partial}{\partial \bar{z}_{l_1}^{k_1}}, \dots, \frac{\partial}{\partial \bar{z}_{l_{d+q-1-j}}^{k_{d+q-1-j}}}, \frac{\partial}{\partial \bar{z}_l^m} \right\}.$$

The indices in the lists $z_{i_1}^0, \dots, z_{i_{q-j}}^0, z_{l'_1}^{k'_1}, \dots, z_{l'_d}^{k'_d}$ and $\bar{z}_{n_1}^0, \dots, \bar{z}_{n_{q-j}}^0, \bar{z}_{l'_1}^{k'_1}, \dots, \bar{z}_{l'_d}^{k'_d}$ should differ by at most one element. A first possibility is

$$z_{i_1}^0, \dots, z_{i_{\alpha-1}}^0, z_{i_{\alpha+1}}^0, \dots, z_{i_{q-j}}^0, z_{l'_1}^{k'_1}, \dots, z_{l'_d}^{k'_d} \text{ and } \bar{z}_{n_1}^0, \dots, \bar{z}_{n_{\alpha-1}}^0, \bar{z}_{n_{\alpha+1}}^0, \dots, \bar{z}_{n_{q-j}}^0, \bar{z}_{l'_1}^{k'_1}, \dots, \bar{z}_{l'_d}^{k'_d}.$$

We obtain a first term $K_1 A_1(s)$ with $K_1 = \left(\frac{i}{2\pi}\right)^{d+q-1-j} (d+q-1-j)!$ and with

$$A_1(s) = \sum_{\alpha, \beta} \sum_I \sum_{\sigma, \tau} \sum_{i_{\alpha}, n_{\beta}} \varepsilon(\sigma) \varepsilon(\tau) (-1)^{\alpha+\beta} v_{1, i_{\sigma(1)}} \dots v_{\alpha-1, i_{\sigma(\alpha-1)}} v_{\alpha, i_{\alpha}} v_{\alpha+1, i_{\sigma(\alpha+1)}} \dots v_{q-j, i_{\sigma(q-j)}} \\ \cdot \bar{v}_{1, i'_{\tau(1)}} \dots \bar{v}_{\beta-1, i'_{\tau(\beta-1)}} \bar{v}_{\beta, n_{\beta}} \bar{v}_{\beta+1, i'_{\tau(\beta+1)}} \dots \bar{v}_{q-j, i'_{\tau(q-j)}} (-1)^{0+0} \tilde{u}_{i_{\alpha}, \hat{0}, n_{\beta}, \hat{0}}(z)$$

where $1 \leq \alpha, \beta \leq q-j$ and $I = (i_1, \dots, i_{\alpha-1}, i_{\alpha+1}, \dots, i_{q-j}) = (i'_1, \dots, i'_{\beta-1}, i'_{\beta+1}, \dots, i'_{q-j})$ is a strictly increasing sequence of $q-j-1$ integers between q and N and where σ, τ are permutations of $\{1, \dots, q-j\}$ such that $\sigma(\alpha) = \alpha$ et $\tau(\beta) = \beta$.

The second possibility is for some $\delta \in \{1, \dots, d\}$

$$z_{i_1}^0, \dots, z_{i_{q-j}}^0, z_{l'_1}^{k'_1}, \dots, z_{l'_{\delta-1}}^{k'_{\delta-1}}, z_{l'_{\delta+1}}^{k'_{\delta+1}}, \dots, z_{l'_d}^{k'_d} \text{ and } \bar{z}_{n_1}^0, \dots, \bar{z}_{n_{q-j}}^0, \bar{z}_{l'_1}^{k'_1}, \dots, \bar{z}_{l'_{\delta-1}}^{k'_{\delta-1}}, \bar{z}_{l'_{\delta+1}}^{k'_{\delta+1}}, \dots, \bar{z}_{l'_d}^{k'_d}.$$

We obtain a second term $K_2 A_2(s)$ with $K_2 = \left(\frac{i}{2\pi}\right)^{d+q-1-j} (d+q-1-j)! = K_1$ and with

$$A_2(s) = \sum_I \sum_{\sigma, \tau} \varepsilon(\sigma) \varepsilon(\tau) v_{1, i_{\sigma(1)}} \dots v_{q-j, i_{\sigma(q-j)}} \bar{v}_{1, i_{\tau(1)}} \dots \bar{v}_{q-j, i_{\tau(q-j)}} \sum_{\substack{1 \leq k' \leq q-1 \\ q \leq l' \leq N}} (-1)^{k'+k'} \tilde{u}_{l' \hat{k}', l' \hat{k}'}(z)$$

where $I = (i_1, \dots, i_{q-j})$ with $q \leq i_1 < \dots < i_{q-j} \leq N$ and σ, τ are permutations of $\{1, \dots, q-j\}$.

Denote by $p_{s^\perp} : \mathbb{C}^{N+1} \rightarrow \mathbb{C}^{N+1}$ the orthogonal projection onto $s^\perp = \text{vect}(e_q, \dots, e_N)$, then

$$p_{s^\perp}^* \left(\frac{i}{2\pi} \partial \bar{\partial} (\|x\|^2) \right)^{q-j-1} = C_1 \sum_{|I|=q-j-1} dx_I \wedge d\bar{x}_I$$

where $I = (i_1, \dots, i_{q-j-1})$ with $q \leq i_1 < \dots < i_{q-j-1} \leq N$ and $C_1 = \frac{i^{(q-j-1)^2}}{(2\pi)^{q-j-1}} (q-j-1)!$ is a constant. We obtain by formula (14)

$$C_1 A_1(s) = \sum_{\alpha, \beta} p_{s^\perp}^* \left(\frac{i}{2\pi} \partial \bar{\partial} (\|x\|^2) \right)_{e_0}^{q-j-1} (v_1, \dots, v_{\alpha-1}, v_{\alpha+1}, \dots, v_{q-j}, \bar{v}_1, \dots, \bar{v}_{\beta-1}, \bar{v}_{\beta+1}, \dots, \bar{v}_{q-j})$$

$$\cdot (-1)^{\alpha+\beta} \int_{t \in \mathbb{S}^{2q-1}} \sum_{i_\alpha, n_\beta} (\pi^* u)_{t,z}(e_{i_\alpha}, e_1, \dots, e_{q-1}, \bar{e}_{n_\beta}, \bar{e}_1, \dots, \bar{e}_{q-1}) v_{\alpha, i_\alpha} \bar{v}_{\beta, n_\beta} \Phi(t).$$

In other words $C_1 A_1(s)$ is the value at $(v_1, \dots, v_{q-j}, \bar{v}_1, \dots, \bar{v}_{q-j})$ of

$$(-1)^{q-j-1} p_{s^\perp}^* \left(\frac{i}{2\pi} \partial \bar{\partial} (\|x\|^2) \right)_x^{q-j-1} \wedge \int_{t \in \mathbb{S}^{2q-1}} (\pi^* u)_{t,z}(., z^1, \dots, z^{q-1}, \bar{z}^1, \dots, \bar{z}^{q-1}) \Phi(t).$$

Note that

$$(\pi^* u)_{t,z}(e_0, e_1, \dots, e_{q-1}, \bar{e}_{n_\beta}, \bar{e}_1, \dots, \bar{e}_{q-1}) = 0 = (\pi^* u)_{t,z}(e_{i_\alpha}, e_1, \dots, e_{q-1}, \bar{e}_0, \bar{e}_1, \dots, \bar{e}_{q-1})$$

since $d\pi_{t,z}(e_0), \dots, d\pi_{t,z}(e_{q-1})$ are linearly dependent, thanks to $t_1 d\pi_{t,z}(e_0) + \dots + t_q d\pi_{t,z}(e_{q-1}) = d\pi_{t,z}(t_1 e_0 + \dots + t_q e_{q-1}) = d\pi_{t,z}(t \cdot z) = 0$.

On the other hand, for the second term, we write

$$p_{s^\perp}^* \left(\frac{i}{2\pi} \partial \bar{\partial} (\|x\|^2) \right)^{q-j} = C_2 \sum_{|I|=q-j} dx_I \wedge d\bar{x}_I$$

where $I = (i_1, \dots, i_{q-j})$ with $q \leq i_1 < \dots < i_{q-j} \leq N$ and $C_2 = \frac{i^{(q-j)^2}}{(2\pi)^{q-j}} (q-j)!$ is a constant. We obtain by (14)

$$C_2 A_2(s) = p_{s^\perp}^* \left(\frac{i}{2\pi} \partial \bar{\partial} (\|x\|^2) \right)_{e_0}^{q-j} (v_1, \dots, v_{q-j}, \bar{v}_1, \dots, \bar{v}_{q-j}) \int_{t \in \mathbb{S}^{2q-1}} \left\{ \sum_{\substack{1 \leq k' \leq q-1 \\ 0 \leq l' \leq N}} (\pi^* u)_{t,z}(e_0, \dots, e_{k'-1}, e_{l'}, e_{k'+1}, \dots, e_{q-1}, \bar{e}_0, \dots, \bar{e}_{k'-1}, \bar{e}_{l'}, \bar{e}_{k'+1}, \dots, \bar{e}_{q-1}) \right\} \Phi(t).$$

In other words $C_2 A_2(s)$ is the value at $(v_1, \dots, v_{q-j}, \bar{v}_1, \dots, \bar{v}_{q-j})$ of

$$p_{s^\perp}^* \left(\frac{i}{2\pi} \partial \bar{\partial} (\|x\|^2) \right)_x^{q-j} \int_{t \in \mathbb{S}^{2q-1}} \text{tr} \left\{ \sum_{1 \leq k' \leq q-1} (\pi^* u)_{t,z}(z^0, \dots, z^{k'-1}, \bar{z}^{k'+1}, \dots, z^{q-1}, \bar{z}^0, \dots, \bar{z}^{k'-1}, \bar{z}^{k'+1}, \dots, \bar{z}^{q-1}) \right\} \Phi(t)$$

where we denote by tr the trace with respect to $\frac{i}{2} \partial \bar{\partial} (\|x\|^2)$ of the differential $(1, 1)$ -form in $\{., \bar{.}\}$.

Now we write $p_{s^\perp}^* \left(\frac{i}{2\pi} \partial \bar{\partial} (\|x\|^2) \right) = \frac{i}{2\pi} \partial \bar{\partial} (\|p_{s^\perp}(x)\|^2) = \frac{i}{2\pi} \partial \bar{\partial} (d(x, s)^2)$ with $d(x, s)$ the distance of x to the vector subspace s of $\mathbb{C}^{N+1}$. We have

$$d(x, s)^2 = \frac{\Delta(x, z^0, \dots, z^{q-1})}{\Delta(z^0, \dots, z^{q-1})} = \sum_{0 \leq i, i' \leq N} x_i \bar{x}_{i'} M_{ii'}$$

where $\Delta(z^0, \dots, z^{q-1})$ is the Gram determinant of $z^0, \dots, z^{q-1}$ and where $(M_{ii'})_{i, i'}$ is a Hermitian matrix depending only on $s = \text{vect}(z^0, \dots, z^{q-1})$. Then

$$\begin{aligned} & p_{s^\perp}^* \left(\frac{i}{2\pi} \partial \bar{\partial} (\|x\|^2) \right)^{q-j-1} \\ &= \left(\frac{i}{2\pi} \right)^{q-j-1} \sum M_{i_1 i'_1} \dots M_{i_{q-j-1} i'_{q-j-1}} dx_{i_1} \wedge d\bar{x}_{i'_1} \wedge \dots \wedge dx_{i_{q-j-1}} \wedge d\bar{x}_{i'_{q-j-1}} \\ &= C_1 \sum_{|I|=|I'|=q-j-1} \det(M_{ii'})_{i \in I, i' \in I'} dx_I \wedge d\bar{x}_{I'} \end{aligned}$$

while

$$\begin{aligned} & \int_{t \in \mathbb{S}^{2q-1}} (\pi^* u)_{t,z}(., z^1, \dots, z^{q-1}, \bar{z}^1, \dots, \bar{z}^{q-1}) \Phi(t) \\ &= \sum_{n, l} \left(\int_{t \in \mathbb{S}^{2q-1}} \left\{ \sum_{|I|=|I'|=q-1} (\pi^* u)_{t,z}(e_n, e_{j_1}, \dots, e_{j_{q-1}}, \bar{e}_l, \bar{e}_{j'_1}, \dots, \bar{e}_{j'_{q-1}}) z_j^{\hat{0}} \bar{z}_{j'}^{\hat{0}} \right\} \Phi(t) \right) dx_n \wedge d\bar{x}_l \\ &= \sum_{n, l} ((-1)^{0+0}) \sum_{|I|=|I'|=q-1} \tilde{u}_{n, l, I'}(z) z_j^{\hat{0}} \bar{z}_{j'}^{\hat{0}} dx_n \wedge d\bar{x}_l \end{aligned}$$

$$= \sum_{n,l} \mathcal{C}_{nl}^{00}(u)(z) dx_n \wedge d\bar{x}_l.$$

With $C'_1 = 1$, we conclude that $C'_1 A_1(s)$ is the value at $(v_1, \dots, v_{q-j}, \bar{v}_1, \dots, \bar{v}_{q-j})$ of

$$\begin{aligned} & (-1)^{q-j-1} \sum_{|I|=|I'|=q-j-1} \det(M_{ii'})_{i \in I, i' \in I'} dx_I \wedge d\bar{x}_{I'} \wedge \sum_{n,l} \mathcal{C}_{nl}^{00}(u)(z) dx_n \wedge d\bar{x}_l \\ &= \sum_{|I|=|I'|=q-j} \left\{ \sum_{1 \leq \alpha, \beta \leq q-j} (-1)^{\alpha+\beta} \det(M_{ii'})_{\substack{i \in I, i \neq i_\alpha \\ i' \in I', i' \neq i'_\beta}} \mathcal{C}_{i_\alpha i'_\beta}^{00}(u)(z) \right\} dx_I \wedge d\bar{x}_{I'} \end{aligned}$$

where $I = (i_1, \dots, i_{q-j})$ with $0 \leq i_1 < \dots < i_{q-j} \leq N$ and where $I' = (i'_1, \dots, i'_{q-j})$ with $0 \leq i'_1 < \dots < i'_{q-j} \leq N$.

For the second term we have

$$\begin{aligned} p_{s^\perp}^* \left(\frac{i}{2\pi} \partial \bar{\partial} (\|x\|^2) \right)^{q-j} &= \left(\frac{i}{2\pi} \right)^{q-j} \sum M_{i_1 i'_1} \dots M_{i_{q-j} i'_{q-j}} dx_{i_1} \wedge d\bar{x}_{i'_1} \wedge \dots \wedge dx_{i_{q-j}} \wedge d\bar{x}_{i'_{q-j}} \\ &= C_2 \sum_{|I|=|I'|=q-j} \det(M_{ii'})_{i \in I, i' \in I'} dx_I \wedge d\bar{x}_{I'} \end{aligned}$$

while

$$\begin{aligned} & \int_{t \in \mathbb{S}^{2q-1}} \text{tr} \left\{ \sum_{1 \leq k' \leq q-1} (\pi^* u)_{t,z}(z^0, \dots, z^{k'-1}, \dots, z^{k'+1}, \dots, z^{q-1}, \bar{z}^0, \dots, \bar{z}^{k'-1}, \dots, \bar{z}^{k'+1}, \dots, \bar{z}^{q-1}) \right\} \Phi(t) \\ &= \int_{t \in \mathbb{S}^{2q-1}} \left\{ \sum_{1 \leq k' \leq q-1} \sum_{0 \leq l \leq N} (-1)^{k'+k'} \sum_{|J|=|J'|=q-1} (\pi^* u)_{t,z}(e_{j_0}, \dots, e_{j_{k'-1}}, e_l, e_{j_{k'+1}}, \dots, e_{j_{q-1}}, \bar{e}_{j'_0}, \dots, \right. \\ & \quad \left. \bar{e}_{j'_{k'-1}}, \bar{e}_l, \bar{e}_{j'_{k'+1}}, \dots, \bar{e}_{j'_{q-1}}) z_j^{k'} \bar{z}_{j'}^{k'} \right\} \Phi(t) \end{aligned}$$

is equal to

$$\sum_{1 \leq k' \leq q-1} \sum_{0 \leq l \leq N} (-1)^{k'+k'} \sum_{|J|=|J'|=q-1} \tilde{u}_{JJ'}(z) z_j^{k'} \bar{z}_{j'}^{k'} = \sum_{1 \leq k' \leq q-1} \sum_{0 \leq l \leq N} \mathcal{C}_{ll}^{k'k'}(u)(z).$$

With $C'_2 = 1$, we conclude that $C'_2 A_2(s)$ is the value at $(v_1, \dots, v_{q-j}, \bar{v}_1, \dots, \bar{v}_{q-j})$ of

$$\left(\sum_{|I|=|I'|=q-j} \det(M_{ii'})_{i \in I, i' \in I'} dx_I \wedge d\bar{x}_{I'} \right) \sum_{\substack{1 \leq k \leq q-1 \\ 0 \leq l \leq N}} \mathcal{C}_{ll}^{kk}(u)(z).$$

As a result of all that, since $K_1 = K_2$, we obtain the following expression

$$\begin{aligned} \pi^*(C^* \mathcal{C}(u))_x &= K' \sum_{|I|=|I'|=q-j} \left(\int_{s \in \varphi^{-1}([x])} \left\{ \sum_{1 \leq \alpha, \beta \leq q-j} (-1)^{\alpha+\beta} \det(M_{ii'})_{\substack{i \in I, i \neq i_\alpha \\ i' \in I', i' \neq i'_\beta}} \mathcal{C}_{i_\alpha i'_\beta}^{00}(u)(z) \right. \right. \\ & \quad \left. \left. + \det(M_{ii'})_{i \in I, i' \in I'} \sum_{\substack{1 \leq k \leq q-1 \\ 0 \leq l \leq N}} \mathcal{C}_{ll}^{kk}(u)(z) \right\} \Omega_s^d \right) dx_I \wedge d\bar{x}_{I'} \end{aligned}$$

where $s \in \varphi^{-1}([x])$ is written $s = \text{vect}(x, z^1, \dots, z^{q-1})$ and where $K' = \frac{K_1}{K} = \frac{K_2}{K}$ is a constant.

Let $F_{I,I'}$ be the smooth function on $G(q, V)$ defined by

$$\begin{aligned} F_{I,I'}(s) &= F_{I,I'}(\text{vect}(x, z^1, \dots, z^{q-1})) = \sum_{1 \leq \alpha, \beta \leq q-j} (-1)^{\alpha+\beta} \det(M_{ii'})_{\substack{i \in I, i \neq i_\alpha \\ i' \in I', i' \neq i'_\beta}} \mathcal{C}_{i_\alpha i'_\beta}^{00}(u)(z) + \\ & \quad \det(M_{ii'})_{i \in I, i' \in I'} \sum_{\substack{1 \leq k \leq q-1 \\ 0 \leq l \leq N}} \mathcal{C}_{ll}^{kk}(u)(z). \end{aligned}$$

Then the expression in coordinates of $C^* \mathcal{C}(u)$ is given by the following result.

Lemma 1. For all $x \in \mathbb{C}^{N+1} = V$ such that $x \neq 0$, we have

$$\begin{aligned} \pi^*(C^* \mathcal{C}(u))_x &= K' \sum_{|I|=|I'|=q-j} dx_I \wedge d\bar{x}_{I'} \int_{s \in \varphi^{-1}([x])} F_{I,I'} \Omega^d \\ &= K' \sum_{|I|=|I'|=q-j} dx_I \wedge d\bar{x}_{I'} ({}^t \mathcal{R}_{q-1})(F_{I,I'})([x]). \end{aligned}$$

where ${}^t \mathcal{R}_{q-1}$ is the dual transform of the projective $(q-1)$ -dimensional Radon transform $\mathcal{R}_{q-1}$.

3.2 Expression of the $F_{I,I'}$ after a change of basis

First we express the $F_{I,I'}$ by means of the transforms $\tilde{u}_{K,K'}$. Note that the relation

$$C_{nl}^{km}(u)(z) = C(u)_s \left(\frac{d}{dt} \text{vect}(z^0, \dots, z^{k-1}, z^k + te_n, z^{k+1}, \dots, z^{q-1}) \Big|_{t=0}, \right. \\ \left. \frac{d}{dt} \text{vect}(z^0, \dots, z^{m-1}, z^m + te_l, z^{m+1}, \dots, z^{q-1}) \Big|_{t=0} \right)$$

implies

$$C_{nl}^{kk}(u)(z) = C_{nl}^{00}(u)(z^k, z^1, \dots, z^{k-1}, z^0, z^{k+1}, \dots, z^{q-1}).$$

On another hand, with $(a_{k,m})_{1 \leq k, m \leq q-1} \in GL_{q-1}(\mathbb{C}) \subset GL_q(\mathbb{C})$, set

$$g(z^0, z^1, \dots, z^{q-1}) = (z^0, \sum_{1 \leq k \leq q-1} a_{k,1} z^k, \dots, \sum_{1 \leq k \leq q-1} a_{k,q-1} z^k).$$

Then we have $\tau(g(z)) = \tau(z)$ thus

$$g^* \sum_{\substack{0 \leq k, m \leq q-1 \\ 0 \leq n, l \leq N}} C_{nl}^{km}(u) dz_n^k \wedge d\bar{z}_l^m = \sum_{\substack{0 \leq k, m \leq q-1 \\ 0 \leq n, l \leq N}} C_{nl}^{km}(u) dz_n^k \wedge d\bar{z}_l^m.$$

We obtain

$$\sum_{0 \leq n, l \leq N} C_{nl}^{00}(u)(g(z)) dz_n^0 \wedge d\bar{z}_l^0 + \dots = \sum_{0 \leq n, l \leq N} C_{nl}^{00}(u)(z) dz_n^0 \wedge d\bar{z}_l^0 + \dots$$

thus the invariance relation $C_{nl}^{00}(u)(g(z)) = C_{nl}^{00}(u)(z)$.

Now we can express the functions $F_{I,I'}$ for $|I| = |I'| = q - j$, in terms of the Radon transforms $\mathcal{R}_{q-1}(f_{K,K'})$ for $|K| = |K'| = q$.

First

$$\sum_{1 \leq \alpha, \beta \leq q-j} (-1)^{\alpha+\beta} \det(M_{ii'})_{\substack{i \in I, i' \neq i_\alpha \\ i' \in I', i' \neq i'_\beta}} C_{i_\alpha i'_\beta}^{00}(u)(z) \\ = \sum_{1 \leq \alpha, \beta \leq q-j} (-1)^{\alpha+\beta} \det(M_{ii'})_{\substack{i \in I, i' \neq i_\alpha \\ i' \in I', i' \neq i'_\beta}} (-1)^{0+0} \sum_{|J|=|J'|=q-1} \tilde{u}_{i_\alpha J, i'_\beta J'}(z) z_J^{\hat{0}} \bar{z}_{J'}^{\hat{0}} \\ = \sum_{|K|=|K'|=q} \tilde{u}_{K,K'}(z) \sum_{i_\alpha \in K, i'_\beta \in K'} (-1)^{\alpha+\beta} (-1)^{\alpha_0+\beta_0} \det(M_{ii'})_{\substack{i \in I, i' \neq i_\alpha \\ i' \in I', i' \neq i'_\beta}} z_{K \setminus \{i_\alpha\}}^{\hat{0}} \bar{z}_{K' \setminus \{i'_\beta\}}^{\hat{0}}$$

where α_0 and β_0 are such that $i_\alpha = k_{\alpha_0}$ and $i'_\beta = k'_{\beta_0}$ when $K = (k_1, \dots, k_q)$ with $0 \leq k_1 < \dots < k_q \leq N$ and $K' = (k'_1, \dots, k'_q)$ with $0 \leq k'_1 < \dots < k'_q \leq N$.

Then

$$\det(M_{ii'})_{i \in I, i' \in I'} \sum_{\substack{1 \leq k \leq q-1 \\ 0 \leq l \leq N}} C_{ll}^{kk}(u)(z) \\ = \sum_{0 \leq l \leq N} \sum_{|J|=|J'|=q-1} \tilde{u}_{IJ, I'J'}(z) \sum_{1 \leq k \leq q-1} \det(M_{ii'})_{i \in I, i' \in I'} z_J^{\hat{k}} \bar{z}_{J'}^{\hat{k}} \\ = \sum_{|K|=|K'|=q} \tilde{u}_{K,K'}(z) \sum_{l \in K \cap K'} (-1)^{\alpha_0+\beta_0} \sum_{1 \leq k \leq q-1} \det(M_{ii'})_{i \in I, i' \in I'} z_{K \setminus \{l\}}^{\hat{k}} \bar{z}_{K' \setminus \{l\}}^{\hat{k}}$$

where α_0 and β_0 are such that $l = k_{\alpha_0} = k'_{\beta_0}$ when $K = (k_1, \dots, k_q)$ with $0 \leq k_1 < \dots < k_q \leq N$ and $K' = (k'_1, \dots, k'_q)$ with $0 \leq k'_1 < \dots < k'_q \leq N$.

With all that together, we conclude that

$$F_{I,I'}(s) = \sum_{|K|=|K'|=q} \mathcal{R}_{q-1}(f_{K,K'})(s) c_{K,K'}^{I,I'}(s) \quad (15)$$

where the coefficients $c_{K,K'}^{I,I'}(s)$ are independent of the smooth differential (q, q) -form u .

Set the matrix $C(s) = (c_{K,K'}^{I,I'}(s))_{II',KK'}$. Note here that (K, K') belongs to a set with $\binom{N+1}{q}^2$ elements, while (I, I') belongs to a set with $\binom{N+1}{q-j}^2 = \binom{N+1}{q}^2$ in both cases $j = 0$ or $j = 2q - N - 1$.

When $g \in U(N+1)$ is an element of the unitary group of $V = \mathbb{C}^{N+1}$, we can express $C(g(s))$ in terms of g and $C(s)$ by means of the formula

$$C(g(s)) = {}^t Q(g)^{-1} C(s) {}^t Q(g) \Leftrightarrow {}^t Q(g) C(g(s)) = C(s) {}^t Q(g) \quad (16)$$

with the matrix $Q(g) = (\det(g_{k,l})_{\substack{k \in K \\ l \in L}} \det(\bar{g}_{k',l'})_{\substack{k' \in K' \\ l' \in L'}})_{KK',LL'}$.

For the proof of this invariance property, let λ_K be the Plücker coordinates of $(\lambda^0, \dots, \lambda^{q-1}) \in (\mathbb{C}^{N+1})^q$ defined by

$$\lambda^0 \wedge \dots \wedge \lambda^{q-1} = \sum_{|K|=q} \lambda_K e_K.$$

This property is a consequence of the fact that the coefficients $c_{K,K'}^{I,I'}(s)$ can be calculated by means of the relation

$$\begin{aligned} \alpha_q \Delta(z) & \sum_{|I|=|I'|=q-j} \sum_{|K|=|K'|=q} \bar{\lambda}_K \lambda_{K'} c_{K,K'}^{I,I'}(s) dx_I \wedge d\bar{x}_{I'} \\ & = D_1 p_{s^+}^* (i\partial\bar{\partial}(\|x\|^2))^{q-j-1} \wedge \partial\bar{\partial}(|\langle x \wedge z^1 \wedge \dots \wedge z^{q-1} | \lambda^0 \wedge \dots \wedge \lambda^{q-1} \rangle|^2) \\ & + D_2 p_{s^+}^* (i\partial\bar{\partial}(\|x\|^2))^{q-j} \sum_{1 \leq k \leq q-1} \text{trace}(|\langle x \wedge z^0 \wedge \dots \wedge \widehat{z^k} \wedge \dots \wedge z^{q-1} | \lambda^0 \wedge \dots \wedge \lambda^{q-1} \rangle|^2) \end{aligned}$$

with $D_1 = (-1)^{q-j-1} (i^{(q-j-1)^2} (q-j-1)!)^{-1}$ and $D_2 = (i^{(q-j)^2} (q-j)!)^{-1}$.

Since the differential $(1, 1)$ -form Ω is invariant by $U(N+1)$, we have also $g^* \pi^* \mathcal{C}^* \mathcal{C}(u) = \pi^* \mathcal{C}^* \mathcal{C}(g^* u)$. But because ${}^t \mathcal{R}_{q-1}$ is not injective (see [14]), we cannot deduce directly from the previous relation the invariance property for the matrix $C(s)$.

Note that $Q(g)$ is the matrix relatively to the basis $(e_L \wedge \bar{e}_{L'})_{LL'}$ of $\wedge^q V \otimes \wedge^q \bar{V}$ of the unitary operator $\wedge^q g \otimes \wedge^q \bar{g}$, therefore $Q(g)$ is unitary.

For $|J| = |J'| = q - j$, set

$$G_{J,J'}(g) = \sum_{|I|=|I'|=q-j} Q_{II',JJ'}(g) F_{I,I'}(g(s)) = \sum_{|L|=|L'|=q} c_{L,L'}^{J,J'}(s) \varphi_{L,L'}(g)$$

thanks to (16), with

$$\varphi_{L,L'}(g) = \sum_{|K|=|K'|=q} \mathcal{R}_{q-1}(f_{K,K'})(g(s)) Q_{KK',LL'}(g).$$

Let the $h_{K,K'}$ be smooth functions on $\mathbb{P}_N = P(V)$ and set

$$\psi_{L,L'}(g) = \sum_{|K|=|K'|=q} \mathcal{R}_{q-1}(h_{K,K'})(g(s)) Q_{KK',LL'}(g).$$

Since $Q(g)$ is unitary, we have

$$\begin{aligned} \sum_{|L|=|L'|=q} (\psi_{L,L'} \bar{G}_{L,L'})(g) & = \sum_{|I|=|I'|=q-j} \sum_{|K|=|K'|=q} (\mathcal{R}_{q-1}(h_{K,K'}) \bar{F}_{I,I'})(g(s)) (Q(g) Q^*(g))_{KK',II'} \\ & = \sum_{|I|=|I'|=q-j} (\mathcal{R}_{q-1}(h_{I,I'}) \bar{F}_{I,I'})(g(s)). \end{aligned}$$

3.3 Determination of $\ker \mathcal{C}^* \mathcal{C}$

First we determine the system of linear P.D.E. satisfied by the functions $\psi_{L,L'}(g) \alpha_q^{-1} \Delta(g(z))^{-1}$ of g . With $s = \text{vect}(z^0, \dots, z^{q-1})$, we have

$$\varphi_{L,L'}(g) \alpha_q^{-1} \Delta(g(z))^{-1}$$

$$\begin{aligned}
&= \sum_{|K|=|K'|=q} \tilde{u}_{K,K'}(g(z)) Q_{KK',LL'}(g) \\
&= \int_{t \in \mathbb{S}^{2q-1}} \langle (\pi^* u)_{t,g(z)}, (\bigwedge^q g \otimes \bigwedge^q \bar{g})(e_L \wedge \bar{e}_{L'}) \rangle \Phi(t) \\
&= \int_{t \in \mathbb{S}^{2q-1}} \langle (\pi^*(g^* u))_{t,z}, e_L \wedge \bar{e}_{L'} \rangle \Phi(t).
\end{aligned}$$

When $z = (z^0, \dots, z^{q-1})$ is fixed, the map

$$g \rightarrow \int_{t \in \mathbb{S}^{2q-1}} (\pi^*(g^* u))_{t,z} \Phi(t) \in \bigwedge^{q,q} V^*$$

is an integral transform of u . By complexifying i.e. by taking $g \in GL(N+1, \mathbb{C})$, this map has the same expression in g and satisfies a system of linear partial differential equations in g .

As a consequence, when the functions $h_{K,K'}$ vary in $\mathcal{C}^\infty(\mathbb{P}_N)$, the functions $\psi_{L,L'}$ vary in a vector space $\mathcal{E}$ of smooth functions in g , characterized by a system of linear partial differential equations, which does not depend on (L, L') .

Lemma 2. If $F_{I,I'} \in \ker({}^t\mathcal{R}_{q-1})$ for all $|I| = |I'| = q-j$, then $F_{I,I'} = 0$ for all $|I| = |I'| = q-j$.

Proof. If $F_{I,I'} \in \ker({}^t\mathcal{R}_{q-1}) = (\text{im } \mathcal{R}_{q-1})^\perp$ for all $|I| = |I'| = q-j$, then $G_{L,L'}$ is orthogonal to each $\psi_{L,L'}$ for all $|L| = |L'| = q$, thus $G_{L,L'} \in \mathcal{E}^\perp$. But since the $G_{J,J'}$ decompose on the $\varphi_{L,L'}$ with constant coefficients, we have $G_{J,J'} \in \mathcal{E}$, therefore $G_{J,J'} = 0$. Since the matrix $Q(g)$ is invertible, this implies $F_{I,I'} = 0$ for all $|I| = |I'| = q-j$. $\square$

In other words, $0 = \mathcal{C}^* \mathcal{C}(u)$ implies

$$0 = \sum_{|I|=|I'|=q-j} F_{I,I'}(\tau(z)) dx_I \wedge d\bar{x}_{I'},$$

which is equivalent to

$$\begin{aligned}
0 &= (q-j) p_{s^\perp}^* (i\partial\bar{\partial}(\|x\|^2))^{q-j-1} \wedge \sum_{0 \leq n, l \leq N} \mathcal{C}_{nl}^{00}(u)(z) id x_n \wedge d\bar{x}_l \\
&\quad + p_{s^\perp}^* (i\partial\bar{\partial}(\|x\|^2))^{q-j} \sum_{\substack{1 \leq k \leq q-1 \\ 0 \leq l \leq N}} \mathcal{C}_{ll}^{kk}(u)(z).
\end{aligned}$$

That is to say $0 = p_{s^\perp}^* (i\partial\bar{\partial}(\|x\|^2))^{q-j-1} \wedge \theta$ with the differential $(1, 1)$ -form θ in $V = \mathbb{C}^{N+1}$ defined by

$$\theta = \sum_{0 \leq n, l \leq N} \{ (q-j) \mathcal{C}_{nl}^{00}(u)(z) + M_{nl} \sum_{\substack{1 \leq k \leq q-1 \\ 0 \leq l' \leq N}} \mathcal{C}_{l'l}^{kk}(u)(z) \} id x_n \wedge d\bar{x}_l.$$

We now prove that necessarily $u = 0$.

Let $b_q, \dots, b_N$ be an orthonormal basis of $s^\perp$. Denote by $b_l^* \in V^*$ the linear form defined by $b_l^*(x) = \langle x | b_l \rangle$. Since $d(x, s)^2 = \sum_{q \leq l \leq N} |\langle x | b_l \rangle|^2$, we have

$$(i\partial\bar{\partial}(d(x, s)^2))^{q-j-1} = (i \sum_{q \leq l \leq N} b_l^* \wedge \bar{b}_l^*)^{q-j-1} = D \sum_{|I|=q-j-1} b_I^* \wedge \bar{b}_I^*$$

where $I = (i_1, \dots, i_{q-j-1})$ with $q \leq i_1 < \dots < i_{q-j-1} \leq N$ and where $D = i^{(q-j-1)^2} (q-j-1)!$.

Let $b_0, \dots, b_{q-1}$ be an orthonormal basis of $s = \text{vect}(z^0, \dots, z^{q-1})$. Writing $\theta = \sum_{0 \leq n, l \leq N} \check{\theta}_{nl} b_n^* \wedge \bar{b}_l^*$ in the basis b_l^* , we have

$$0 = \sum_{|I|=q-j-1} b_I^* \wedge \bar{b}_I^* \wedge \sum_{0 \leq n, l \leq N} \check{\theta}_{nl} b_n^* \wedge \bar{b}_l^* = (-1)^{q-j-1} \sum_{|I|=q-j-1} \sum_{n, l \neq I} \check{\theta}_{nl} b_n^* \wedge \bar{b}_l^*.$$

We set $J = \{n, i_1, \dots, i_{q-j-1}\}$ and $J' = \{l, i_1, \dots, i_{q-j-1}\}$ in such a way that $\text{card}(J \cap J') \geq q-j-1$. Then $\text{card}(J \cap J') = q-j-1 \Leftrightarrow n \neq l$ and $J = J' \Leftrightarrow n = l$.

In the first case $I = J \cap J'$ and $\{n\} = J \setminus J \cap J'$ while $\{l\} = J' \setminus J \cap J'$: we obtain $\check{\theta}_{nl} = 0 = \theta(b_n, \bar{b}_l)$ if $n \neq l$. On another hand we have

$$\sum_{|l|=q-j-1} \sum_{l \notin I} \check{\theta}_{ll} b_{ll}^* \wedge \bar{b}_{ll}^* = \sum_{|l|=q-j-1} \sum_{\substack{l \notin I \\ l < q}} \check{\theta}_{ll} b_{ll}^* \wedge \bar{b}_{ll}^* + \sum_{|l|=q-j} \left(\sum_{l \in J} \check{\theta}_{ll} \right) b_J^* \wedge \bar{b}_J^*$$

where $J = (i_1, \dots, i_{q-j})$ with $q \leq i_1 < \dots < i_{q-j} \leq N$. In the second case, we obtain $\check{\theta}_{ll} = 0$ if $0 \leq l \leq q-1$, while $\sum_{l \in J} \check{\theta}_{ll} = 0$ for all such J . It remains $N - q + 1$ unknown variables $\check{\theta}_{ll}$ satisfying $\binom{N-q+1}{q-j}$ relations.

If $j = 0 \Leftrightarrow q \leq N - q + 1$, we have $\binom{N-q+1}{q-j} = \binom{N-q+1}{q} \geq \binom{N-q+1}{1}$ i.e. $\geq N - q + 1$. The number of equations being bigger, we obtain $\theta = 0$.

If $j = 2q - N - 1 \Leftrightarrow q \geq N - q + 1$, we have $\theta(b_n, \bar{b}_l) = 0$ for all (n, l) except for $n = l \geq q$, for which we only have $\sum_{q \leq l \leq N} \theta(b_l, \bar{b}_l) = 0$.

Recall that

$$\begin{aligned} \frac{1}{i} \theta(b_n, \bar{b}_l) &= \int_{t \in \mathbb{S}^{2q-1}} \{ (q-j)(\pi^* u)_{t \cdot z}(b_n, z^1, \dots, z^{q-1}, \bar{b}_l, \bar{z}^1, \dots, \bar{z}^{q-1}) + {}^t b_n(M_{i,i'}) \bar{b}_l \\ &\cdot \sum_{\substack{1 \leq k \leq q-1 \\ 0 \leq l' \leq N}} (\pi^* u)_{t \cdot z}(e_{l'}, z^0, \dots, z^{k-1}, z^{k+1}, \dots, z^{q-1}, \bar{e}_{l'}, \bar{z}^0, \dots, \bar{z}^{k-1}, \bar{z}^{k+1}, \dots, \bar{z}^{q-1}) \} \Phi(t). \end{aligned}$$

Thus we obtain the condition

$$\begin{aligned} 0 &= (q-j)(\pi^* u)_x(b_n, z^1, \dots, z^{q-1}, \bar{b}_l, \bar{z}^1, \dots, \bar{z}^{q-1}) \\ &+ {}^t b_n(M_{i,i'}) \bar{b}_l \sum_{\substack{1 \leq k \leq q-1 \\ 0 \leq l' \leq N}} (\pi^* u)_x(e_{l'}, z^0, \dots, z^{k-1}, z^{k+1}, \dots, z^{q-1}, \bar{e}_{l'}, \bar{z}^0, \dots, \bar{z}^{k-1}, \bar{z}^{k+1}, \dots, \bar{z}^{q-1}) \end{aligned}$$

where $x = t \cdot z = t_1 z^0 + \dots + t_q z^{q-1}$, for every $t = (t_1, \dots, t_q) \in \mathbb{S}^{2q-1}$ and for every $z = (z^0, \dots, z^{q-1}) \in (\mathbb{C}^{N+1})^q$. We have denoted by ${}^t b_n(M_{i,i'}) \bar{b}_l = (\partial \bar{\partial}(d(x, s)^2))(b_n \wedge \bar{b}_l)$ the value at (b_n, b_l) of the Hermitian form associated to the matrix $(M_{i,i'})_{i,i'}$.

By taking $z^0 = x$ and by using that the radial contractions of $(\pi^* u)_x$ are equal to 0, it remains

$$(\pi^* u)_x(b_n, z^1, \dots, z^{q-1}, \bar{b}_l, \bar{z}^1, \dots, \bar{z}^{q-1}) = 0 \quad (17)$$

for all n, l , therefore $(\pi^* u)_x(e_n, z^1, \dots, z^{q-1}, \bar{e}_l, \bar{z}^1, \dots, \bar{z}^{q-1}) = 0$ for all x and all $z^1, \dots, z^{q-1}$. We can assume the differential (q, q) -form u real, then we obtain $(\pi^* u)_x = 0$ i.e. $u = 0$.

If $j = 2q - N - 1$, the relation (17) is valid for $n \neq l$ and for $n = l < q$. Therefore we can write

$$(\pi^* u)_x(., z^1, \dots, z^{q-1}, ., \bar{z}^1, \dots, \bar{z}^{q-1}) = \sum_{q \leq l \leq N} \lambda_l(x, z^1, \dots, z^{q-1}) b_l^* \wedge \bar{b}_l^* \quad (18)$$

with functions $\lambda_l(x, z^1, \dots, z^{q-1})$ of $x, z^1, \dots, z^{q-1}$. But the family $(b_q, \dots, b_N)$ is any orthonormal basis of $\text{vect}(x, z^1, \dots, z^{q-1})^\perp$, thus the function $\lambda_l(x, z^1, \dots, z^{q-1}) = \lambda(x, z^1, \dots, z^{q-1})$ does not depend on l . The relation (18) becomes

$$(\pi^* u)_x(., z^1, \dots, z^{q-1}, ., \bar{z}^1, \dots, \bar{z}^{q-1}) = \lambda(x, z^1, \dots, z^{q-1}) \partial \bar{\partial}(d(., s)^2),$$

which gives $\theta(b_l, \bar{b}_l)$ independent of $q \leq l \leq N$. We obtain that $\theta(b_l, \bar{b}_l) = 0$ thus $\theta = 0$ and that the same conclusion is valid as for the case $j = 0$ i.e. $u = 0$.

Theorem 1. *The integral transformation*

$$C^* C : \{\text{currents of bidegree } (q, q) \text{ on } \mathbb{P}_N\} \rightarrow \{\text{currents of bidegree } (q-j, q-j) \text{ on } \mathbb{P}_N\}$$

is injective.

Remark 1. If u is a smooth differential (q, q) -form on $\mathbb{P}_N$ satisfying

$$0 = \sum_{|I|=|J|=q-1} \tilde{u}_{nI,lJ}(z) \bar{z}_I^{\hat{0}} \bar{z}_J^{\hat{0}} \Leftrightarrow 0 = \int_{t \in \mathbb{S}^{2q-1}} (\pi^* u)_{t,z}(e_n, z^1, \dots, z^{q-1}, \bar{e}_l, \bar{z}^1, \dots, \bar{z}^{q-1}) \Phi(t)$$

on $(\mathbb{C}^{N+1})^q$ for all $0 \leq n, l \leq N$, then $u = 0$.

In effect, for $0 \leq n, l \leq N$, we define a smooth differential $(q-1, q-1)$ -form $\varphi = \varphi_{n,l}$ on $\mathbb{P}_N = P(V)$ by

$$\varphi_{[x]} = \|x\|^2 u_{[x]}(d\pi_x(e_n), \dots, \overline{d\pi_x(e_l)}, \dots)$$

for all $x \in V \setminus \{0\}$. In other words, we have

$$(\pi^* u)_x(e_n, z^1, \dots, z^{q-1}, \bar{e}_l, \bar{z}^1, \dots, \bar{z}^{q-1}) = \|x\|^{-2} (\pi^* \varphi)_x(z^1, \dots, z^{q-1}, \bar{z}^1, \dots, \bar{z}^{q-1})$$

with

$$(-1)^{q-1} q^2 (\pi^* u)_x = \|x\|^{-2} \sum_{0 \leq n, l \leq N} (\pi^* \varphi_{n,l})_x \wedge dx_n \wedge d\bar{x}_l.$$

Let us now express the Chow transform $\mathcal{C}(\varphi)$ i.e. let us calculate

$$\int_{P(\tau(z))} \varphi$$

with $\tau(z) = \text{vect}(z^0, \dots, z^{q-1}) \in G(q, V)$. To this end, we introduce the function $f \in C^\infty(P(\tau(z)))$ such that

$$\varphi|_{P(\tau(z))} = f \omega_{P(\tau(z))}^{q-1}.$$

For $x \in \tau(z) \setminus \{0\}$ and $0 \leq k \leq q-1$, we have $d\pi_x(z^k) \in T_{[x]}P(\tau(z))$ and so

$$\begin{aligned} & \varphi_{[x]}(d\pi_x(z^1), \dots, d\pi_x(z^{q-1}), \overline{d\pi_x(z^1)}, \dots, \overline{d\pi_x(z^{q-1})}) \\ &= f([x]) \omega_{[x]}^{q-1}(d\pi_x(z^1), \dots, d\pi_x(z^{q-1}), \overline{d\pi_x(z^1)}, \dots, \overline{d\pi_x(z^{q-1})}). \end{aligned}$$

With $C = \frac{i^{(q-1)^2} (q-1)!}{(2\pi)^{q-1}}$, we use that

$$\begin{aligned} & \omega_{[x]}^{q-1}(d\pi_x(z^1), \dots, d\pi_x(z^{q-1}), \overline{d\pi_x(z^1)}, \dots, \overline{d\pi_x(z^{q-1})}) \\ &= C \|x\|^{-2q+2} \left\| \left(z^1 - \frac{\langle z^1 | x \rangle}{\|x\|^2} x \right) \wedge \dots \wedge \left(z^{q-1} - \frac{\langle z^{q-1} | x \rangle}{\|x\|^2} x \right) \right\|^2 \\ &= C \|x\|^{-2q} |\langle x | z^0 \rangle|^2 \end{aligned}$$

by assuming here that $(z^0, \dots, z^{q-1})$ is an orthonormal basis of $\tau(z)$. First we obtain

$$\begin{aligned} \int_{P(\tau(z))} \frac{|\langle x | z^0 \rangle|^2}{\|x\|^2} \varphi &= \int_{P(\tau(z))} \frac{|\langle x | z^0 \rangle|^2}{\|x\|^2} f \omega_{P(\tau(z))}^{q-1} \\ &= \int_{t \in \mathbb{S}^{2q-1}} (|\langle x | z^0 \rangle|^2 \|x\|^{-2q-2} f([x]))|_{x=t \cdot z} \Phi(t) \end{aligned}$$

which is equal to

$$\begin{aligned} & C^{-1} \int_{t \in \mathbb{S}^{2q-1}} \{ \|x\|^{-2} \varphi_{[x]}(d\pi_x(z^1), \dots, d\pi_x(z^{q-1}), \overline{d\pi_x(z^1)}, \dots, \overline{d\pi_x(z^{q-1})}) \}|_{x=t \cdot z} \Phi(t) \\ &= C^{-1} \int_{t \in \mathbb{S}^{2q-1}} (\pi^* u)_{t,z}(e_n, z^1, \dots, z^{q-1}, \bar{e}_l, \bar{z}^1, \dots, \bar{z}^{q-1}) \Phi(t) = 0. \end{aligned}$$

As a consequence, we have

$$\int_{P(\tau(z))} \frac{|\langle x | z^k \rangle|^2}{\|x\|^2} \varphi = 0$$

for all $0 \leq k \leq q-1$, thus by summing $\int_{P(s)} \varphi = 0$ for all $s \in G(q, V)$. This implies $\varphi \in \text{im } \partial + \text{im } \bar{\partial}$. But at the same time, we have

$$\frac{x_{n'} \bar{x}_{l'}}{\|x\|^2} \varphi \in \text{im } \partial + \text{im } \bar{\partial}$$

for all $0 \leq n', l' \leq N$. As a conclusion, $\varphi_{n,l} = \varphi = 0$ and $u = 0$ as expected.

3.4 Integral kernel on $\mathbb{P}_N \times \mathbb{P}_N$ of the transform $\mathcal{C}^* \mathcal{C}$

We now calculate the integral kernel $\mathcal{K}$ on $\mathbb{P}_N \times \mathbb{P}_N = P(V) \times P(V)$ of $\mathcal{C}^* \mathcal{C}$ i.e. calculate $\mathcal{K}$ such that

$$\mathcal{C}^* \mathcal{C}(u) = \text{pr}_{1*}(\mathcal{K} \wedge \text{pr}_2^* u)$$

for every smooth differential (q, q) -form u on $P(V) = \mathbb{P}_N$. To this end we use a base change (see [16])

$$\begin{array}{ccc} \Gamma' & \xrightarrow{p_2} & \Gamma \\ p_1 \downarrow & & \downarrow \psi \\ \Gamma & \xrightarrow{\psi} & G(q, V) \end{array}$$

where $\Gamma' = \{(a, b) \in \Gamma^2, \psi(a) = \psi(b)\}$ and where $p_1 : \Gamma' \rightarrow \Gamma$ and $p_2 : \Gamma' \rightarrow \Gamma$ are the restrictions of the canonical projections.

Lemma 3. For u a smooth differential (q, q) -form on $P(V)$, we have $\psi^* \psi_* \varphi^* u = p_{1*} p_2^* \varphi^* u$.

Proof. For $([x], s) \in \Gamma$ and $(\eta, \sigma) \in T_{([x], s)} \Gamma \subset T_{[x]} P(V) \times T_s G(q, V)$, we have on the one hand

$$(\psi^* \psi_* \varphi^* u)_{([x], s)}((\eta, \sigma), (\bar{\eta}, \bar{\sigma})) = (\psi_* \varphi^* u)_s(\sigma, \bar{\sigma}) = \int_{[x'] \in P(s)} u_{[x']}(\xi_{[x']}, \bar{\xi}_{[x']}, \dots)$$

where $\xi_{[x']} \in T_{[x']} P(V)$ is such that $(\xi_{[x']}, \sigma) \in T_{([x'], s)} \Gamma$ for every $[x'] \in P(s)$. But

$$\Gamma' = \{([x], [x'], s) \in P(V) \times P(V) \times G(q, V), \text{vect}(x, x') \subset s\}$$

and $(\eta, \eta', \sigma) \in T_{[x]} P(V) \times T_{[x']} P(V) \times T_s G(q, V)$ is in $T_{([x], [x'], s)} \Gamma'$ when $(\eta, \sigma) \in T_{([x], s)} \Gamma$ and $(\eta', \sigma) \in T_{([x'], s)} \Gamma$.

On the other hand, we have

$$\begin{aligned} & (p_{1*} p_2^* \varphi^* u)_{([x], s)}((\eta, \sigma), (\bar{\eta}, \bar{\sigma})) \\ &= \int_{[x'] \in P(s)} u_{[x']}(\mathbf{d}(\varphi \circ p_2)_{([x], [x'], s)}(\eta, \eta', \sigma), \overline{\mathbf{d}(\varphi \circ p_2)_{([x], [x'], s)}(\eta, \eta', \sigma)}, \dots) \\ &= \int_{[x'] \in P(s)} u_{[x']}(\eta', \bar{\eta}', \dots) \end{aligned}$$

where $\eta' \in T_{[x']} P(V)$ is such that $(\eta, \eta', \sigma) \in T_{([x], [x'], s)} \Gamma'$ i.e. $(\eta', \sigma) \in T_{([x'], s)} \Gamma$. □

Then we use a projection formula

$$\begin{aligned} & \psi^* \Omega^{d+q-1-j} \wedge \psi^* \psi_* \varphi^* u = \psi^* \Omega^{d+q-1-j} \wedge p_{1*} p_2^* \varphi^* u \\ &= p_{1*} (p_1^* \psi^* \Omega^{d+q-1-j} \wedge p_2^* \varphi^* u) = p_{1*} ((\psi \circ p_1)^* \Omega^{d+q-1-j} \wedge (\varphi \circ p_2)^* u) \end{aligned}$$

and, with

$$q : \Gamma' \rightarrow P(V) \times P(V)$$

defined by $q([x], [x'], s) = ([x], [x'])$, the commutation formula $\varphi \circ p_i = \text{pr}_i \circ q$. Thanks to (12), we conclude that

$$\mathcal{C}^* \mathcal{C}(u) = (\varphi \circ p_1)_* ((\psi \circ p_1)^* \Omega^{d+q-1-j} \wedge (\varphi \circ p_2)^* u)$$

$$\begin{aligned}
&= (\text{pr}_1 \circ q)_* ((\psi \circ p_1)^* \Omega^{d+q-1-j} \wedge (\text{pr}_2 \circ q)^* u) \\
&= \text{pr}_{1*} \{ q_* ((\psi \circ p_1)^* \Omega^{d+q-1-j}) \wedge \text{pr}_2^* u \}.
\end{aligned}$$

Proposition 5. *The integral kernel $\mathcal{K}(\cdot, \cdot)$ of $\mathcal{C}^* \mathcal{C}$ is given by*

$$\mathcal{K} = q_* ((\psi \circ p_1)^* \Omega^{d+q-1-j})$$

and it is a differential form of bidegree $(N-j, N-j)$, which is smooth in $(P(V) \times P(V)) \setminus D_{P(V)}$, where $D_{P(V)}$ is the diagonal of $P(V) \times P(V)$.

Proof. The restriction $q : \Gamma' \setminus q^{-1}(D_{P(V)}) \rightarrow (P(V) \times P(V)) \setminus D_{P(V)}$ is a submersion. $\square$

We now determine the order of singularity of $\mathcal{K}$ near the diagonal $D_{P(V)}$. Let $[x], [x']$ be points in $P(V)$ and let (η_k, η'_k) be a vector in $T_{[x]}P(V) \times T_{[x']}P(V)$ for each $1 \leq k \leq N-j$. First we have

$$\begin{aligned}
&\mathcal{K}_{([x], [x'])}((\eta_1, \eta'_1), \dots, (\eta_{N-j}, \eta'_{N-j}), (\bar{\eta}_1, \bar{\eta}'_1), \dots, (\bar{\eta}_{N-j}, \bar{\eta}'_{N-j})) \\
&= \int_{s \in \text{vect}(x, x')} ((\psi \circ p_1)^* \Omega^{d+q-1-j})_{([x], [x'], s)}((\eta_1, \eta'_1, \sigma_1), \dots, (\eta_{N-j}, \eta'_{N-j}, \sigma_{N-j}), \\
&\quad (\bar{\eta}_1, \bar{\eta}'_1, \bar{\sigma}_1), \dots, (\bar{\eta}_{N-j}, \bar{\eta}'_{N-j}, \bar{\sigma}_{N-j}), \dots) \\
&= \int_{s \in \text{vect}(x, x')} \Omega_s^{d+q-1-j}(\sigma_1, \dots, \sigma_{N-j}, \bar{\sigma}_1, \dots, \bar{\sigma}_{N-j}, \dots) \tag{19}
\end{aligned}$$

where $\sigma_k \in T_s G(q, V)$ is such that $(\eta_k, \eta'_k, \sigma_k) \in T_{([x], [x'], s)} \Gamma'$ and where the fiber

$$q^{-1}([x], [x']) = \{s \in \text{vect}(x, x')\}$$

is identified with $G(q-2, V/(\mathbb{C}x \oplus \mathbb{C}x'))$, which is of dimension $(q-2)(N+1-q)$, when $[x] \neq [x']$.

With $T_{[x]}P(V) = \text{Hom}(\mathbb{C}x, V/\mathbb{C}x)$ and $T_s G(q, V) = \text{Hom}(s, V/s)$, the condition $(\eta, \sigma) \in T_{([x], s)} \Gamma'$ reduces to

$$\sigma(x) = -\eta(x) \bmod s$$

when $x \in s$. Thus we have $\sigma_k(x) = -\eta_k(x) \bmod s$ and $\sigma_k(x') = -\eta'_k(x') \bmod s$ when $\text{vect}(x, x') \subset s$. Set

$$y = \frac{x'}{\|x'\|} - \frac{\langle x'|x \rangle x}{\|x'\| \cdot \|x\|^2}$$

in such a way that $\|y\|^2 = \frac{\|x \wedge x'\|^2}{\|x\|^2 \|x'\|^2} = 1 - \frac{|\langle x'|x \rangle|^2}{\|x\|^2 \|x'\|^2}$. Then we take the lifting σ_k defined by

$$\|y\| \sigma_k = -\|y\| \frac{\langle \cdot | x \rangle}{\|x\|^2} \otimes (\eta_k(x) \bmod s) + \frac{\langle \cdot | y \rangle}{\|y\|} \otimes \left\{ \left(-\frac{\eta'_k(x')}{\|x'\|} + \frac{\langle x'|x \rangle}{\|x'\| \cdot \|x\|} \frac{\eta_k(x)}{\|x\|} \right) \bmod s \right\}.$$

In particular $\|y\| \sigma_k$ is bounded when $\|y\|$ converges to 0. In formula (19), by developping

$$\sigma_1 \wedge \dots \wedge \sigma_{N-j} \wedge \bar{\sigma}_1 \wedge \dots \wedge \bar{\sigma}_{N-j}$$

we choose in σ_k the first term $-\frac{\langle \cdot | x \rangle}{\|x\|^2} \otimes (\eta_k(x) \bmod s)$ or the second. But $\dim(V/s) = N+1-q$, so we choose at most $2(N+1-q)$ times the second. As a consequence, when $\|y\| \rightarrow 0$, we see that

$$\|y\|^{2(N+1-q)} \mathcal{K}_{([x], [x'])}((\eta_1, \eta'_1), \dots, (\eta_{N-j}, \eta'_{N-j}), (\bar{\eta}_1, \bar{\eta}'_1), \dots, (\bar{\eta}_{N-j}, \bar{\eta}'_{N-j}))$$

is bounded.

Theorem 2. *There is a linear differential operator with smooth coefficients*

$$\mathcal{P} : \{\text{currents of bidegree } (q-j, q-j) \text{ on } \mathbb{P}_N\} \rightarrow \{\text{currents of bidegree } (q, q) \text{ on } \mathbb{P}_N\}$$

of order $\leq 2q-2$ such that $\text{id} = \mathcal{P} \mathcal{C}^* \mathcal{C}$.

Proof. Let Q be a homogeneous polynomial function on $\mathbb{R}^m$ of degree b . With $\partial = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m})$, let $Q(\partial)$ be the linear differential operator with constant coefficients associated to Q . If u is a smooth function with compact support on $\mathbb{R}^m$, a solution v of $u = Q(\partial)v$ is given by $v(x) = \int_{z \in \mathbb{R}^m} K(x, z)u(z)dz$ with an integral kernel $K(x, z) = E(x - z)$ where the distribution E satisfies

$$Q(\partial)(E) = \delta \Leftrightarrow (\mathcal{F}E)(\xi) = \frac{1}{Q(2\pi i\xi)} \Leftrightarrow E = \overline{\mathcal{F}}\left(\frac{1}{Q(2\pi i\xi)}\right).$$

Then we have

$$E(\xi) = \int_{\mathbb{R}^m} \frac{1}{Q(2\pi i\xi)} e^{2\pi i\langle x|\xi \rangle} dx = \int_{\mathbb{R}^m} \frac{1}{Q(\frac{2\pi i x}{\|\xi\|})} e^{2\pi i\langle x|\frac{\xi}{\|\xi\|} \rangle} \frac{dx}{\|\xi\|^m} = \frac{1}{\|\xi\|^{m-b}} E\left(\frac{\xi}{\|\xi\|}\right)$$

thus the singularity of K is of order $m - b$ along the diagonal of $\mathbb{R}^m \times \mathbb{R}^m$. So we have here $2N - b = 2(N + 1 - q)$. $\square$

3.5 Approximation of closed currents of bidegree $(1, 1)$ on $G(q, V)$ by divisors

By using an inversion formula similar to that of Theorem 2, we prove now the following approximation property for closed $(1, 1)$ -currents on $G(q, V) = \mathbb{G}_{q-1, N}$ (see also [6]).

Proposition 6. *Every closed $(1, 1)$ -current Θ on $G(q, V)$ can be written*

$$\Theta = \int_{H \in \text{Div } G(q, V)} \lambda(H)[H]$$

where λ is a distribution on the space $\text{Div } G(q, V)$ of divisors of $G(q, V)$ and H is an effective divisor of $G(q, V)$. Moreover λ can be calculated in the following way :

$$\lambda(H) = \int_{s \in G(q, V)} \mathcal{M}(K(s, H)) \wedge \Theta(s)$$

where $\mathcal{M}$ is a linear differential operator with smooth coefficients in H and where the integral kernel K is such that $K(., H)$ weakly converges to $K(., H_0)$ in $G(q, V)$ when H converges to H_0 .

Proof. Let $d_G = q(N + 1 - q)$ be the dimension of $G(q, V)$ and let

$$\mathcal{R} : \mathcal{C}_{d_G-1, d_G-1}^\infty(G(q, V)) \rightarrow \mathcal{C}^\infty(\text{Div } G(q, V))$$

be the Radon transform defined by

$$(\mathcal{R}\varphi)(H) = \int_H \varphi$$

when $\varphi \in \mathcal{C}_{d_G-1, d_G-1}^\infty(G(q, V))$ and H is an effective divisor of $G(q, V)$.

Then $\mathcal{R}\varphi = 0$ if and only if $\varphi = \partial u + \bar{\partial} v$ with $u \in \mathcal{C}_{d_G-2, d_G-1}^\infty(G(q, V))$ and $v \in \mathcal{C}_{d_G-1, d_G-2}^\infty(G(q, V))$.

We identify a current of maximum bidegree on an irreducible component of $\text{Div } G(q, V)$ with a distribution and we consider the transposed map

$${}^t\mathcal{R} : \{\text{distributions on } \text{Div } G(q, V)\} \rightarrow \{(1, 1)\text{-currents on } G(q, V)\}.$$

Its image is

$$\text{im } ({}^t\mathcal{R}) = (\ker \mathcal{R})^\perp = (\text{im } \partial)^\perp \cap (\text{im } \bar{\partial})^\perp = \ker \partial \cap \ker \bar{\partial}.$$

In other words, since $\partial\Theta = 0$ and $\bar{\partial}\Theta = 0$, we can write $\Theta = ({}^t\mathcal{R})(\lambda)$ for some distribution λ on $\text{Div } G(q, V)$.

Moreover, since ${}^t\mathcal{R}$ is an integral transform, by [18] and by analogy with Theorem 2, we have an inversion formula

$$\lambda = (\mathcal{M}\mathcal{I}{}^t\mathcal{R})(\lambda) \bmod \ker ({}^t\mathcal{R})$$

where $\mathcal{M}$ is a linear differential operator with smooth coefficients in H and where $\mathcal{I}$ is an integral transform. Thus we can express

$$\mathcal{I}(\Theta)(H) = \int_{s \in G(q, V)} K(s, H) \wedge \Theta(s)$$

with an integral kernel K satisfying : $K(., H)$ weakly converges to $K(., H_0)$ in $G(q, V)$ when H converges to H_0 .

As a conclusion, we can take

$$\lambda(H) = (\mathcal{M}\mathcal{I})(\Theta)(H) = \int_{s \in G(q, V)} \mathcal{M}(K(s, H)) \wedge \Theta(s). \quad \square$$

4 Chow transformation for $\mathbb{P}_N$ and parabolic subgroups of $SL(N+1, \mathbb{C})$

With the point of view of homogeneous spaces (see [13, 19]), we obtain again the equations (8) and (9) characterizing the closed $(1, 1)$ -currents on $G(q, V)$ which are in the image of the Chow transformation.

So we write

$$\mathbb{P}_N = GL(N+1, \mathbb{C})/GL(1, N, \mathbb{C}) = SL(N+1, \mathbb{C})/(GL(1, N, \mathbb{C}) \cap SL(N+1, \mathbb{C}))$$

where $GL(1, N, \mathbb{C}) = \{g \in GL(N+1, \mathbb{C}) \text{ such that } g(e_0) \text{ proportional to } e_0\}$.

Also we have

$$\mathbb{P}_N = U(N+1)/(U(N+1) \cap GL(1, N, \mathbb{C})) = U(N+1)/K$$

with

$$K = U(N+1) \cap GL(1, N, \mathbb{C}) = U(1) \times U(N).$$

We write an element M in $gl(N+1, \mathbb{C}) = \mathcal{M}_{N+1, N+1}(\mathbb{C})$ in the following manner

$$M = \begin{pmatrix} A & C \\ D & B \end{pmatrix} \quad (20)$$

where $A \in gl(1, \mathbb{C})$, $B \in gl(N, \mathbb{C})$, $C \in n_- = \mathcal{M}_{1, N}(\mathbb{C})$ and $D \in n_+ = \mathcal{M}_{N, 1}(\mathbb{C})$.

We denote by a the complex vector space of the $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ in such a way that $gl(N+1, \mathbb{C}) = a \oplus n_- \oplus n_+$.

The Lie algebra of $U(N+1)$ is

$$\{M \in gl(N+1, \mathbb{C}), M^* = -M \text{ i.e. } M \text{ anti-Hermitian}\}. \quad (21)$$

All $M \in gl(N+1, \mathbb{C})$ can be written in an unique way $M = M_1 + iM_2$ where M_1 and M_2 are anti-Hermitian, thus $gl(N+1, \mathbb{C})$ is the complexified Lie algebra of the Lie algebra of $U(N+1)$.

Moreover M given by (20) is anti-Hermitian when $A^* = -A$, $B^* = -B$ and $C = -D^*$. Thus the Lie algebra of $U(N+1)$ is $a_0 \oplus t_0$ where a_0 is the real vector space of the $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ with A, B anti-Hermitian and where t_0

is the real vector space of the $\begin{pmatrix} 0 & -D^* \\ D & 0 \end{pmatrix}$.

All $\begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}$ can be written in an unique way $\begin{pmatrix} 0 & -D_1^* \\ D_1 & 0 \end{pmatrix} + i \begin{pmatrix} 0 & -D_2^* \\ D_2 & 0 \end{pmatrix}$, thus we have $t_0 \otimes_{\mathbb{R}} \mathbb{C} = n_+ \oplus n_-$.

Note that a_0 is the Lie algebra of K and $a_0 \otimes_{\mathbb{R}} \mathbb{C} = a$. We set

$$p : U(N+1) \rightarrow \mathbb{P}_N$$

the canonical map, which is given by $p(g) = [g(e_0)]$. By differentiating at I we obtain a $\mathbb{R}$ -linear map

$$a_0 \oplus t_0 = T_I U(N+1) \rightarrow T_{[e_0]}(\mathbb{P}_N)_{\mathbb{R}}$$

given by

$$M \rightarrow \frac{d}{dt}[e^{tM}(e_0)]|_{t=0} = \frac{d}{dt}[e_0 + tM(e_0)]|_{t=0} = dp_I(M)$$

where we take $t \in \mathbb{R}$. By complexifying we deduce a $\mathbb{C}$ -linear map

$$gl(N+1, \mathbb{C}) = T_I U(N+1) \otimes_{\mathbb{R}} \mathbb{C} \rightarrow T_{[e_0]}(\mathbb{P}_N)_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = T_{[e_0]} \mathbb{P}_N \oplus \overline{T_{[e_0]} \mathbb{P}_N}$$

given by

$$M \rightarrow \left(\frac{d}{dt}[e_0 + tM(e_0)]|_{t=0}, \overline{\frac{d}{dt}[e_0 - tM^*(e_0)]|_{t=0}} \right)$$

where we take $t \in \mathbb{C}$. Its kernel is a and it gives the identifications

$$n_+ \simeq T_{[e_0]} \mathbb{P}_N \text{ and } n_- \simeq \overline{T_{[e_0]} \mathbb{P}_N}.$$

For $g \in U(N+1)$, we set

$$\lambda_g : (U(N+1), I) \rightarrow (U(N+1), g)$$

the translation map given by $f \rightarrow gf$. The induced $\mathbb{C}$ -linear map

$$T_I U(N+1) \otimes_{\mathbb{R}} \mathbb{C} \rightarrow T_g U(N+1) \otimes_{\mathbb{R}} \mathbb{C} \rightarrow T_{[g(e_0)]} \mathbb{P}_N \oplus \overline{T_{[g(e_0)]} \mathbb{P}_N}$$

gives the identifications

$$n_+ \simeq T_{[g(e_0)]} \mathbb{P}_N \text{ and } n_- \simeq \overline{T_{[g(e_0)]} \mathbb{P}_N}.$$

Let u be a smooth differential $(q-1, q-1)$ -form on $\mathbb{P}_N$. Therefore, from

$$u_{[g(e_0)]} \in \bigwedge^{q-1} T_{[g(e_0)]}^* \mathbb{P}_N \otimes \bigwedge^{q-1} \overline{T_{[g(e_0)]}^* \mathbb{P}_N}$$

we deduce an element $((p \circ \lambda_g)^* u)(I) = \tilde{u}(g) \in \wedge^{q-1} n_+^* \otimes \wedge^{q-1} n_-^*$.

For $k \in K$ and $M \in gl(N+1, \mathbb{C})$, we set

$$ad(k)(M) = \frac{d}{dt}(ke^{tM}k^{-1})|_{t=0} = kMk^{-1}$$

which satisfies $ad(k)(n_+) = n_+$ and $ad(k)(n_-) = n_-$. Since $[k(e_0)] = [e_0]$, we have $p \circ \lambda_{gk} = p \circ \lambda_g \circ ad(k)$ and we can evaluate

$$\tilde{u}(gk) = (ad(k)^*(p \circ \lambda_g)^* u)(I) = ad(k)^* \{((p \circ \lambda_g)^* u)(I)\} = ad(k)^* \tilde{u}(g). \quad (22)$$

Proposition 7. A smooth differential $(q-1, q-1)$ -form u on $\mathbb{P}_N$ corresponds to a smooth function $\tilde{u}$ on $U(N+1)$ with values in $\wedge^{q-1} n_+^* \otimes \wedge^{q-1} n_-^*$, which satisfies the property (22) of right invariance by K .

For $h \in U(N+1)$, let us remark that

$$\widetilde{h^* u}(g) = \tilde{u}(hg)$$

because $((p \circ \lambda_g)^* h^* u)(I) = ((p \circ \lambda_{hg})^* u)(I)$.

We now express in terms of $\tilde{u}$ the Chow transform $\mathcal{C}(u)$ of u , which is a function on $\mathbb{G}_{q-1, N} = G(q, V)$.

Every $s \in G(q, V)$ can be written $s = \text{vect}(g(e_0), \dots, g(e_{q-1})) = g(\mathbb{C}^q)$ for some $g \in U(N+1)$ and $\text{vect}(g'(e_0), \dots, g'(e_{q-1})) = \text{vect}(g(e_0), \dots, g(e_{q-1}))$ if and only if

$$g^{-1}g' \in U(\text{vect}(e_0, \dots, e_{q-1})) \times U(\text{vect}(e_0, \dots, e_{q-1})^\perp) = U(q) \times U(N+1-q).$$

Therefore we have

$$\mathbb{G}_{q-1, N} = U(N+1)/(U(q) \times U(N+1-q)).$$

But $\mathcal{C}(u)$ is obtained by integration of u on the projective subspaces of $\mathbb{P}_N$ of dimension $q - 1$ i.e.

$$\mathcal{C}(u)(\text{vect}(h(e_0), \dots, h(e_{q-1}))) = \int_{P(\text{vect}(h(e_0), \dots, h(e_{q-1})))} u = \int_{\mathbb{P}_{q-1}} h^* u$$

with $P(\text{vect}(e_0, \dots, e_{q-1})) = P(\mathbb{C}^q)$ equal to

$$\mathbb{P}_{q-1} = (U(q) \times U(N+1-q)) / (U(1) \times U(q-1) \times U(N+1-q)). \quad (23)$$

Let $F \in \mathcal{C}^\infty(\mathbb{P}_{q-1})$ be such that $(h^* u)|_{\mathbb{P}_{q-1}} = F \omega|_{\mathbb{P}_{q-1}}^{q-1}$ with ω the Fubini-Study form on $\mathbb{P}_N = P(V)$. A point in $\mathbb{P}_{q-1}$ can be written $[g(e_0)]$ for some $g \in U(q) \times U(N+1-q)$ and

$$T_{[g(e_0)]} \mathbb{P}_{q-1} = \text{Hom}(\mathbb{C}g(e_0), \mathbb{C}^q / \mathbb{C}g(e_0)).$$

Since $g(e_0), \dots, g(e_{q-1})$ is an orthonormal basis of $\mathbb{C}^q$, an orthonormal basis of $T_{[g(e_0)]} \mathbb{P}_{q-1}$ is given by the

$$w_k = g(e_0)^{-1} \otimes (g(e_k) \bmod \mathbb{C}g(e_0)) = \frac{d}{dt} [g(e_0) + tg(e_k)]|_{t=0} = \frac{d}{dt} [g(I + tM_k)(e_0)]|_{t=0}$$

for $1 \leq k \leq q-1$ and with $M_k = e_0^* \otimes e_k \in n_+$ and $M_k^* = e_k^* \otimes e_0 \in n_-$. Thus

$$\begin{aligned} \widetilde{h^* u}(g)(M_1, \dots, M_{q-1}, -M_1^*, \dots, -M_{q-1}^*) &= (h^* u)_{[g(e_0)]}(w_1, \dots, w_{q-1}, \overline{w}_1, \dots, \overline{w}_{q-1}) \\ &= F([g(e_0)]) \omega_{[g(e_0)]}^{q-1}(w_1, \dots, w_{q-1}, \overline{w}_1, \dots, \overline{w}_{q-1}) \end{aligned}$$

which is equal to

$$\begin{aligned} &F([g(e_0)]) \omega_{[e_0]}^{q-1} \left(\frac{d}{dt} [e_0 + te_1]|_{t=0}, \dots, \frac{d}{dt} [e_0 + te_{q-1}]|_{t=0}, \overline{\frac{d}{dt} [e_0 + te_1]|_{t=0}}, \dots, \overline{\frac{d}{dt} [e_0 + te_{q-1}]|_{t=0}} \right) \\ &= \frac{i^{(q-1)^2}}{(2\pi)^{q-1}} (q-1)! F([g(e_0)]). \end{aligned}$$

Lemma 4. The smooth function F on $\mathbb{P}_{q-1}$ such that $(h^* u)|_{\mathbb{P}_{q-1}} = F \omega|_{\mathbb{P}_{q-1}}^{q-1}$ is given by

$$F([g(e_0)]) = C^{-1} \tilde{u}(hg)(M_1, \dots, M_{q-1}, M_1^*, \dots, M_{q-1}^*)$$

with $C = \frac{i^{q^2-1}}{(2\pi)^{q-1}} (q-1)!$ for every $g \in U(q) \times U(N+1-q)$.

As a consequence, if we write $s = h(\mathbb{C}^q)$ with $h \in U(N+1)$, we have

$$\begin{aligned} \mathcal{C}(u)(s) &= \int_{g \in U(q) \times U(N+1-q)} F([g(e_0)]) d\mu(g) \\ &= C^{-1} \int_{g \in U(q) \times U(N+1-q)} \tilde{u}(hg)(M_1, \dots, M_{q-1}, M_1^*, \dots, M_{q-1}^*) d\mu(g) \end{aligned}$$

where μ is the left invariant measure on $U(q) \times U(N+1-q)$ with total mass equal to 1.

Note that we also have

$$\mathbb{G}_{q-1, N} = GL(N+1, \mathbb{C}) / GL(q, N+1-q, \mathbb{C})$$

where $GL(q, N+1-q, \mathbb{C}) = \{g \in GL(N+1, \mathbb{C}) \text{ such that } g(\text{vect}(e_0, \dots, e_{q-1})) = \text{vect}(e_0, \dots, e_{q-1})\}$.

Moreover $p(g) = [g(e_0)]$ is still defined as a holomorphic map

$$p : GL(N+1, \mathbb{C}) \rightarrow \mathbb{P}_N$$

with dp_g the differential at g obtained from

$$M \rightarrow \frac{d}{dt} [g(I + tM)(e_0)]|_{t=0}$$

and which still gives the identification $n_+ \simeq T_{[g(e_0)]} \mathbb{P}_N$. Therefore the vectorial function $\tilde{u}(g)$ is still defined for $g \in GL(N+1, \mathbb{C})$.

Lemma 5. For $s = h(\mathbb{C}^q)$ with $h \in GL(N+1, \mathbb{C})$, we have

$$\mathcal{C}(u)(s) = C^{-1} \int_{g \in U(q) \times U(N+1-q)} \tilde{u}(hg)(M_1, \dots, M_{q-1}, M_1^*, \dots, M_{q-1}^*) d\mu(g).$$

By writing $s = \text{vect}(h(e_0), \dots, h(e_{q-1}))$, we denote by $(h_{ik})_{0 \leq i \leq N}$ the coordinates of $h(e_k)$ with respect to the basis $e_0, \dots, e_N$. For $I = (i_0, \dots, i_{q-1})$ with $0 \leq i_0 < \dots < i_{q-1} \leq N$, we consider the differential operator

$$\det\left(\frac{\partial}{\partial h_{ik}}\right)_{\substack{i \in I \\ 0 \leq k \leq q-1}} = \sum_{\sigma} \varepsilon(\sigma) \frac{\partial^q}{\partial h_{i_{\sigma_0}, 0} \dots \partial h_{i_{\sigma_{q-1}}, q-1}}$$

where σ is each permutation of $\{0, \dots, q-1\}$. We evaluate on the previous function $\mathcal{C}(u)$ of h the differential operator

$$\det\left(\frac{\partial}{\partial h_{ik}}\right)_{\substack{i \in I \\ 0 \leq k \leq q-1}} \det\left(\frac{\partial}{\partial h_{ik}}\right)_{\substack{i \in J \\ 0 \leq k \leq q-1}}.$$

To this end we use that

$$\frac{\partial^q}{\partial h_{i_{\sigma_0}, 0} \dots \partial h_{i_{\sigma_{q-1}}, q-1}} (\tilde{u}(hg)) = \sum_{0 \leq m_0, \dots, m_{q-1} \leq N} \frac{\partial^q \tilde{u}}{\partial h'_{i_{\sigma_0}, m_0} \dots \partial h'_{i_{\sigma_{q-1}}, m_{q-1}}} (hg) g_{0, m_0} \dots g_{q-1, m_{q-1}}$$

which gives

$$\left\{ \det\left(\frac{\partial}{\partial h_{ik}}\right)_{\substack{i \in I \\ 0 \leq k \leq q-1}} \right\} (\tilde{u}(hg)) = \sum_{|M|=q} \left(\left\{ \det\left(\frac{\partial}{\partial h'_{im}}\right)_{\substack{i \in I \\ m \in M}} \right\} \tilde{u}(hg) \det(g_{km})_{\substack{0 \leq k \leq q-1 \\ m \in M}} \right)$$

with $M = (m_0, \dots, m_{q-1})$ where $0 \leq m_0 < \dots < m_{q-1} \leq N$. Since $g \in U(q) \times U(N+1-q)$ we can take $0 \leq m_k \leq q-1$ thus necessarily $m_k = k$. So we conclude that

$$\left\{ \det\left(\frac{\partial}{\partial h_{ik}}\right)_{\substack{i \in I \\ 0 \leq k \leq q-1}} \det\left(\frac{\partial}{\partial h_{ik}}\right)_{\substack{i \in J \\ 0 \leq k \leq q-1}} \right\} (\tilde{u}(hg)) = \left(\left\{ \det\left(\frac{\partial}{\partial h'_{ik}}\right)_{\substack{i \in I \\ 0 \leq k \leq q-1}} \det\left(\frac{\partial}{\partial h'_{ik}}\right)_{\substack{i \in J \\ 0 \leq k \leq q-1}} \right\} \tilde{u}(hg) \right) \quad (24)$$

since

$$|\det(g_{km})_{\substack{0 \leq k \leq q-1 \\ 0 \leq m \leq q-1}}|^2 = 1.$$

Lemma 6. The smooth function on $U(N+1)$

$$g \rightarrow \left(\left\{ \det\left(\frac{\partial}{\partial h'_{il}}\right)_{\substack{i \in I \\ 0 \leq l \leq q-1}} \det\left(\frac{\partial}{\partial h'_{il}}\right)_{\substack{i \in J \\ 0 \leq l \leq q-1}} \right\} \tilde{u}(g) \right) (M_1, \dots, M_{q-1}, M_1^*, \dots, M_{q-1}^*) = \tilde{f}_{IJ}(g)$$

is invariant by the right action of $K = U(1) \times U(N)$ and thus induces a smooth function f_{IJ} on $\mathbb{P}_N$.

Proof. For all $g \in GL(N+1, \mathbb{C})$ and all $k \in K$, we rewrite formula (22) i.e.

$$\tilde{u}(gk) = ad(k)^* \tilde{u}(g).$$

By applying the following differential operator

$$\det\left(\frac{\partial}{\partial g_{il}}\right)_{\substack{i \in I \\ 0 \leq l \leq q-1}} \det\left(\frac{\partial}{\partial g_{il}}\right)_{\substack{i \in J \\ 0 \leq l \leq q-1}}$$

to both members, we obtain for all $k \in K$ the relation

$$\begin{aligned} \sum_{|M|=|M'|=q} \left(\left\{ \det\left(\frac{\partial}{\partial h'_{im}}\right)_{\substack{i \in I \\ m \in M}} \det\left(\frac{\partial}{\partial h'_{im}}\right)_{\substack{i \in J \\ m \in M'}} \right\} \tilde{u}(gk) \det(k_{lm})_{\substack{0 \leq l \leq q-1 \\ m \in M}} \det(\bar{k}_{lm})_{\substack{0 \leq l \leq q-1 \\ m \in M'}} \right) \\ = ad(k)^* \left(\left\{ \det\left(\frac{\partial}{\partial h'_{il}}\right)_{\substack{i \in I \\ 0 \leq l \leq q-1}} \det\left(\frac{\partial}{\partial h'_{il}}\right)_{\substack{i \in J \\ 0 \leq l \leq q-1}} \right\} \tilde{u}(g) \right). \end{aligned}$$

By evaluating on $M_1 \wedge \dots \wedge M_{q-1} \wedge M_1^* \wedge \dots \wedge M_{q-1}^*$ and by identifying, it comes

$$\left(\left\{ \det\left(\frac{\partial}{\partial h'_{il}}\right)_{\substack{i \in I \\ 0 \leq l \leq q-1}} \det\left(\frac{\partial}{\partial h'_{il}}\right)_{\substack{i \in J \\ 0 \leq l \leq q-1}} \right\} \tilde{u}(gk) (M_1, \dots, M_{q-1}, M_1^*, \dots, M_{q-1}^*) \right) |\det(k_{lm})_{\substack{0 \leq l \leq q-1 \\ 0 \leq m \leq q-1}}|^2$$

$$= \left(\left\{ \det\left(\frac{\partial}{\partial h'_{il}}\right)_{\substack{j \in I \\ 0 \leq l \leq q-1}} \det\left(\frac{\partial}{\partial h'_{jl}}\right)_{\substack{j \in J \\ 0 \leq l \leq q-1}} \right\} \tilde{u} \right)(g)(M_1, \dots, M_{q-1}, M_1^*, \dots, M_{q-1}^*) \left| \det(k_{lm})_{\substack{1 \leq l \leq q-1 \\ 1 \leq m \leq q-1}} \right|^2$$

thanks to the development

$$(e_0^* \otimes k(e_1)) \wedge \dots \wedge (e_0^* \otimes k(e_{q-1})) = \sum_{0 \leq m_1, \dots, m_{q-1} \leq N} k_{m_1,1} \dots k_{m_{q-1},q-1} (e_0^* \otimes e_{m_1}) \wedge \dots \wedge (e_0^* \otimes e_{m_{q-1}}).$$

□

As a conclusion, for $h \in GL(N+1, \mathbb{C})$ we obtain

$$\left\{ \det\left(\frac{\partial}{\partial h_{ik}}\right)_{\substack{j \in I \\ 0 \leq k \leq q-1}} \det\left(\frac{\partial}{\partial h_{jk}}\right)_{\substack{j \in J \\ 0 \leq k \leq q-1}} \right\} (\mathcal{C}(u)(h(\mathbb{C}^q))) = C^{-1} \int_{g \in U(q) \times U(N+1-q)} \tilde{f}_{IJ}(hg) d\mu(g).$$

On the other hand, the projective $(q-1)$ -dimensional Radon transform of f_{IJ} is given by

$$\mathcal{R}_{q-1}(f_{IJ})(s) = \int_{P(s)} f_{IJ} \omega^{q-1} = \int_{\mathbb{P}^{q-1}} (h^* f_{IJ}) \omega^{q-1} = \int_{g \in U(q) \times U(N+1-q)} \tilde{f}_{IJ}(hg) d\mu(g) \quad (25)$$

by writing $s = h(\mathbb{C}^q)$ with $h \in U(N+1)$ and by using (23).

To retrieve equations (8) and (9), it remains to evaluate on $\mathcal{R}_{q-1}(f_{IJ})$ the differential operator

$$\frac{\partial^2}{\partial h_{ik} \partial h_{i'k'}} - \frac{\partial^2}{\partial h_{i'k} \partial h_{ik'}}$$

with $0 \leq k, k' \leq q-1$ and $0 \leq i, i' \leq N$. To this end we use that

$$\frac{\partial^2}{\partial h_{ik} \partial h_{i'k'}} (\tilde{f}_{IJ}(hg)) = \sum_{0 \leq m, m' \leq N} \left(\frac{\partial^2 \tilde{f}_{IJ}}{\partial h'_{im} \partial h'_{i'm'}} \right) (hg) g_{km} g_{k'm'}$$

with by definition

$$\left(\frac{\partial^2 \tilde{f}_{IJ}}{\partial h'_{im} \partial h'_{i'm'}} \right) (g) = \frac{\partial^2}{\partial t \partial s} (\tilde{f}_{IJ}(g + te_m^* \otimes e_i + se_{m'}^* \otimes e_{i'}))|_{(t,s)=(0,0)}.$$

But by Lemma 6 we can express

$$\tilde{f}_{IJ}(g + te_m^* \otimes e_i + se_{m'}^* \otimes e_{i'}) = f_{IJ}([g(e_0) + t\delta_{m,0}e_i + s\delta_{m',0}e_{i'}]).$$

Necessarily $m = 0 = m'$, so it is equal to $f_{IJ}([g(e_0) + te_i + se_{i'}])$ and the derivative of order 2 in $(0,0)$ is invariant by permuting i and i' .

Theorem 3. The Chow transform $U = \mathcal{C}(u)$ of a smooth differential $(q-1, q-1)$ -form u on $\mathbb{P}_N$ is a smooth function on $\mathbb{G}_{q-1,N}$. The smooth function $\tilde{U}(h) = U(h(\mathbb{C}^q))$ for $h \in GL(N+1, \mathbb{C})$ satisfies the linear differential equations

$$\left(\frac{\partial^2}{\partial h_{ik} \partial h_{i'k'}} - \frac{\partial^2}{\partial h_{i'k} \partial h_{ik'}} \right) \left\{ \det\left(\frac{\partial}{\partial h_{jl}}\right)_{\substack{j \in I \\ 0 \leq l \leq q-1}} \det\left(\frac{\partial}{\partial h_{jl}}\right)_{\substack{j \in J \\ 0 \leq l \leq q-1}} \right\} \tilde{U} = 0$$

and

$$\left(\frac{\partial^2}{\partial \bar{h}_{ik} \partial \bar{h}_{i'k'}} - \frac{\partial^2}{\partial \bar{h}_{i'k} \partial \bar{h}_{ik'}} \right) \left\{ \det\left(\frac{\partial}{\partial \bar{h}_{jl}}\right)_{\substack{j \in I \\ 0 \leq l \leq q-1}} \det\left(\frac{\partial}{\partial \bar{h}_{jl}}\right)_{\substack{j \in J \\ 0 \leq l \leq q-1}} \right\} \tilde{U} = 0$$

for all $|I| = |J| = q$, all $0 \leq k, k' \leq q-1$ and all $0 \leq i, i' \leq N$.

5 Chow transformation and geometry of the cotangent bundle

5.1 Chow transform of the conormal of a current

Here we use the geometry of the cotangent bundle (see [11, 17]) to study the injectivity of the Chow transformation.

For $([x], s) \in \Gamma$, the fiber of the conormal to Γ in $P(V) \times G(q, V)$ at $([x], s)$ is

$$N_{([x], s)}^* \Gamma = \left((T_{[x]} P(V) \times T_s G(q, V)) / T_{([x], s)} \Gamma \right)^*$$

i.e. the space of linear forms on $T_{[x]} P(V) \times T_s G(q, V)$ which are equal to 0 on $T_{([x], s)} \Gamma$. Therefore

$$N^* \Gamma \hookrightarrow T^*(P(V) \times G(q, V))|_{\Gamma} \hookrightarrow T^* P(V) \times T^* G(q, V).$$

We still denote by $\pi : V \setminus \{0\} \rightarrow P(V)$ the natural map.

Lemma 7. *Let $(\xi, u) \in T_{[x]}^* P(V) \times T_s^* G(q, V)$ be such that $\xi \neq 0$ and $u \neq 0$. There is $c \in \mathbb{C}^*$ such that $(\xi, cu) \in N_{([x], s)}^* \Gamma$ if and only if*

- (i) $x \in s$,
- (ii) $s \subset d\pi_x^{-1}(\ker \xi) \stackrel{\text{def}}{=} H \subset V$,
- (iii) $\ker u = \{\sigma \in \text{Hom}(s, V/s), \sigma(x) \in H/s\}$.

Proof. Let E be the holomorphic vector bundle on $P(V) \times G(q, V)$ defined by

$$E_{([x], s)} = \text{Hom}(\mathbb{C}x, V/s)$$

and let h be the holomorphic section of E equal at $([x], s)$ to the natural morphism $\mathbb{C}x \rightarrow V/s$. The set of zeroes of h is Γ . With $(e_0, \dots, e_{N-q})$ a holomorphic local frame of E , we can write $h = \sum_i h_i e_i$. At a point $([x], s)$ of Γ , then $\sum_i dh_i \otimes e_i$ is a surjective morphism

$$T_{([x], s)}(P(V) \times G(q, V)) \rightarrow E_{([x], s)}$$

independent of the choice of the holomorphic local frame. The kernel of this morphism is $T_{([x], s)} \Gamma$, so the normal bundle to Γ is isomorphic to $E|_{\Gamma}$ and

$$N_{([x], s)}^* \Gamma = \text{Hom}(V/s, \mathbb{C}x).$$

We consider $\xi \in \text{Hom}(V/\mathbb{C}x, \mathbb{C}x)$ and $u \in \text{Hom}(V/s, s)$. There is $c \in \mathbb{C}^*$ such that $(\xi, cu) \in N_{([x], s)}^* \Gamma$ if and only if there is $\zeta \in \text{Hom}(V/s, \mathbb{C}x)$ satisfying

$$\xi = \zeta \circ p \text{ and } u \text{ proportional to } i \circ \zeta$$

with $p : V/\mathbb{C}x \rightarrow V/s$ and $i : \mathbb{C}x \rightarrow s$ the natural maps.

The existence of ζ such that $\xi = \zeta \circ p$ is equivalent to $\ker p \subset \ker \xi$ i.e. $s \subset H$ and then we have $\zeta(y \bmod s) = \xi(y \bmod \mathbb{C}x)$ for all $y \in V$. By writing $\zeta = x \otimes \lambda$ with $\lambda \in (V/s)^*$, we have for $\sigma \in \text{Hom}(s, V/s)$ the equality $\langle i \circ \zeta, \sigma \rangle = \lambda(\sigma(x))$. In such a way, the hyperplane orthogonal to $i \circ \zeta$ is

$$\ker u = \{\sigma \in \text{Hom}(s, V/s), \sigma(x) \in \ker \lambda\} = \{\sigma \in \text{Hom}(s, V/s), \sigma(x) \in H/s\}. \quad \square$$

We denote by Φ respectively Ψ the restriction to the conormal $N^* \Gamma$ of the canonical projection $T^* P(V) \times T^* G(q, V) \rightarrow T^* P(V)$ respectively $T^* P(V) \times T^* G(q, V) \rightarrow T^* G(q, V)$.

Let $Z \subset P(V)$ be an irreducible algebraic subset of codimension q . Then

$$N^* Z = (TP(V)|_Z / TZ)^* \hookrightarrow T^* P(V)|_Z \hookrightarrow T^* P(V).$$

We use the double fibration

$$\begin{array}{ccc} & N^* \Gamma & \\ \swarrow & & \searrow \\ T^* P(V) & & T^* G(q, V) \end{array}$$

to transform $N^* Z$.

Proposition 8. For $Z \subset P(V)$ algebraic with pure codimension q , let $\Sigma = \mathcal{C}(Z)$ be the Chow divisor of Z . Then we have $\Psi_*\Phi^*(N^*Z) = N^*\Sigma$ and conversely $\Phi_*\Psi^*(N^*\Sigma) = N^*Z$.

Proof. For the first equality, let $u \in T_s^*G(q, V)$ be such that $u \in \Psi_*\Phi^*(N^*Z)$. There is $\xi \in N^*Z$ such that $(\xi, u) \in N^*\Gamma$, then $\xi \in T_{[x]}^*P(V)$ with $[x] \in Z$. By Lemma 7, x belongs to s so $P(s)$ cuts Z and then $s \in \Sigma$. Since ξ is equal to 0 on $T_{[x]}Z + T_{[x]}P(s)$ which is a hyperplane in $T_{[x]}P(V)$, we have $\ker \xi = T_{[x]}Z + T_{[x]}P(s)$ i.e. $H = d\pi_x^{-1}(\ker \xi) = s + d\pi_x^{-1}(T_{[x]}Z)$. By [25] we can assume $Z \cap P(s) = \{[x]\}$ and then

$$T_s\Sigma = \{\sigma \in \text{Hom}(s, V/s), \sigma(x) \in (s + T_x\pi^{-1}(Z))/s\}. \quad (26)$$

In such a way, by Lemma 7, we obtain $\ker u = T_s\Sigma$, thus $u \in N^*\Sigma$.

For the second equality, let $\xi \in T_{[x]}^*P(V)$ be such that $\xi \in \Phi_*\Psi^*(N^*\Sigma)$. There is $u \in N^*\Sigma$ such that $(\xi, u) \in N^*\Gamma$. Then $u \in T_s^*G(q, V)$ with some $s \in \Sigma$. With $\{[x]\} = Z \cap P(s)$, we apply (26) to obtain

$$\ker u = \{\sigma \in \text{Hom}(s, V/s), \sigma(x) \in (s + T_x\pi^{-1}(Z))/s\}.$$

Then by Lemma 7, we have $\ker \xi \supset T_{[x]}Z$, thus $\xi \in N^*Z$. □

We now extend the formula

$$N^*Z = \Phi_*\Psi^*\Psi_*\Phi^*(N^*Z) \quad (27)$$

resulting of Proposition 8, by replacing Z by a closed positive current T of bidegree (q, q) on $P(V)$ and by writing as in Proposition 3

$$T = \lim_{\nu} Z_{\nu} \text{ and } T^* = \lim_{\nu} Z_{\nu}^*$$

with Z_{ν} effective algebraic cycle of codimension q in $P(V)$ and Z_{ν}^* the dual cycle of Z_{ν} in $P(V^*)$.

First we recall that there is a surjective morphism

$$\alpha : T^*P(V) \setminus 0_{P(V)} \rightarrow \{([x], [\lambda]) \in P(V) \times P(V^*) \text{ such that } \lambda(x) = 0\}$$

which associates $\alpha(\xi) = ([x], [\lambda])$ to $\xi \in T_{[x]}^*P(V) = \text{Hom}(V/\mathbb{C}x, \mathbb{C}x)$ equal to $\lambda \otimes x$ with $\lambda(x) = 0$.

Then $\xi \in N_{[x]}^*Z$ if and only if $\ker \lambda \supset d\pi_x^{-1}(T_{[x]}Z)$ i.e. $\alpha(\xi)$ belongs to W_Z with

$$W_Z = \{([x], [\lambda]) \in P(V) \times P(V^*) \text{ such that } [x] \in Z \text{ and } P(\ker \lambda) \supset \mathbb{T}_{[x]}Z\}$$

where $\mathbb{T}_{[x]}Z = P(d\pi_x^{-1}(T_{[x]}Z)) \subset P(V)$.

In other words $(N^*Z) \setminus 0_Z = \alpha^{-1}(W_Z)$ and W_Z is an algebraic subvariety of $P(V) \times P(V^*)$ with $\dim W_Z = N - 1$. The image by pr_2 of W_Z is $\text{pr}_2(W_Z) = Z^* \subset P(V^*)$, for which we expect that

$$\text{codim } Z^* = N + 1 - q \Leftrightarrow \dim Z^* = q - 1 = \text{codim } Z - 1$$

by Proposition 3. We equip $P(V^*)$ with the dual metric ω^* and $P(V) \times P(V^*)$ with the induced metric.

Proposition 9. In the sense of currents, we have the following equalities

$$(i) [N^*Z] = \alpha^*[W_Z],$$

$$(ii) \binom{N-1}{q-1} \text{pr}_{2*} \left(\frac{[W_Z]}{\deg W_Z} \wedge \text{pr}_1^* \omega^{N-q} \right) = \frac{[Z^*]}{\deg Z^*} \text{ where } \deg W_Z = \binom{N-1}{q-1} \deg Z \text{ is the degree of } W_Z.$$

Proof. With β the restriction of pr_2 to W_Z , the following diagram

$$\begin{array}{ccc} W_Z & \xrightarrow{i_0} & P(V) \times P(V^*) \\ \beta \downarrow & & \downarrow \text{pr}_2 \\ Z^* & \xrightarrow{j} & P(V^*) \end{array}$$

is commutative, where i_0 and j are the natural injections. With these notations, we have

$$\text{pr}_{2*}([W_Z] \wedge \text{pr}_1^* \omega^{N-q}) = \text{pr}_{2*} i_{0*} i_0^* \text{pr}_1^* \omega^{N-q} = j_* \beta_* ((\text{pr}_1^* \omega^{N-q})|_{W_Z}).$$

The fibers of β are of dimension $N - q = \dim W_Z - \dim Z^*$ and the function

$$\beta_*((\mathrm{pr}_1^* \omega^{N-q})|_{W_Z})([\lambda]) = \int_{\beta^{-1}([\lambda])} \omega^{N-q}$$

is continuous with respect to $[\lambda] \in Z^*$. Since it is a positive integer, when Z is irreducible, this function is a constant M . We obtain

$$\mathrm{pr}_{2*}([W_Z] \wedge \mathrm{pr}_1^* \omega^{N-q}) = M[Z^*].$$

Then by testing on $(\omega^*)^{q-1}$ it comes

$$M \deg Z^* = M \int_{Z^*} (\omega^*)^{q-1} = \int_{W_Z} \mathrm{pr}_1^* \omega^{N-q} \wedge \mathrm{pr}_2^* (\omega^*)^{q-1}$$

and it remains to evaluate $\deg W_Z$. By the binomial formula, for $0 \leq k \leq N-1$, we have to calculate

$$\int_{W_Z} \mathrm{pr}_1^* \omega^{N-1-k} \wedge \mathrm{pr}_2^* (\omega^*)^k = \int_{P(V^*)} \mathrm{pr}_{2*}([W_Z] \wedge \mathrm{pr}_1^* \omega^{N-1-k}) \wedge (\omega^*)^k.$$

But in a similar manner

$$\mathrm{pr}_{2*}([W_Z] \wedge \mathrm{pr}_1^* \omega^{N-1-k}) = j_* \beta_*((\mathrm{pr}_1^* \omega^{N-1-k})|_{W_Z})$$

and $\beta_*((\mathrm{pr}_1^* \omega^{N-1-k})|_{W_Z})$ is of bidegree $\dim Z^* - k$, therefore $k \leq q-1$. On the other hand we have

$$\int_{W_Z} \mathrm{pr}_1^* \omega^{N-1-k} \wedge \mathrm{pr}_2^* (\omega^*)^k = \int_{P(V)} \mathrm{pr}_{1*}([W_Z] \wedge \mathrm{pr}_2^* (\omega^*)^k) \wedge \omega^{N-1-k}$$

and with γ the restriction of pr_1 to W_Z we use the commutative diagram

$$\begin{array}{ccc} W_Z & \xrightarrow{i_0} & P(V) \times P(V^*) \\ \gamma \downarrow & & \downarrow \mathrm{pr}_1 \\ Z & \xrightarrow{j_1} & P(V) \end{array}$$

which allows to express

$$\mathrm{pr}_{1*}([W_Z] \wedge \mathrm{pr}_2^* (\omega^*)^k) = j_{1*} \gamma_*((\mathrm{pr}_2^* (\omega^*)^k)|_{W_Z}).$$

Now $\gamma_*((\mathrm{pr}_2^* (\omega^*)^k)|_{W_Z})$ is of bidegree $k - (q-1)$, therefore $k \geq q-1$ thus $k = q-1$. We conclude that

$$\binom{N-1}{q-1} \deg W_Z = \int_{W_Z} \mathrm{pr}_1^* \omega^{N-q} \wedge \mathrm{pr}_2^* (\omega^*)^{q-1} = \int_Z \omega^{N-q} \gamma_*((\mathrm{pr}_2^* (\omega^*)^{q-1})|_{W_Z}).$$

Since the fibers of γ are projective subspaces of dimension $q-1$, we have

$$\gamma_*((\mathrm{pr}_2^* (\omega^*)^{q-1})|_{W_Z}) = 1$$

thus the equalities $M \deg Z^* = \binom{N-1}{q-1} \deg W_Z = \deg Z$. □

When $T = \lim_{\nu} Z_{\nu}$, then $W_{Z_{\nu}}$ has bounded degree by Proposition 9. Therefore we can assume that $W_{Z_{\nu}}$ converges to a closed positive current on $P(V) \times P(V^*)$, with support in the incidence manifold

$$\{([x], [\lambda]) \in P(V) \times P(V^*) \text{ such that } \lambda(x) = 0\}.$$

This current is a direct image of a closed positive current W_T defined on this manifold.

Let us now prove that W_T does not depend on the sequence Z_{ν} . In effect, the map $W_Z \rightarrow Z^*$ is injective, since

$$Z^* = 0 \Rightarrow Z = 0 \Rightarrow W_Z = 0.$$

Thus there is a continuous linear operator S such that $W_Z = S(Z^*)$, inverting the formula (ii) of Proposition 9. Since $Z_{\nu}^* \rightarrow T^*$ by Proposition 3, then $W_{Z_{\nu}}$ converges to $S(T^*)$. So $W_T = S(T^*)$ is independent of Z_{ν} .

Theorem 4. For all closed positive (q, q) -currents T on $P(V)$, the conormal current $\text{con}(T)$ is a closed positive current on $T^*P(V)$ defined by

$$\text{con}(T) = \alpha^*(W_T)$$

and satisfying the inversion formula for the Chow transformation

$$\text{con}(T) = \Phi_* \Psi^* \Psi_* \Phi^*(\text{con}(T)).$$

5.2 Expression of $\mathcal{C}^*\mathcal{C}$ from the Crofton formula

For u a smooth differential (q, q) -form on $P(V)$, we now calculate $\mathcal{C}^*\mathcal{C}(u)$ from formula (12) by expressing $\Omega^{d+q-1-j}$ thanks to a Crofton formula.

Recall that $d = (q-1)(N+1-q) = d_G - N - 1 + q$ with $d_G = \dim G(q, V)$, therefore $d + q - 1 - j = d_G - (N + 2 + j - 2q)$. Since $j = \max(0, 2q - N - 1)$, the bidimension of $\Omega^{d+q-1-j}$ is

$$\begin{aligned} & (1, 1) && \text{if } 2q \geq N + 1 \\ & (N + 2 - 2q, N + 2 - 2q) && \text{if } 2q \leq N + 1 \end{aligned}$$

and when $j \neq 0$ the cohomology class $\{\Omega\}^{d+q-1-j} = \{\Omega\}^{d_G-1}$ on $G(q, V)$ contains a multiple of a Schubert cycle σ (see [12, 30]).

In effect, if $2q \geq N + 1$, let $V_{q-1} \subset V_{q+1}$ be vectorial subspaces of $V = \mathbb{C}^{N+1}$ of respective dimensions $q-1$ and $q+1$. Then we can take

$$\sigma = \sigma_{V_{q-1}, V_{q+1}} = \{s \in G(q, V), V_{q-1} \subset s \subset V_{q+1}\} = P(V_{q+1}/V_{q-1}) \simeq \mathbb{P}_1$$

because $V_{q-1} \subset s \subset V_{q+1}$ is equivalent to a system of homogeneous equations of degree 1.

In this case the Crofton formula can be stated as

$$\Omega^{d_G-1} = K \int_{V_{q-1} \subset V_{q+1}} [\sigma] d\nu = K \int_{V_{q-1} \subset V_{q+1}} [\sigma_{V_{q-1}, V_{q+1}}] d\nu(V_{q-1}, V_{q+1})$$

where K is a constant factor and where ν is the measure on

$$\{(V_{q-1}, V_{q+1}) \in G(q-1, V) \times G(q+1, V) \text{ such that } V_{q-1} \subset V_{q+1}\}$$

associated to the Hermitian metric induced by that of $G(q-1, V) \times G(q+1, V)$. For the proof, we note that

$$\int_{V_{q-1} \subset V_{q+1}} [\sigma] d\nu$$

is invariant by the unitary group $U(N+1) = U(V)$, thus is harmonic. Since it is cohomologous on $G(q, V)$ to a multiple of Ω^{d_G-1} , it is necessarily equal to a multiple of Ω^{d_G-1} .

With $i: \sigma \rightarrow G(q, V)$ the canonical injection, we have the expected expression

$$K^{-1} \mathcal{C}^* \mathcal{C}(u) = \int_{V_{q-1} \subset V_{q+1}} \varphi_* \psi^* ([\sigma] \wedge \psi_* \varphi^* u) d\nu = \int_{V_{q-1} \subset V_{q+1}} \varphi_* \psi^* i_* i^* \psi_* \varphi^* u d\nu$$

where $\varphi_* \psi^* i_*$ is the transposed map of $i^* \psi_* \varphi^*$.

Now we replace u by the integration current $[Z]$ associated to an effective algebraic cycle Z of $P(V)$ of codimension q and of degree $\deg Z$. First

$$\mathcal{C}(Z) = \{s \in G(q, V), P(s) \cap Z \neq \emptyset\} \Rightarrow i^* \mathcal{C}(Z) = \{s \in G(q, V), V_{q-1} \subset s \subset V_{q+1} \text{ and } P(s) \cap Z \neq \emptyset\}.$$

We can assume that $P(V_{q+1}) \cap Z$ is a finite set of $\deg Z$ points $[x_1], [x_2], \dots$ and so

$$i^* \mathcal{C}(Z) = \{V_{q-1} \oplus \mathbb{C}x_1, V_{q-1} \oplus \mathbb{C}x_2, \dots\}.$$

As a consequence we obtain the formula

$$\varphi_* \psi^* i_* i^* \mathcal{C}(Z) = \sum_{1 \leq l \leq \deg Z} [P(V_{q-1} \oplus \mathbb{C}x_l)]$$

which allows to reprove directly the injectivity of the map $Z \rightarrow \mathcal{C}^* \mathcal{C}([Z])$ obtained by the following result.

Proposition 10. *For Z an effective algebraic cycle of $P(V)$ of codimension q , the singular support of the current $\mathcal{C}^* \mathcal{C}([Z])$ is*

$$\text{sing supp } (\mathcal{C}^* \mathcal{C}([Z])) = \text{supp } Z.$$

Proof. With $\Sigma = \mathcal{C}(Z)$ the Chow divisor of Z and $\psi^{-1} \Sigma$ the inverse image divisor in Γ , we have

$$\mathcal{C}^* \mathcal{C}([Z]) = \varphi_* \psi^* (\Omega^{d+q-1-j} \wedge [\Sigma]) = (\varphi|_{\psi^{-1} \Sigma})_* ((\psi^* \Omega^{d+q-1-j})|_{\psi^{-1} \Sigma}).$$

For $[x] \in P(V)$, let us evaluate the dimension of the fiber $(\varphi|_{\psi^{-1} \Sigma})^{-1}([x]) = \varphi^{-1}([x]) \cap \psi^{-1} \Sigma$.

If $[x] \in Z$, then (see [25]) we have

$$\varphi^{-1}([x]) \subset \psi^{-1} \Sigma.$$

If $[x] \notin Z$, then $\varphi^{-1}([x]) \not\subset \psi^{-1} \Sigma$ and therefore $\varphi^{-1}([x]) \cap \psi^{-1} \Sigma$ is a divisor in $\varphi^{-1}([x])$ and is of constant dimension. As a consequence, $\mathcal{C}^* \mathcal{C}([Z])$ is smooth in $P(V) \setminus Z$.

On the other hand, by Theorem 2 we have $\mathcal{P} \mathcal{C}^* \mathcal{C}([Z]) = [Z]$, thus $\mathcal{C}^* \mathcal{C}([Z])$ is not smooth near $[x]$ if $[x] \in Z$. $\square$

So $\mathcal{C}^* \mathcal{C}([Z]) = \mathcal{C}^* \mathcal{C}([Z']) \Rightarrow Z = Z'$ when Z and Z' are irreducible. Further there is a continuous linear operator $\mathcal{P}$ such that $[Z] = (\mathcal{P} \mathcal{C}^* \mathcal{C})([Z])$ when Z is irreducible. By linearity, this equality can be extended to each algebraic cycle Z .

In other words, with

$$P(V_{q+1}) \cap Z' = \{[x'_1], [x'_2], \dots\},$$

we have

$$\sum_l \int_{V_{q-1} \subset V_{q+1}} [P(V_{q-1} \oplus \mathbb{C}x_l)] d\nu = \sum_l \int_{V_{q-1} \subset V_{q+1}} [P(V_{q-1} \oplus \mathbb{C}x'_l)] d\nu \Rightarrow [Z]_{|P(V_{q+1})} = [Z']_{|P(V_{q+1})}$$

for all $V_{q+1} \in G(q+1, V)$ i.e. $[Z] = [Z']$.

If $2q \leq N+1$, let $V_{q-1} \subset V_{N+2-q}$ be vectorial subspaces of $V = \mathbb{C}^{N+1}$ of respective dimensions $q-1$ and $N+2-q$. Then we can take the Schubert cycle

$$\sigma = \sigma_{V_{q-1}, V_{N+2-q}} = \{s \in G(q, V), V_{q-1} \subset s \subset V_{N+2-q}\} = P(V_{N+2-q}/V_{q-1}) \simeq \mathbb{P}_{N+2-2q}.$$

In that case the Crofton formula can be stated as

$$\mathcal{B} = \int_{V_{q-1} \subset V_{N+2-q}} [\sigma] d\nu = \int_{V_{q-1} \subset V_{N+2-q}} [\sigma_{V_{q-1}, V_{N+2-q}}] d\nu(V_{q-1}, V_{N+2-q})$$

where $\mathcal{B}$ is a harmonic differential form of bidegree $(d_G - N - 2 + 2q, d_G - N - 2 + 2q)$ and where ν is the measure on

$$\{(V_{q-1}, V_{N+2-q}) \in G(q-1, V) \times G(N+2-q, V) \text{ such that } V_{q-1} \subset V_{N+2-q}\}$$

associated to the Hermitian metric induced by that of $G(q-1, V) \times G(N+2-q, V)$.

With $i: \sigma \rightarrow G(q, V)$ the canonical injection, when we replace u by $[Z]$ in the transform

$$\int_{V_{q-1} \subset V_{N+2-q}} \varphi_* \psi^* i_* i^* \psi_* \varphi^* u d\nu$$

of u , we use that

$$i^* \mathcal{C}(Z) = \{s \in G(q, V), V_{q-1} \subset s \subset V_{N+2-q} \text{ and } P(s) \cap Z \neq \emptyset\}$$

and we still denote by $\pi : V \setminus \{0\} \rightarrow P(V)$ the canonical map. We extend $\pi^{-1}(Z)$ as an analytic subset of V . Then for $s \in G(q, V)$, we have $s \in \text{supp } i^* \mathcal{C}(Z)$ if and only if $s = V_{q-1} \oplus \mathbb{C}x$ with

$$x \in (V_{q-1} + \pi^{-1}(Z)) \cap (V_{N+2-q} \setminus V_{q-1}).$$

As a consequence, we obtain that $\varphi_* \psi^* i_* i^* \mathcal{C}(Z)$ is the image by π of

$$(V_{q-1} + ((V_{q-1} + \pi^{-1}(Z)) \cap V_{N+2-q})) \setminus \{0\}.$$

The injectivity of the previous transform means that if

$$(V_{q-1} + \pi^{-1}(Z)) \cap V_{N+2-q} = (V_{q-1} + \pi^{-1}(Z')) \cap V_{N+2-q}$$

for all $V_{q-1} \subset V_{N+2-q}$, then $Z = Z'$.

5.3 Skoda potential of a closed positive current

For T a closed positive (q, q) -current on $\mathbb{P}_N = P(V)$, we now express the Skoda potential U associated to T (see [5, 28, 29]) by means of the Chow transform $\mathcal{C}(T)$. This function is quasi plurisubharmonic on $\mathbb{P}_N$ and

$$(\deg T)\omega + dd^c U$$

is obtained from $\mathcal{C}(T)$ with an integral transform. Precisely we set

$$T_1 = \varphi_* \psi^* (\mathcal{A} \wedge \mathcal{C}(T)) = \varphi_* \psi^* (\mathcal{A} \wedge \psi_* \varphi^* T)$$

where $\mathcal{A}$ is a positive harmonic differential (d, d) -form on $G(q, V)$. Then T_1 is a closed positive $(1, 1)$ -current on $\mathbb{P}_N$ and

$$T_1 = \text{pr}_{1*} (K \wedge \text{pr}_2^* T)$$

where the integral kernel K on $\mathbb{P}_N \times \mathbb{P}_N$ is given as in Proposition 5 by

$$K = q_* ((\psi \circ p_1)^* \mathcal{A}).$$

We express $\mathcal{A}$ thanks to a Crofton formula. First for $V_{q-1} \in G(q-1, V)$ we consider the Schubert cycle σ defined by

$$\sigma = \sigma_{V_{q-1}} = \{s \in G(q, V), V_{q-1} \subset s\} = P(V/V_{q-1}) \simeq \mathbb{P}_{p+1}.$$

The condition $V_{q-1} \subset s$ is equivalent to a system of homogeneous equations of degree 1. The Crofton formula gives that

$$\mathcal{A} = \int_{V_{q-1} \in G(q-1, V)} [\sigma] d\nu = \int_{V_{q-1} \in G(q-1, V)} [\sigma_{V_{q-1}}] d\nu(V_{q-1})$$

is harmonic, where ν is the positive measure on $G(q-1, V)$ which is invariant by $U(N+1)$ and of total mass equal to 1. As a consequence

$$K = \int_{V_{q-1} \in G(q-1, V)} q_* ((\psi \circ p_1)^* [\sigma]) d\nu$$

with

$$\begin{aligned} q_* ((\psi \circ p_1)^* \sigma) &= q(\{([x], [x'], s) \text{ such that } x \in s, x' \in s, V_{q-1} \subset s\}) \\ &= \{([x], [x']) \text{ such that } \mathbb{C}x \oplus V_{q-1} = \mathbb{C}x' \oplus V_{q-1}\}. \end{aligned}$$

Let $\pi_{V_{q-1}} : P(V) \setminus P(V_{q-1}) \rightarrow P(V_{q-1}^\perp)$ be the projective map associated to the orthogonal projection $V \rightarrow V_{q-1}^\perp$. Then

$$q_* ((\psi \circ p_1)^* \sigma) = (\pi_{V_{q-1}} \times \pi_{V_{q-1}})^* D_{P(V_{q-1}^\perp)}$$

where $D_{P(V_{q-1}^\perp)}$ is the diagonal of $P(V_{q-1}^\perp) \times P(V_{q-1}^\perp)$. In other words the integral kernel is

$$K = \int_{V_{q-1} \in G(q-1, V)} (\pi_{V_{q-1}} \times \pi_{V_{q-1}})^* [D_{P(V_{q-1}^\perp)}] d\nu$$

and we know it is enough to prove the following result.

Theorem 5. *The closed positive $(1, 1)$ -current T_1 on $\mathbb{P}_N$ satisfies $T_1 = (\deg T)\omega + dd^c U$ with U the Skoda potential associated to T . Thus we have the equalities:*

- (i) $\deg T_1 = \deg T$ for the degrees with respect to ω ,
- (ii) $\nu(T_1, [x]) = \nu(T, [x])$ for the Lelong numbers at each point $[x] \in \mathbb{P}_N$.

6 Chow transform defined on the space of cycles

Let X be a complex projective manifold of dimension d_X and let T be a smooth differential form on X of bidimension (p, p) with $p = d_X - q$. The Chow transform $\hat{C}(T)$ of T is a current of bidegree $(1, 1)$ on the space $C_{q-1}(X)$ of effective algebraic cycles in X of dimension $q - 1$, obtained in the following way.

Let $\hat{F}$ be the incidence variety i.e. the subvariety in $X \times C_{q-1}(X)$ of (x, c) satisfying $c \ni x$. With $\hat{\varphi} : \hat{F} \rightarrow X$ and $\hat{\psi} : \hat{F} \rightarrow C_{q-1}(X)$ the restrictions to $\hat{F}$ of the canonical projections, we set

$$\hat{C}(T) = \hat{\psi}_* \hat{\varphi}^* T.$$

The differential form $\hat{C}(T)$ can be calculated by integration along the fibers of $\hat{\psi}$. For $c \in C_{q-1}(X)$ and $\sigma \in T_c C_{q-1}(X)$ a tangent vector, we have

$$\hat{C}(T)_c(\sigma, \bar{\sigma}) = \int_{x \in c} T_x(\xi_x, \bar{\xi}_x, \dots)$$

where $\xi_x \in T_x X$ is the projection of a lifting of σ by $d\hat{\psi}_{(x,c)}$ i.e. ξ_x satisfies $(\xi_x, \sigma) \in T_{(x,c)} \hat{F}$. We denote by $N_X c$ the normal bundle to c in X and we use the linear map (see [4, 10, 23])

$$T_c C_{q-1}(X) \rightarrow H^0(c, N_X c)$$

which associates to σ the infinitesimal moving $\tilde{\sigma}$. Then the condition for ξ_x is

$$\tilde{\sigma}(x) = -\xi_x \bmod T_x c \in T_x X / T_x c$$

for all $x \in c$.

6.1 A commutative diagram

We consider $j : X \rightarrow P(V)$ an embedding of X into $\mathbb{P}_N = P(V)$ and the induced meromorphic map

$$\rho : G(p+1, V^*) \rightarrow C_{q-1}(X)$$

which associates $X \cap P(\ker \lambda_0) \cap \dots \cap P(\ker \lambda_p)$ to $\text{vect}(\lambda_0, \dots, \lambda_p) \in G(p+1, V^*)$.

Lemma 8. *For T a smooth differential form of bidimension (p, p) on X , we have the equality $\rho^* \hat{C}(T) = C(j_* T)$ between $(1, 1)$ -currents on $G(p+1, V^*) = G(N-p, V)$.*

Proof. This is equivalent to

$$\langle \hat{C}(T), \rho_* v \rangle = \langle C(j_* T), v \rangle \Leftrightarrow \langle T, \hat{\varphi}_* \hat{\psi}^* \rho_* v \rangle = \langle T, (\varphi_* \psi^* v)|_X \rangle$$

for each smooth differential form v of bidimension $(1, 1)$ on $G(p+1, V^*)$. With $r(x, s) = (x, X \cap P(s))$, the diagram

$$\begin{array}{ccc} \Gamma \cap (X \times G(N-p, V)) & \xrightarrow{r} & \hat{\Gamma} \\ \psi \downarrow & & \downarrow \hat{\psi} \\ G(N-p, V) & \xrightarrow{\rho} & C_{q-1}(X) \end{array}$$

is commutative. Therefore

$$\hat{\varphi}_* \hat{\psi}^* \rho_* v = \hat{\varphi}_* r_* ((\psi^* v)|_{\varphi^{-1}(X)}) = (\varphi|_{\varphi^{-1}(X)})_* ((\psi^* v)|_{\varphi^{-1}(X)}) = (\varphi_* \psi^* v)|_X. \quad \square$$

Proposition 11. *The transformation of integration along the fibers*

$$\hat{\mathcal{C}} : \{\text{smooth differential } (q, q)\text{-forms on } X\} \rightarrow \{(1, 1)\text{-currents on } C_{q-1}(X)\} \quad (28)$$

is injective.

Proof. This is a consequence of Lemma 8 and of the injectivity of $\mathcal{C}$. $\square$

Remark 2. *The transformation $\hat{\mathcal{C}}$ can be used to solve the problem of approximating closed (q, q) -currents by algebraic cycles.*

In effect, if T is closed, $\hat{\mathcal{C}}(T)$ is a closed $(1, 1)$ -current on $C_{q-1}(X)$ of order 0 and if moreover $\{T\}$ is rational, $\hat{\mathcal{C}}(T)$ is a weak limit of divisors with complex coefficients. Then we can write

$$\hat{\mathcal{C}}(T) = \int_{D \in \text{Div } C_{q-1}(X)} \lambda(D) [D]$$

where $\text{Div } C_{q-1}(X)$ is the space of divisors of $C_{q-1}(X)$ and λ is a measure on $\text{Div } C_{q-1}(X)$. The Lebesgue-Nikodym decomposition of λ is

$$\lambda = \lambda_0 + \lambda_1$$

with λ_0 a L^1_{loc} function and λ_1 a measure such that $\text{supp } \lambda_1$ is negligible. We now determine $\text{supp } \lambda_1$ by using the injection $C_p(X) \hookrightarrow \text{Div } C_{q-1}(X)$ which associates to $Z \in C_p(X)$ the incidence divisor

$$D_Z = \{c \in C_{q-1}(X), Z \cap c \neq \emptyset\}$$

of Z . We fix $D_0 \in \text{Div } C_{q-1}(X)$ and we consider w a current of bidimension $(1, 1)$ on $C_{q-1}(X)$ such that $D \rightarrow \int_D w$ is the Dirac measure at D_0 . Thus

$$\langle T, \hat{\varphi}_* \hat{\psi}^* w \rangle = \int_{D \in \text{Div } C_{q-1}(X)} \lambda(D) \int_D w$$

is a value of $\int_{\text{Div } C_{q-1}(X)} \lambda \delta_{D_0}$ in a generalized sense. We set

$$Z_0 = \{x \in X, \hat{\varphi}^{-1}(x) \subset \hat{\psi}^{-1}(D_0)\} = \{x \in X, c \ni x \Rightarrow c \in D_0\} \subset X$$

which is of dimension $\leq p$ and satisfies $\dim Z_0 = p$ when D_0 is an incidence divisor.

Assume $D_0 \notin C_p(X)$ i.e. $\dim Z_0 < p$, then the singularities of the current $\hat{\varphi}_* \hat{\psi}^* w$ are of a lower order and $\int_{\text{Div } C_{q-1}(X)} \lambda \delta_{D_0}$ is the effective value of λ at D_0 . In other words $D_0 \notin \text{supp } \lambda_1$.

Conclusion : $\text{supp } \lambda_1 \subset C_p(X)$, which is enough to prove that T is a weak limit of algebraic cycles with complex coefficients.

6.2 A kinematic formula

Here we retrieve the equations (8) and (9), which characterize the Chow transform of a smooth differential $(q-1, q-1)$ -form u on $P(V) = \mathbb{P}_N$, by using the following kinematic formula (see [15])

$$\frac{d}{dt} \left(\int_{c_t} u \right)_{|t=0} = \frac{d}{dt} \left((\hat{\psi}_* \hat{\varphi}^* u)(c_t) \right)_{|t=0} = (\hat{\psi}_* \hat{\varphi}^* du)_c \left(\frac{dc_t}{dt} \right)_{|t=0} \quad (29)$$

where $c = c_0$ with $c_t \in C_{q-1}(\mathbb{P}_N)$ an one-parameter family of effective algebraic cycles of $\mathbb{P}_N$ of dimension $q - 1$, which is differentiable at $t = 0$.

In particular, we consider the family of projective subspaces of $\mathbb{P}_N = P(V)$ of dimension $q - 1$, which is parametrized by $G(q, V) = \mathbb{G}_{q-1, N}$. We use homogeneous coordinates on $G(q, V)$ i.e. we write $s \in G(q, V)$ as being

$$s = \tau(z) = \text{vect}(z^0, \dots, z^{q-1})$$

for $z = (z^0, \dots, z^{q-1}) \in V^q$. The associated Plücker coordinates are defined by

$$z^0 \wedge \dots \wedge z^{q-1} = \sum_{|I|=q} z_I e_I \in \bigwedge^q V \quad (30)$$

where $e_I = e_{i_1} \wedge \dots \wedge e_{i_q}$ for $I = (i_1, \dots, i_q)$ with $0 \leq i_1 < \dots < i_q \leq N$, denoting by $e_0, \dots, e_N$ an orthonormal basis of V .

We define the differential operator in the homogeneous coordinates

$$\partial_I = \det \left(\frac{\partial}{\partial z_j^k} \right)_{\substack{0 \leq k \leq q-1 \\ j \in I}} = \frac{\partial}{\partial z_I}$$

and for $|I| = |J| = q$, we evaluate $\partial_I \bar{\partial}_J U$ where

$$U(z) = \int_{P(\tau(z))} u$$

by means of formula (29). In effect, first we have

$$(\partial_I \bar{\partial}_J U)(z) = (\psi_* \varphi^* (\partial \bar{\partial} u))_s \left(\frac{\partial s}{\partial z_I}, \overline{\frac{\partial s}{\partial z_J}} \right).$$

We introduce $i : G(q, V) \rightarrow P(\bigwedge^q V)$ the Plücker embedding, which allows to calculate

$$di_s \left(\frac{\partial s}{\partial z_I} \right) = \frac{\partial}{\partial z_I} (i(s)) = (z^0 \wedge \dots \wedge z^{q-1})^{-1} \otimes (e_I \bmod \mathbb{C}(z^0 \wedge \dots \wedge z^{q-1})) \in T_{[z^0 \wedge \dots \wedge z^{q-1}]} P(\bigwedge^q V)$$

by means of (30). We set $v = \partial \bar{\partial} u$ and we use the previous formula for the Chow transform

$$\tau^* C(v) = \sum_{|I|=|J|=q} \tilde{v}_{IJ}(z) dz_I \wedge d\bar{z}_J$$

where $\tilde{v}_{IJ}(z)$ is given by (14). Since the inverse image in $\bigwedge^q V$ of $di_s \left(\frac{\partial s}{\partial z_I} \right)$ is constant equal to e_I , the function $\partial_I \bar{\partial}_J U$ decompose on the $\tilde{v}_{IJ}$ with constant coefficients. Therefore $\partial_I \bar{\partial}_J U$ is itself of the form $\tilde{v}$. But we know that such functions satisfy the John differential equations (see the proof of Theorem 3 for example)

$$0 = \left(\frac{\partial^2}{\partial z_m^l \partial z_{m'}^{l'}} - \frac{\partial^2}{\partial z_{m'}^l \partial z_m^{l'}} \right) \tilde{v} \text{ and } 0 = \left(\frac{\partial^2}{\partial \bar{z}_m^l \partial \bar{z}_{m'}^{l'}} - \frac{\partial^2}{\partial \bar{z}_{m'}^l \partial \bar{z}_m^{l'}} \right) \tilde{v}$$

for all $0 \leq l, l' \leq q - 1$ and all $0 \leq m, m' \leq N$.

As a conclusion, we find again that U satisfies the differential equations

$$0 = \left(\frac{\partial^2}{\partial z_m^l \partial z_{m'}^{l'}} - \frac{\partial^2}{\partial z_{m'}^l \partial z_m^{l'}} \right) \partial_I \bar{\partial}_J U \text{ and } 0 = \left(\frac{\partial^2}{\partial \bar{z}_m^l \partial \bar{z}_{m'}^{l'}} - \frac{\partial^2}{\partial \bar{z}_{m'}^l \partial \bar{z}_m^{l'}} \right) \partial_I \bar{\partial}_J U$$

for all $|I| = |J| = q$, all $0 \leq l, l' \leq q - 1$ and all $0 \leq m, m' \leq N$.

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