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Contact metric manifolds with large automorphism group and (κ, μ) -spaces

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Abstract: We discuss the classification of simply connected, complete (κ, μ) -spaces from the point of view of homogeneous spaces. In particular, we exhibit new models of (κ, μ) -spaces having Boeckx invariant -1. Finally, we prove that the number $\frac{(n+1)(n+2)}{2}$ is the maximum dimension of the automorphism group of a contact metric manifold of dimension $2n + 1$, $n \geq 2$, whose symmetric operator h has rank at least 3 at some point; if this dimension is attained, and the dimension of the manifold is not 7, it must be a (κ, μ) -space. The same conclusion holds also in dimension 7 provided the almost CR structure of the contact metric manifold under consideration is integrable.

Keywords: contact metric manifold, (κ, μ) -space

MSC: Primary 53C25, 53D10; Secondary 53C30

1 Introduction

Among the contact metric spaces $(M, \varphi, \xi, \eta, g)$, the so-called (κ, μ) -spaces form a special and significant class with remarkable geometric properties; they were originally defined by Blair, Koufogiorgos and Papanthiou in 1995, by the following curvature condition:

$$R(X, Y)\xi = (\kappa Id + \mu h)(\eta(Y)X - \eta(X)Y), \quad (1)$$

where $h = \frac{1}{2}\mathcal{L}_\xi\varphi$ and κ, μ are real numbers; see [2]. For the notation and basic facts of contact metric geometry we refer the reader to Blair's book [1]. Of course the above condition is satisfied by the Sasakian manifolds (corresponding to $h = 0$ and $\kappa = 1$); in all that follows we shall deal exclusively with non Sasakian (κ, μ) -spaces.

The main motivation for studying these manifolds was a previous result of Blair, stating that the unique simply connected, complete contact metric manifold of dimension $2n + 1$, $n \geq 2$, satisfying $R(X, Y)\xi = 0$ is the Riemannian product $\mathbb{S}^n \times \mathbb{R}^{n+1}$, where the metric on the sphere is chosen of curvature 4; this is the unique contact metric non Sasakian orientable hypersurface of $\mathbb{C}^{n+1}$ (see [17]).

In the paper [2], it was established that every (κ, μ) -space is a CR manifold, and that its curvature tensor is completely determined by (1). Blair, Koufogiorgos and Papanthiou also exhibited models of type $T_1N(c)$, i.e. tangent sphere bundles over Riemannian manifolds $N(c)$ with constant curvature $c \in \mathbb{R}$, $c \neq 1$. Moreover, they provided a complete classification in the 3-dimensional case.

Later other characterizations of the (κ, μ) -spaces and related geometric results appeared in the literature. See for instance [5], [6], [12], [10], [11].

Concerning the classification problem in higher dimension, in [3] and [4] Boeckx proved that every (κ, μ) -space is locally homogeneous, and showed that, up to equivalence and D -homothetic deformations, the fam-

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ily of simply connected, complete (κ, μ) -spaces of fixed dimension is parametrized by $\mathbb{R}$. Indeed, he determined a number I_M which is invariant under D -homothetic deformations, namely

$$I_M := \frac{1 - \frac{\mu}{2}}{\sqrt{1 - \kappa}},$$

which completely determines locally a (κ, μ) -manifold M , up to equivalence:

Theorem 1 (Boeckx). *Let $(M_i, \varphi_i, \xi_i, \eta_i, g_i)$, $i = 1, 2$, be two (κ, μ) -spaces of the same dimension. Then $I_{M_1} = I_{M_2}$ if and only if, up to a D -homothetic deformation of the contact metric structure, the two spaces are locally equivalent as contact metric spaces.*

The tangent sphere bundles $T_1N(c)$, as c varies in $\mathbb{R} - \{1\}$, provide examples exhausting all possible values $I_M > -1$. The case $I_M \leq -1$ seemed to lead to models of different nature. Namely, Boeckx himself found examples of contact metric (κ, μ) -spaces, for every value of the invariant ≤ -1 , namely a two parameter family of abstractly constructed Lie groups with a left-invariant contact metric structure. However, he gave no geometric description of these examples. A geometrical construction of models with $I_M \leq -1$ has been provided more recently by E. Loiudice and the present author in [15] and [16]. Namely, in [15] they are obtained as tangent hyperquadric bundles

$$T_{-1}N(c) = \{(p, v) \in TN(c) : g_p(v, v) = -1\}.$$

over a Lorentzian space form $N(c)$ with curvature $c \neq -1$, endowed with a suitable contact metric structure, which is a pseudohermitian structure associated to its natural CR structure, inherited from the standard almost complex structure of the tangent bundle $TN(c)$ (concerning this CR structure, see also [18]). Moreover, in [16] all the simply connected (κ, μ) -spaces M with $I_M \neq \pm 1$ of dimension at least five have been classified in a uniform way as homogeneous contact metric manifolds. Below we give a classification table, including also the cases $I_M = \pm 1$. Consider the following Stiefel manifolds:

$V_0(2, n+2) :=$ set of orthonormal frames $\{u, v\}$ of the Euclidean space $\mathbb{R}_0^{n+2}$:

$$u \cdot u = 1, \quad v \cdot v = 1, \quad u \cdot v = 0;$$

$V_1(2, n+2) :=$ set of frames $\{u, v\}$ of the Minkowski space $\mathbb{R}_1^{n+2}$ such that:

$$u \cdot u = -1, \quad v \cdot v = 1, \quad u \cdot v = 0;$$

$V_2(2, n+2) :=$ set of frames $\{u, v\}$ of the index 2 space $\mathbb{R}_2^{n+2}$ such that:

$$u \cdot u = -1, \quad v \cdot v = -1, \quad u \cdot v = 0.$$

Theorem 2. *Up to equivalence and D -homothetic deformations, the simply connected, complete, (κ, μ) -spaces of dimension at least five are:*

Boeckx invariant	Homogeneous model M , $\dim(M) = 2n + 1 \geq 5$	Alternative interpretation
$I_M > 1$	$V_0(2, n+2) = SO(n+2)/SO(n)$	$T_1(\mathbb{S}^{n+1})$
$-1 < I_M < 1$	$V_1(2, n+2) = SO(1, n+1)/SO(n)$	$T_1(\mathbb{H}^{n+1})$
$I_M < -1$	$V_2(2, n+2) = SO(2, n)/SO(n)$	$T_{-1}(\mathbb{H}_1^{n+1})$
$I_M = 1$	$\mathbb{S}^n \times \mathbb{R}^{n+1} = SO(n+1) \times \mathbb{R}^{n+1}/SO(n)$	$T_1(\mathbb{R}^{n+1})$
$I_M = -1$	$\mathbb{H}^n \times \mathbb{R}_1^{n+1} = SO(1, n) \times \mathbb{R}^{n+1}/SO(n)$	$T_{-1}(\mathbb{R}_1^{n+1})$

Each of the Stiefel manifolds listed in the table carries a one parameter family of inequivalent invariant contact metrics structures satisfying the (κ, μ) condition, whose Boeckx invariants cover the range indicated in the first column.

For $I_M \neq \pm 1$, this classification was achieved in [16] by using the fact the (κ, μ) -spaces can be characterized as (locally) CR -symmetric spaces in the sense of Kaup-Zaitsev [13]; namely, a non K -contact, contact metric space M is a (κ, μ) -space if and only if each point p admits a local isometric CR automorphism $\sigma_p : M \rightarrow M$ such that:

$$\sigma_p(p) = p, \quad (d\sigma)_p(X) = -X, \quad \forall X \in D_p,$$

where D is the contact distribution (cf. [12]). This yields that the simply connected, complete (κ, μ) -spaces admit a transitive Lie group G of automorphisms whose Lie algebra $\mathfrak{g}$ has a canonical symmetric decomposition (see Theorem 4 in the next section). These facts lead to the conclusion that the base space M/ξ of the canonical fibration of M according to Boothby-Wang [8] is an affine symmetric space, which turns out to be a suitable Grassmannian. For details concerning the proof see [16].

Here we shall concentrate on the case $I_M = -1$ (not included in [16]), giving an explicit description of the $SO(1, n) \ltimes \mathbb{R}^{n+1}$ -invariant contact metric structure on the manifold $\mathbb{H}^n \times \mathbb{R}_1^{n+1}$ appearing in the above table (see Theorem 5 in the next section).

Looking at the same classification table, one can observe that the spaces under discussion have a rich amount of symmetry, as expected from their above description as CR -symmetric spaces. Namely, for each of the models M in the list, letting $\dim(M) = 2n + 1$, we see that:

$$\dim(\text{Aut}(M)) \geq \frac{(n+1)(n+2)}{2}.$$

In this connection, in this paper we prove the following result, yielding that the above inequality is in fact an equality and showing that the (κ, μ) -spaces are the only manifolds having largest automorphism group, within the class of contact metric manifolds of prescribed dimension, whose operator h has rank at least 3 at some point:

Theorem 3. *Let $(M, \varphi, \xi, \eta, g)$ be a connected contact metric manifold of dimension $2n + 1$, $n \geq 2$. Assume that at some point $p \in M$ the operator h_p has rank > 2 . Then:*

- a) $\dim(\text{Aut}(M)) \leq \frac{(n+1)(n+2)}{2}$.
- b) If $\dim(\text{Aut}(M)) = \frac{(n+1)(n+2)}{2}$, then M is a (κ, μ) -space, provided $n \neq 3$ or $n = 3$ and the almost CR structure of M is integrable.

We remark that for the entire class of contact metric manifolds of dimension $2n + 1$, Tanno proved that the maximum dimension of $\text{Aut}(M)$ is $(n + 1)^2$, which is attained only by three types of homogeneous Sasakian manifolds with constant φ -sectional curvature [20].

2 A Lie-theoretic characterization of (κ, μ) -spaces

Given a contact metric manifold $(M, \varphi, \xi, \eta, g)$, we shall denote by $\text{Aut}(M)$ be the group of all automorphisms of M , i.e. the group consisting of all isometries $f : M \rightarrow M$ with respect to g , preserving the contact form η . It follows that such an f also preserves ξ and φ . Clearly, $\text{Aut}(M)$ is a closed Lie subgroup of the isometry group $I(M, g)$. The contact distribution $\ker(\eta)$ of M will be denoted by D .

Assume now that $(M, \varphi, \xi, \eta, g)$ is a homogeneous contact metric manifold, so that $M = G/H$ where G is a Lie subgroup of $\text{Aut}(M)$ acting transitively on M .

Fix a reductive decomposition of $\mathfrak{g} := \text{Lie}(G)$ (it is known that such a decomposition always exists, being G a group of isometries, cf. e.g. [19]):

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad \mathfrak{h} = \text{Lie}(H), \quad \mathfrak{m} \simeq T_o M, \quad o = H. \quad (2)$$

Set:

$$\zeta := \xi_o, \quad \mathfrak{b} := D_o,$$

so that

$$\mathfrak{m} = \mathfrak{b} \oplus \mathbb{R}\zeta.$$

We have another decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \bar{\mathfrak{h}} \oplus \mathfrak{b}, \quad \bar{\mathfrak{h}} := \mathfrak{h} \oplus \mathbb{R}\zeta. \quad (3)$$

Keeping this notation, the following characterization of the (κ, μ) -spaces has been obtained in [16]:

Theorem 4. *Let $(M, \varphi, \xi, \eta, g)$ be a simply connected, complete, contact metric manifold. Assume M is not K -contact. Then the following conditions are equivalent:*

1) M is a (κ, μ) -space.

2) M admits a transitive, effective Lie group of automorphisms G whose Lie algebra $\mathfrak{g}$ is a symmetric Lie algebra with respect to the decomposition $\mathfrak{g} = \bar{\mathfrak{h}} \oplus \mathfrak{b}$.

Here, the condition concerning the Lie algebra $\mathfrak{g}$ stated in 2) means that the following usual Cartan relations hold:

$$[\bar{\mathfrak{h}}, \bar{\mathfrak{h}}] \subset \bar{\mathfrak{h}}, \quad [\bar{\mathfrak{h}}, \mathfrak{b}] \subset \mathfrak{b}, \quad [\mathfrak{b}, \mathfrak{b}] \subset \bar{\mathfrak{h}}.$$

The proof of $1) \Rightarrow 2)$ is based on the existence of CR -symmetries ([12]), while $2) \Rightarrow 1)$ relies on the characterization of (κ, μ) -spaces by η -parallelism of h , due to Boeckx and Cho [5].

As an application, we shall discuss the simply connected, complete (κ, μ) -spaces with Boeckx invariant -1 .

Actually, among the (κ, μ) -spaces, those having Boeckx invariant $I_M = \pm 1$ are particular: they can be characterized by using the so-called *Pang invariants* $\Pi^\pm$ of the Legendrian foliations $D^+ = D(\lambda)$ and $D^- = D(-\lambda)$, see [9]. Here D^+ and D^- are the (integrable) eigendistributions of the symmetric operator h relative to its non null eigenvalues; recall that for every (κ, μ) -space, the spectrum of h is $\{\lambda, -\lambda, 0\}$, where $\lambda = \sqrt{1 - \kappa}$ (cf. e.g. [1]).

By definition, the Pang invariant of a Legendrian foliation $\mathfrak{F}$ on a contact manifold (M, η) is the symmetric tensor field:

$$\Pi^\mathfrak{F}(X, Y) = 2d\eta([\xi, X], Y),$$

where X, Y are sections of $\mathfrak{F}$. In the case of a (κ, μ) -space, one has [9]:

$$I_M = 1 \iff \Pi^- = 0, \quad \Pi^+ \text{ is positive definite}$$

$$I_M = -1 \iff \Pi^+ = 0, \quad \Pi^- \text{ is negative definite.}$$

Recall that the model having Boeckx invariant $I_M = 1$ is the Riemannian product $\mathbb{S}^n \times \mathbb{R}^{n+1}$ (where $\mathbb{S}^n$ has curvature 4), i.e., the tangent sphere bundle of the Euclidean space $\mathbb{R}^{n+1}$, which is in a natural way a homogeneous space under the action of the subgroup $SO(n+1) \ltimes \mathbb{R}^{n+1}$ of the group of isometries of the Euclidean space $\mathbb{R}^{n+1}$.

To construct a model with $I_M = -1$, it is natural to think it should be the “dual” $M = \mathbb{H}^n \times \mathbb{R}^{n+1}$, where $\mathbb{H}^n$ is an hyperbolic space. Of course, this cannot be a Riemannian product, due to the fact that, by a well-know

result of Boeckx and Cho [7], $\mathbb{S}^n \times \mathbb{R}^{n+1}$ is the unique non Sasakian, simply connected Riemannian symmetric contact metric manifold of dimension $2n + 1$.

We shall think to our product manifold M as:

$$M = \mathbb{H}^n \times \mathbb{R}_1^{n+1}$$

where $\mathbb{R}_1^{n+1}$ is the Minkowski space, i.e. $M = T_{-1}(\mathbb{R}_1^{n+1})$, and we shall construct a homogeneous contact metric structure by using the natural transitive action of the Poincaré group $E(1, n)$; actually we consider motions of the form:

$$x \mapsto Ax + v, \quad A \in SO(1, n), \quad v \in \mathbb{R}^{n+1}.$$

Then $M = G/H$, where $G = SO(1, n) \ltimes \mathbb{R}^{n+1}$ consists of the matrices:

$$\begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix}, \quad A \in SO(1, n), \quad v \in \mathbb{R}^{n+1},$$

while H consists of the matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B \in SO(n),$$

so that $H \cong SO(n)$.

Accordingly, the corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ can be described as follows:

$$\text{Generic element of } \mathfrak{g}: \begin{pmatrix} 0 & {}^t w & \alpha \\ w & A & u \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{array}{l} A \in \mathfrak{so}(n) \\ w, u \in \mathbb{R}^n \\ \alpha \in \mathbb{R} \end{array}$$

$$\text{Generic element of } \mathfrak{h}: \begin{pmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A \in \mathfrak{so}(n).$$

We have a natural reductive decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad \mathfrak{m} := \left\{ \begin{pmatrix} 0 & {}^t w & \alpha \\ w & 0 & u \\ 0 & 0 & 0 \end{pmatrix} : w, u \in \mathbb{R}^n, \alpha \in \mathbb{R} \right\}.$$

We shall denote a matrix belonging to the vector space $\mathfrak{m}$ by using the notation $(w \ u \ \alpha)$. A G -invariant contact metric structure can be introduced in a natural way by using the splitting:

$$\mathfrak{m} = \mathfrak{b} \oplus \mathbb{R}\zeta, \quad \zeta := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathfrak{b} := \left\{ \begin{pmatrix} 0 & {}^t w & 0 \\ w & 0 & u \\ 0 & 0 & 0 \end{pmatrix} : w, u \in \mathbb{R}^n \right\}.$$

We consider the G -invariant 1-form η and the G -invariant Riemannian metric g on M , determined at the base point $o = H$ by setting:

$$\eta_o|_{\mathfrak{b}} = 0, \quad \eta_o(\zeta) = 1, \quad g_o((w, u, \alpha), (\bar{w}, \bar{u}, \bar{\alpha})) = \frac{1}{2}(\langle w, \bar{w} \rangle + \langle u, \bar{u} \rangle) + \alpha\bar{\alpha},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$. Then a routine check yields that η is a contact form whose Reeb vector field ξ satisfies $\xi_o = \zeta$, and g is an associated metric.

Moreover, easy matrix computations show:

- $\mathfrak{g} = (\mathfrak{h} \oplus \mathbb{R}\zeta) \oplus \mathfrak{b}$ is a symmetric decomposition of the Lie algebra $\mathfrak{g}$.

- The spectrum of the operator h is $\{\frac{1}{2}, -\frac{1}{2}, 0\}$, and the eigendistributions D^+ and D^- of h relative to the non null eigenvalues are the G -invariant distributions whose determinations at o are given by the following subspaces of $\mathfrak{t}$:

$$\mathfrak{b}_+ = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & u \\ 0 & 0 & 0 \end{pmatrix} : u \in \mathbb{R}^n \right\}, \quad \mathfrak{b}_- = \left\{ \begin{pmatrix} 0 & {}^t w & 0 \\ w & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : w \in \mathbb{R}^n \right\}.$$

The first fact yields that M is a (κ, μ) -space according to Theorem 4. In order to verify that $I_M = -1$, it is convenient to compute the Pang invariants of D^+ and D^- .

Lemma 1. *For a G -invariant Legendre foliation $\mathfrak{F}$ on a homogeneous reductive contact metric manifold $M = G/H$, with a given reductive decomposition (2) of the Lie algebra $\mathfrak{g} = \text{Lie}(G)$, the Pang invariant $\Pi^{\mathfrak{F}}$ is given in terms of Lie algebra bracket as follows:*

$$\Pi_o^{\mathfrak{F}}(X, Y) = 2 \langle [\zeta, X]_{\mathfrak{m}}, \varphi Y \rangle, \quad X, Y \in \mathfrak{e},$$

where $\mathfrak{e} \subset \mathfrak{m}$ is the determination of $\mathfrak{F}$ at the base point $o = H$, $\zeta = \xi_o$ and $\langle, \rangle = g_o$.

Proof. We shall make use of the canonical G -invariant linear connection $\tilde{\nabla}$ on M relative to the fixed reductive decomposition of $\mathfrak{g}$. Recall that all the G -invariant tensor fields on M are $\tilde{\nabla}$ -parallel (cf. [14, Prop. 2.7, Ch. X, p. 193]). In particular $\tilde{\nabla}\xi = 0$, whence, for every X, Y smooth sections of $\mathfrak{F}$:

$$\Pi^{\mathfrak{F}}(X, Y) = 2d\eta([\xi, X], Y) = 2d\eta(\tilde{\nabla}_\xi X, Y) - 2d\eta(\tilde{T}(\xi, X), Y)$$

where $\tilde{T}$ denotes the torsion of $\tilde{\nabla}$. Moreover, since $\mathfrak{F}$ is G -invariant, $\tilde{\nabla}_\xi X$ is also tangent to $\mathfrak{F}$, so that $d\eta(\tilde{\nabla}_\xi X, Y) = 0$; hence

$$\Pi^{\mathfrak{F}}(X, Y) = -2d\eta(\tilde{T}(\xi, X), Y), \quad \forall X, Y \in \Gamma\mathfrak{F}.$$

Evaluating at o , this gives

$$\Pi_o^{\mathfrak{F}}(X, Y) = 2d\eta([\zeta, X]_{\mathfrak{m}}, Y), \quad X, Y \in \mathfrak{e}$$

taking into account the expression of $\tilde{T}$ given in [14, Prop. 2.3, Ch. X, p. 192]. □

In order to apply this lemma in our case, we observe that:

$$ad_\zeta(w \cup 0) = (0 \quad -w \quad 0), \quad \varphi(w \cup 0) = (-u \quad w \quad 0),$$

from which we get

$$\Pi^+ = 0, \quad \Pi^- = -2g|_{D^- \times D^-}$$

yielding $I_M = -1$.

In conclusion, we have proved

Theorem 5. *Up to equivalence and D -homothetic deformations, the simply connected, complete (κ, μ) -space of dimension $2n + 1 \geq 5$ and with Boeckx invariant -1 is the homogeneous space $\mathbb{H}^n \times \mathbb{R}_1^{n+1} = SO(1, n) \ltimes \mathbb{R}^{n+1}/SO(n)$, endowed with an invariant contact metric structure.*

3 Proof of Theorem 3

We start the proof with the following algebraic remark:

Lemma 2. *Let $(V, \langle, \rangle)$ be an Euclidean vector space of dimension $2n$, $n \geq 2$, endowed with an orthogonal complex structure $J : V \rightarrow V$. Let $h : V \rightarrow V$ be a symmetric endomorphism anticommuting with J . Then the group $U(h, J) \subset O(V)$ consisting of the orthogonal transformations commuting with both J and h is isomorphic to*

$$O(k_1) \times O(k_2) \times \cdots \times O(k_m) \times U(s)$$

where the integers k_i are the multiplicities of the distinct positive eigenvalues $\alpha_1, \dots, \alpha_m$ of h , and $2s$ is the dimension of $\text{Ker}(h)$.

Proof. Since h anti-commutes with J , the spectrum of h must be of the form

$$\{\alpha_1, \dots, \alpha_m, -\alpha_1, \dots, -\alpha_m\}, \quad \alpha_i > 0, \alpha_i \neq \alpha_j$$

provided h is non singular, or of the form

$$\{\alpha_1, \dots, \alpha_m, -\alpha_1, \dots, -\alpha_m, 0\}$$

if $s \geq 1$. Moreover, we have $J(\text{Ker}(h)) = \text{Ker}(h)$ and the eigenspaces $V(\alpha_i)$ and $V(-\alpha_i)$ of h satisfy:

$$J(V(\alpha_i)) = V(-\alpha_i).$$

Hence V admits an orthonormal basis of the form

$$\{v_1^i, \dots, v_{k_i}^i, Jv_1^i, \dots, Jv_{k_i}^i\}$$

where $i = 1, \dots, m$, or of the form

$$\{v_1^i, \dots, v_{k_i}^i, Jv_1^i, \dots, Jv_{k_i}^i, u_1, \dots, u_s, Ju_1, \dots, Ju_s\}$$

where $h(v_j^i) = \alpha_i v_j^i$ and $h(u_t) = 0$. With respect to such a basis, every automorphism $L \in U(h, J)$ is represented by a block diagonal matrix A of the form

$$A = \text{diag}(A_1, \dots, A_m, A_1, \dots, A_m) \quad (4)$$

or of the form

$$A = \text{diag}(A_1, \dots, A_m, A_1, \dots, A_m, B),$$

where $A_i \in O(k_i)$ and B belongs to the subgroup of $O(2s)$ consisting of the orthogonal matrices commuting with the matrix

$$\begin{pmatrix} 0 & -I_s \\ I_s & 0 \end{pmatrix}$$

corresponding to the standard complex structure of $\mathbb{R}^{2s}$, which is naturally isomorphic to $U(s)$. This proves the assertion. $\square$

Now we can start proving Theorem 3. It is known that since $\text{Aut}(M)$ is a closed subgroup of the isometry group $I(M, g)$, the $\text{Aut}(M)$ -orbit S of p is a closed submanifold of M , diffeomorphic to $\text{Aut}(M)/\text{Aut}_p(M)$, where $\text{Aut}_p(M)$ denotes the isotropy subgroup of $\text{Aut}(M)$ relative to the point p .

Consider the linear isotropy representation $\rho : \text{Aut}_p(M) \rightarrow O(T_p M, g_p)$. Observe that every automorphism of the contact metric structure must also preserve the tensor field h . Hence, denoting again by D the contact distribution and by $\bar{\rho} : \text{Aut}_p(M) \rightarrow O(D_p, g_p)$ the representation induced by ρ , i.e.

$$\bar{\rho}(F) := (dF)_p|_{D_p},$$

and keeping the notation of Lemma 2, we have that $\text{Im}(\bar{\rho})$ is contained in the group $U(h_p, J_p) \subset O(D_p, g_p)$ determined by the symmetric operator $h_p : D_p \rightarrow D_p$ and by the complex structure $J_p : D_p \rightarrow D_p$ given by the restriction of φ_p . According to the assumption on the rank of h_p and the same lemma, this group has dimension at most $\frac{n(n-1)}{2} = \dim O(n)$. This yields a).

To prove b), assume that $\dim \text{Aut}(M) = \frac{(n+1)(n+2)}{2} = 2n + 1 + \frac{n(n-1)}{2}$; then $\text{Aut}_p(M)$ has dimension $\frac{n(n-1)}{2}$ and $\dim(S) = 2n + 1$; this yields $S = M$ since S is closed. Hence in this case $\text{Aut}(M)$ is transitive on M , i.e. M is a homogeneous contact metric manifold. Moreover, being $\dim U(h_p, J_p) \geq \dim O(n)$, using again Lemma 2, we see that $h_p : D_p \rightarrow D_p$ must be non singular and must admit exactly one positive eigenvalue $\alpha = \alpha_1$. By homogeneity, this actually holds at every point of M . Observe also that actually $\dim U(h_p, J_p) = \dim O(n)$, and since $\text{Aut}_p(M)$ is compact, $\text{Im}(\bar{\rho})$ contains the connected component of the identity of $U(h_p, J_p)$.

Denote by D^+ and D^- the eigendistributions of h relative to the constant eigenvalues α and $-\alpha$. In order to complete the proof, we shall verify that M is a η -parallel contact metric manifold in the sense of [5]. Denote by Θ the $(0, 3)$ tensor field on M defined by

$$\Theta(X, Y, Z) := g((\nabla_X h)Y, Z),$$

where ∇ is the Levi-Civita connection. We need to show that $\Theta(X, Y, Z) = 0$ whenever the vector fields X, Y, Z belong to D .

First of all, we remark that this automatically holds if Y and Z both belong to D^+ or to D^- . Hence, since Θ is symmetric in Y and Z , it suffices to prove that:

$$\Theta(X, Y, JZ) = \Theta(JX, Y, JZ) = 0 \quad X, Y, Z \in \Gamma D^+$$

and, by homogeneity, since every $F \in \text{Aut}(M)$ clearly preserves Θ , this in turn reduces to:

$$\Theta_p(X, Y, JZ) = \Theta_p(JX, Y, JZ) = 0 \quad X, Y, Z \in \mathcal{B}, \quad (5)$$

where $\mathcal{B} = \{X_1, \dots, X_n\}$ is a fixed orthonormal basis of D_p^+ .

For the sake of convenience, let us put, for every $X, Y, Z \in D_p^+$:

$$\Psi(X, Y, Z) := \Theta_p(X, Y, JZ), \quad \Psi'(X, Y, Z) := \Theta_p(JX, Y, JZ),$$

so that Ψ and Ψ' are $(0, 3)$ tensors on the vector space D_p^+ .

Now, assuming $n \neq 3$, using the matrix description (4) of the automorphisms of D_p belonging to the group $U(h_p, J_p)$, with $m = 1$, we readily see that given X, Y, Z in $\mathcal{B}$, there always exists a linear transformation $L : D_p \rightarrow D_p$ belonging to the connected component of the identity $U^0(h_p, J_p) \cong SO(D_p^+) \cong SO(n)$ of $U(h_p, J_p)$, and such that:

$$L(X) = -X, \quad L(Y) = -Y, \quad L(Z) = -Z.$$

Since $U^0(h_p, J_p)$ is contained in $\text{Im}(\bar{\rho})$, there exists an automorphism $F \in \text{Aut}_p(M)$ such that $(dF)_p|_{D_p} = L$. Hence, taking again into account the $\text{Aut}(M)$ -invariance of Θ and J , we get:

$$\Psi(X, Y, Z) = \Psi(LX, LY, LZ) = -\Psi(X, Y, Z)$$

and similarly for Ψ' . This proves (5) and thus M is a (κ, μ) -space.

Concerning the case $n = 3$, assume henceforth that the almost CR structure (D, J) , $J = \varphi|_D$, is integrable. Then, according to [1, Theorem 6.7], we have that φ is η -parallel:

$$g((\nabla_X \varphi)Y, Z) = 0$$

for every X, Y, Z sections of D . Using this, the symmetry of $\nabla_X h$ and of $\nabla_{JX} h$ and the fact that h anticommutes with φ , it is immediate to verify that Ψ and Ψ' are symmetric in Y and Z . On the other hand, the same argument as above yields that both Ψ and Ψ' are preserved by every orthogonal transformation in $SO(D_p^+)$, hence they must be 3-forms on D_p^+ . Thus we arrive again at $\Psi = \Psi' = 0$, proving our assertion.

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