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Algebraic Structure of Holomorphic Poisson Cohomology on Nilmanifolds

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Abstract: It is proved that on nilmanifolds with abelian complex structure, there exists a canonically constructed non-trivial holomorphic Poisson structure. We identify the necessary and sufficient condition for its associated cohomology to be isomorphic to the cohomology associated to trivial (zero) holomorphic Poisson structure. We also identify a sufficient condition for this isomorphism to be at the level of Gerstenhaber algebras.

1 Introduction

The study of deformation of real Poisson bracket and its related cohomology theory was initiated by Lichnerowicz et. al. long ago [7] [19]. Deformation of Poisson brackets in complex algebraic category emerged 20 years ago [22]. There has been associated deformation theory within the category of complex analytic geometry [1] [15] [25].

In the past fifteen years, holomorphic Poisson structure is also conceived as an object in the realm of generalized geometry [12] [13] [14]. In this vein, one studies deformation within the framework of Lie bi-algebroids [20]. There has been work along this line by various authors [9] [10] [11] [16] [17] [18] [24]. In this note, the authors continue to treat holomorphic Poisson structures within the realm of generalized complex structures, and extend the work in [24].

A holomorphic bi-vector field Λ is Poisson if its self bracket is equal to zero: $[\![\Lambda, \Lambda]\!] = 0$. When $\Lambda = 0$, it is a trivial case and we get nothing but a complex manifold. As the condition $[\![\Lambda, \Lambda]\!] = 0$ is homogeneous, when $\Lambda \neq 0$, for any complex number t , $t\Lambda$ is a holomorphic Poisson structure. When $t \neq 0$, they are all equivalent and could be considered as a deformation of the trivial holomorphic Poisson structure given by $t = 0$. Associated to each non-trivial holomorphic Poisson structure, one has a holomorphic version of Lichnerowicz's differentials $\bar{\partial}_{t\Lambda}$ [3] [18] [19]. They act on the space of sections of exterior bundles of $(1, 0)$ -vectors and $(0, 1)$ -forms.

$$\bar{\partial}_{t\Lambda} : \wedge^k(T^{1,0}M \oplus T^{*(0,1)}M) \rightarrow \wedge^{k+1}(T^{1,0}M \oplus T^{*(0,1)}M),$$

for any non-negative integer k . They form an elliptic complex, and yield cohomology spaces. $\bar{\partial}_{t\Lambda}$ is a perturbation of the classical operator $\bar{\partial}$. When $t = 0$, the resulting cohomology spaces are

$$H_0^k = \oplus_{p+q=k} H^q(M, \Theta^p)$$

where Θ^p is the sheaf of germs of sections of the p -th exterior power of the holomorphic tangent bundle of the underlying complex manifold M . For each $t \neq 0$, we have cohomology spaces $H_{t\Lambda}^k$. They are all isomorphic to H_A^k as demonstrated in an appendix.

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Similar to the classical Lichnerowicz differential, $\bar{\partial}_\Lambda = \bar{\partial} + \text{ad}_\Lambda$ where ad_Λ is the Schouten bracket of the Poisson structure Λ with tensorial objects. In the holomorphic setting, the cohomology H_Λ^k is the hypercohomology of a spectral sequence of a bi-complex. ad_Λ is also the operator on the first page of the spectral sequence:

$$\text{ad}_\Lambda = d_1^{p,q} : H^q(M, \Theta^p) \rightarrow H^q(M, \Theta^{p+1}).$$

The general goal of this piece of work is to understand the relation between the algebraic structures on H_Λ^k and those on H_0^k . Our focus is on nilmanifolds with abelian complex structures. As found in [4] [5] [26] and explained in Section 3 in this paper, on such a manifold there exists a global frame of $(1, 0)$ -forms $\{\omega^1, \dots, \omega^n\}$ such that for each k , $d\omega^k$ is type- $(1, 1)$ and it is in the exterior algebra generated by $\{\omega^1, \dots, \omega^{k-1}, \bar{\omega}^1, \dots, \bar{\omega}^{k-1}\}$. We call such a basis an ascending basis. Let $\{V_1, \dots, V_n\}$ be the dual basis. Its complex linear span is denoted by $\mathfrak{g}^{1,0}$. As to be explained in Definition 1 below, we find that $\Lambda = V_n \wedge V_{n-1}$ is a canonically defined holomorphic Poisson structure.

Our first main result is to identify a necessary and sufficient condition for the map $d_1^{p,q}$ of the spectral sequence associated to this particular holomorphic Poisson structure to be identically zero.

Theorem *Let M be a nilmanifold with abelian complex structure. Let Λ be the canonical holomorphic Poisson structure associated to an ascending basis of the complex structure. The map $d_1^{p,q}$ is identically zero for all p, q if and only if there exists a vector X in $\mathfrak{g}^{1,0}$ such that $\text{ad}_\Lambda \bar{\omega}^n = \bar{\partial}X$. Whenever the vector X exists, we call it a potential vector.*

The above theorem appears as Theorem 3 in Section 4. Additional effort is needed to prove that the degeneracy of the spectral sequence actually lead to a vector space isomorphism

$$\phi : \oplus_{p+q=k} H^q(M, \Theta^p) \rightarrow H_\Lambda^k \quad (1)$$

as stated in Theorem 4. The proof of these two results occupies much of Section 4. They generalize a result in [24].

As explained in Proposition 1, given $\Lambda = V_n \wedge V_{n-1}$, a characterization of the potential vector X is the set of equations below.

$$\iota_X d\omega^k = 0, \quad \text{for all } 1 \leq k \leq n-1, \quad \text{and} \quad \iota_X d\omega^n = -\iota_{V_{n-1}} d\bar{\omega}^n.$$

This characterization also corrects an error in Proposition 9 of [24].

After the conditions on a vector space isomorphism is clarified, we complete an analysis in Theorem 5 on whether the isomorphism ϕ in (1) is an isomorphism of algebras. We state the result below.

Theorem *Suppose that Λ is the canonical holomorphic Poisson structure associated to an ascending basis of an abelian complex structure on a nilmanifold. If X is a potential vector, then the map ϕ is an isomorphism of Gerstenhaber algebras when $\text{ad}_X \bar{\omega}^k = \llbracket X, \bar{\omega}^k \rrbracket = 0$ for all $1 \leq k \leq n-1$.*

The above result upgrades our past observation regarding isomorphism on the level of vector spaces. As an application of the above results, we prove Theorem 6 in Section 6.1 as stated below.

Theorem *On a two-step nilmanifold with abelian complex structure, if $d\omega^n$ is a non-degenerate map then the canonical Poisson cohomology with respect to an ascending basis is isomorphic to the one associated to the trivial (zero) Poisson structure as Gerstenhaber algebras.*

The above result recovers an ad hoc computation by the first author in his paper [23] on Kodaira surfaces, and give the computation a theoretical understanding. It also extends and explains various examples of two-steps nilmanifolds found in [24]. At the end of this article we provide abundant examples beyond two-step nilmanifolds on which the conditions for Theorem 5 are satisfied. We also found examples for which the conclusion for Theorem 3 does not hold. Together, they present a rich subject for further investigation.

2 Algebraic Structures on Poisson Cohomology

Let M be a manifold with an integrable complex structure J . The complexified tangent bundle of M splits into direct sum of the bundle of $(1, 0)$ -vectors $TM^{1,0}$ and the bundle of $(0, 1)$ -vectors $TM^{0,1}$. We denote their dual bundles by $TM^{*(1,0)}$ and $TM^{*(0,1)}$ respectively. Their p -th exterior bundles are denoted by $TM^{p,0}$, $TM^{0,p}$, $TM^{*(p,0)}$ and $TM^{*(0,p)}$ respectively. Define $L = TM^{1,0} \oplus TM^{*(0,1)}$. Via a natural pairing between forms and vectors, its complex conjugation $\bar{L}$ is complex linearly isomorphic to its dual L^* . The pair L and $\bar{L}$ forms a complex Lie bi-algebroid with respect to the Courant bracket [12] [20]. It follows the theory of Lie bi-algebroid [20] [21] and there exists a natural differential

$$\bar{\partial} : C^\infty(M, L) \rightarrow C^\infty(M, \wedge^2 L). \quad (2)$$

When the Courant bracket is restricted to L , it is the Schouten bracket [21], also known as Frölicher-Nijenhuis bracket in classical complex deformation theory. It is extended to a differential of exterior algebras:

$$\bar{\partial} : C^\infty(M, \wedge^k L) \rightarrow C^\infty(M, \wedge^{k+1} L). \quad (3)$$

The differential $\bar{\partial}$ has the properties such that when it is restricted to $(0, 1)$ -forms, it is the classical $\bar{\partial}$ -operator in complex manifold theory; meaning that it is the $(0, 2)$ -component of the exterior differential [11]. Similarly, when the Lie algebroid differential is restricted to $(1, 0)$ -vector fields, it is the Cauchy-Riemann operator as seen in [8]. The differential $\bar{\partial}$, the bracket $\llbracket -, - \rrbracket$ and the exterior product $\wedge$ together equip the space of sections of the exterior bundle $C^\infty(M, \wedge^\bullet L)$ with the structure of a differential Gerstenhaber algebra. In particular, if a is a smooth section of $\wedge^{|a|} L$, where $|a|$ denotes the degree of a in the exterior algebra, and b is a smooth section of $\wedge^{|b|} L$, then

$$\bar{\partial} \llbracket a, b \rrbracket = \llbracket \bar{\partial} a, b \rrbracket + (-1)^{|a|+1} \llbracket a, \bar{\partial} b \rrbracket; \quad (4)$$

$$\bar{\partial}(a \wedge b) = (\bar{\partial} a) \wedge b + (-1)^{|a|} a \wedge (\bar{\partial} b), \quad (5)$$

Since $\bar{\partial} \circ \bar{\partial} = 0$, one obtains the Dolbeault cohomology with coefficients in holomorphic polyvector fields. Denoting the sheaf of germs of sections of the p^{th} exterior power of the holomorphic tangent bundle by Θ^p , we have

$$H_0^\bullet = H_0^\bullet(M, \wedge^\bullet L) \cong \bigoplus_{p,q \geq 0} H^q(M, \Theta^p).$$

In subsequent computations, when $p = 0$, Θ^p represents the structure sheaf $\mathcal{O}$ of the complex manifold M .

Due to the compatibility between $\bar{\partial}$ and the Schouten bracket $\llbracket -, - \rrbracket$, and the compatibility between $\bar{\partial}$ and the exterior product $\wedge$ as noted above, the Schouten bracket and exterior product descend to the cohomology space $H^\bullet(M, \Theta^\bullet)$. In other words, the triple

$$(\oplus_{p,q} H^q(M, \Theta^p), \llbracket -, - \rrbracket, \wedge) \quad (6)$$

forms a Gerstenhaber algebra.

Suppose that Λ is a holomorphic Poisson bivector field, then $\llbracket \Lambda, \Lambda \rrbracket = 0$ and $\bar{\partial} \Lambda = 0$. Consider the map

$$\text{ad}_\Lambda : C^\infty(M, \wedge^k L) \rightarrow C^\infty(M, \wedge^{k+1} L) \quad (7)$$

defined by $\text{ad}_\Lambda(\Phi) = \llbracket \Lambda, \Phi \rrbracket$ where $\Phi \in C^\infty(M, \wedge^k L)$. The identity $\llbracket \Lambda, \Lambda \rrbracket = 0$ is translated into $\text{ad}_\Lambda \circ \text{ad}_\Lambda = 0$. Since the complex structure is integrable, we have $\bar{\partial} \circ \bar{\partial} = 0$. Therefore, the map

$$\bar{\partial}_\Lambda = \text{ad}_\Lambda + \bar{\partial} : C^\infty(M, \wedge^k L) \rightarrow C^\infty(M, \wedge^{k+1} L) \quad (8)$$

satisfies the condition $\bar{\partial}_\Lambda \circ \bar{\partial}_\Lambda = 0$. It generates a complex, and hence a cohomology space. We denote it by $H_\Lambda^\bullet(M)$, and call it the (associated) Poisson cohomology. Furthermore, as Λ is a degree-2 element in the differential Gerstenhaber algebra $(C^\infty(M, \wedge^\bullet L), \bar{\partial}, \llbracket -, - \rrbracket, \wedge)$, the operator ad_Λ satisfies the identities:

$$\begin{aligned} \text{ad}_\Lambda \llbracket a, b \rrbracket &= \llbracket \text{ad}_\Lambda a, b \rrbracket + (-1)^{|a|+1} \llbracket a, \text{ad}_\Lambda b \rrbracket; \\ \text{ad}_\Lambda(a \wedge b) &= (\text{ad}_\Lambda a) \wedge b + (-1)^{|a|} a \wedge (\text{ad}_\Lambda b). \end{aligned}$$

So does $\bar{\partial}_A$.

$$\bar{\partial}_A \llbracket a, b \rrbracket = \llbracket \bar{\partial}_A a, b \rrbracket + (-1)^{|a|+1} \llbracket a, \bar{\partial}_A b \rrbracket; \quad (9)$$

$$\bar{\partial}_A(a \wedge b) = (\bar{\partial}_A a) \wedge b + (-1)^{|a|} a \wedge (\bar{\partial}_A b). \quad (10)$$

It follows that $(C^\infty(M, \wedge^\bullet L), \bar{\partial}_A, \llbracket -, - \rrbracket, \wedge)$ is a differential Gerstenhaber algebra, and $(H_A^\bullet(M), \llbracket -, - \rrbracket, \wedge)$ inherits the structure of a Gerstenhaber algebra. Observe that

$$\wedge^k L = \bigoplus_{p+q=k} (TM^{p,0} \otimes TM^{\star(0,q)})$$

and

$$\text{ad}_A : C^\infty(M, TM^{p,0} \otimes TM^{\star(0,q)}) \rightarrow C^\infty(M, TM^{p+1,0} \otimes TM^{\star(0,q)}), \quad (11)$$

$$\bar{\partial} : C^\infty(M, TM^{p,0} \otimes TM^{\star(0,q)}) \rightarrow C^\infty(M, TM^{p,0} \otimes TM^{\star(0,q+1)}). \quad (12)$$

Since $\bar{\partial} \circ \bar{\partial} = 0$, $\text{ad}_A \circ \bar{\partial} + \bar{\partial} \circ \text{ad}_A = 0$, and $\text{ad}_A \circ \text{ad}_A = 0$, the cohomology $H_A^\bullet(M)$ is given by the hypercohomology of the bi-complex $C^\infty(M, TM^{\bullet,0} \otimes TM^{\star(0,\bullet)})$. The first page of related spectral sequence is given by the classical Dolbeault cohomology of $(0, q)$ -forms with coefficients in the sheaf of germs of holomorphic p -vector fields [27].

$$E_1^{p,q} = H^q(M, \Theta^p). \quad (13)$$

For $p = 0$, $E_1^{0,q} = H^q(M, \mathcal{O})$, the space of holomorphic $(0, q)$ -forms. The map $d_1^{p,q}$ in the first page of the spectral sequence is the map ad_A , which descends from (11) to cohomology level.

$$\text{ad}_A : H^q(M, \Theta^p) \rightarrow H^q(M, \Theta^{p+1}). \quad (14)$$

3 Abelian Complex Structure on Nilmanifolds

After reviewing some fundamental observations on complex structures on nilmanifolds as given in [26], we set up several technical observations to facilitate the computation in subsequent sections.

A nilmanifold M is a co-compact quotient of a simply connected nilpotent Lie group G . Denote the Lie algebra of the group G by $\mathfrak{g}$. We assume that $\dim_R \mathfrak{g} = 2n$. Denote the dual space by $\mathfrak{g}^*$. One has the Chevalley-Eilenberg differential d . If α is in $\mathfrak{g}^*$, X, Y are in $\mathfrak{g}$, then by definition

$$d\alpha(X, Y) = -\alpha(\llbracket X, Y \rrbracket). \quad (15)$$

It is extended to the Chevalley-Eilenberg complex,

$$0 \rightarrow \mathfrak{g}^* \xrightarrow{d} \wedge^2 \mathfrak{g}^* \xrightarrow{d} \cdots \xrightarrow{d} \wedge^{2n} \mathfrak{g}^* \rightarrow 0.$$

The descending central series of $\mathfrak{g}$ is the chain of ideals defined inductively by $\mathfrak{g}^0 = \mathfrak{g}$ and $\mathfrak{g}^j = [\mathfrak{g}^{j-1}, \mathfrak{g}]$ for all $j \geq 1$. By definition, the Lie algebra $\mathfrak{g}$ is s -step nilpotent if $\mathfrak{g}^s = \{0\}$ and $\mathfrak{g}^{s-1} \neq \{0\}$. In particular, the chain of ideals terminates,

$$\mathfrak{g} = \mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \cdots \supseteq \mathfrak{g}^{s-1} \supseteq \mathfrak{g}^s = \{0\}.$$

A real linear map $J : \mathfrak{g} \rightarrow \mathfrak{g}$ with the properties that $J \circ J = -\text{identity}$ and for all $X, Y \in \mathfrak{g}$,

$$\llbracket JX, JY \rrbracket = \llbracket X, Y \rrbracket \quad (16)$$

is an abelian complex structure. Denote the spaces of $(1, 0)$ -vectors and $(0, 1)$ -vectors respectively by $\mathfrak{g}^{1,0}$ and $\mathfrak{g}^{0,1}$. The identity (16) above implies that $\mathfrak{g}^{1,0}$ is an abelian complex algebra. Therefore, the only Lie bracket among elements of $\mathfrak{g}_C$ is of the form $\llbracket X, \bar{Y} \rrbracket$ and its complex conjugate, where $X \in \mathfrak{g}^{1,0}$ and $\bar{Y} \in \mathfrak{g}^{0,1}$.

Taking the adjoint of J , one obtains a map, also denoted by J , from $\mathfrak{g}^*$ to itself. Denote the space of $(1, 0)$ -forms by

$$\mathfrak{g}^{\star(1,0)} = \{\alpha - iJ\alpha : \alpha \in \mathfrak{g}^*\}.$$

Its complex conjugation is the space of $(0, 1)$ -forms. It is denoted by $\mathfrak{g}^{*(0,1)}$.

Their p -th exterior products are respectively denoted by $\mathfrak{g}^{p,0}$, $\mathfrak{g}^{0,p}$, $\mathfrak{g}^{*(p,0)}$ and $\mathfrak{g}^{*(0,p)}$. In particular, by taking invariant polyvector fields and invariant forms, we have an inclusion map

$$\iota : \mathfrak{g}^{p,0} \otimes \mathfrak{g}^{*(0,q)} \rightarrow C^\infty(M, TM^{p,0} \otimes TM^{*(0,q)}). \quad (17)$$

Theorem 1. [2] *The inclusion map ι is a quasi-isomorphism. In other words, it induces an isomorphism of cohomology*

$$\iota : H^q(\mathfrak{g}^{p,0}) \cong H^q(M, \Theta^p).$$

The Chevelley-Eilenberg complex is complex linearly extended. When the complex structure is abelian as defined in (16), then $d\alpha$ is a type- $(1,1)$ form when α is a type- $(1,0)$ form. In the rest of this article, the following observation will play a crucial role.

Theorem 2. [4] [5] [26] *If $\mathfrak{g}$ is a nilpotent Lie algebra with an abelian complex structure, then there exists a basis $\{\omega^1, \dots, \omega^n\}$ of $\mathfrak{g}^{*(1,0)}$ such that*

$$d\omega^{j+1} \in I(\omega^1, \dots, \omega^j) \wedge I(\bar{\omega}^1, \dots, \bar{\omega}^j),$$

where $I(\omega^1, \dots, \omega^j)$ denotes the ideal generated by $\{\omega^1, \dots, \omega^j\}$.

In other words, the structure constants $A_{k\ell}^j$ are such that for each j ,

$$d\omega^j = \sum_{1 \leq k, \ell \leq j-1} A_{k\ell}^j \omega^k \wedge \bar{\omega}^\ell. \quad (18)$$

Taking complex conjugation and rearranging indices, we get

$$d\bar{\omega}^j = - \sum_{1 \leq k, \ell \leq j-1} \bar{A}_{\ell k}^j \omega^k \wedge \bar{\omega}^\ell. \quad (19)$$

In particular,

$$d\omega^n = \sum_{1 \leq k, \ell \leq n-1} A_{k\ell}^n \omega^k \wedge \bar{\omega}^\ell, \quad \text{and} \quad d\bar{\omega}^n = - \sum_{1 \leq k, \ell \leq n-1} \bar{A}_{\ell k}^n \omega^k \wedge \bar{\omega}^\ell. \quad (20)$$

As noted in [26], one obtains the basis $\{\omega^1, \dots, \omega^n\}$ inductively starting with $d\omega^1 = 0$. Therefore, we address this basis for $\mathfrak{g}^{1,0}$ as an *ascending basis*. Apparently, an ascending basis is not necessarily unique. Let $\{V_1, \dots, V_n\}$ be the dual basis for $\mathfrak{g}^{1,0}$. We will refer to it as the (dual) ascending basis for $\mathfrak{g}^{1,0}$.

We will also use the following notations,

$$\mathfrak{t}^{1,0} = \text{span of } \{V_1, \dots, V_{n-1}\}, \quad \mathfrak{c}^{1,0} = \text{span of } \{V_n\}, \quad (21)$$

$$\mathfrak{t}^{*(1,0)} = \text{span of } \{\omega^1, \dots, \omega^{n-1}\}, \quad \mathfrak{c}^{*(1,0)} = \text{span of } \{\omega^n\}. \quad (22)$$

Their conjugated counterparts $\mathfrak{t}^{0,1}$, $\mathfrak{c}^{0,1}$, $\mathfrak{t}^{*(0,1)}$ and $\mathfrak{c}^{*(0,1)}$ are naturally defined. With these notations, we consider the contraction of $d\omega^n$ and $d\bar{\omega}^n$ as linear maps.

$$d\omega^n : \mathfrak{t}^{1,0} \rightarrow \mathfrak{t}^{*(0,1)}, \quad d\bar{\omega}^n : \mathfrak{t}^{1,0} \rightarrow \mathfrak{t}^{*(0,1)}. \quad (23)$$

Moreover, with the given ordered bases, the matrix representation A^n of $d\omega^n$ is $A_{k\ell}^n$ acting on row vectors from the right. i.e. if $V = c^1 V_1 + \dots + c^{n-1} V_{n-1}$, then

$$\iota_V d\omega^n = \sum_{1 \leq k, \ell \leq n-1} c^k A_{k\ell}^n \bar{\omega}^\ell.$$

For instance,

$$\iota_{V_{n-1}} d\omega^n = \sum_{1 \leq \ell \leq n-1} A_{n-1, \ell}^n \bar{\omega}^\ell. \quad (24)$$

The matrix representation of $d\bar{\omega}^n$ is $-\bar{A}^n{}^T$.

From the structure equations (18) and (19), one finds the expression of structure equations in the basis $\{V_1, \dots, V_n\} \cup \{\bar{V}_1, \dots, \bar{V}_n\}$ for the complexified Lie algebra $\mathfrak{g}_\mathbb{C}$. Namely,

$$[V_k, \bar{V}_\ell] = - \sum_{k, \ell \leq j-1} A_{k\ell}^j V_j + \sum_{k, \ell \leq j-1} \bar{A}_{\ell k}^j \bar{V}_j. \quad (25)$$

It is now apparent that V_n is in the center of $\mathfrak{g}_\mathbb{C}$. Moreover, for each ω^k in the ascending basis and any $\bar{Y}$ in $\mathfrak{g}^{0,1}$,

$$(\bar{\partial}X)(\omega^k, \bar{Y}) = X([V_n, \omega^k]) = X(\iota_{\bar{Y}} d\omega^k) = -d\omega^k(X, \bar{Y}) = -(\iota_X d\omega^k)(\bar{Y}).$$

It follows that

$$\bar{\partial}X = - \sum_{k=1}^n V_k \wedge (\iota_X d\omega^k). \quad (26)$$

As a result of (24),

$$\bar{\partial}V_n = 0, \quad \text{and} \quad \bar{\partial}V_{n-1} = -V_n \wedge \iota_{V_{n-1}} d\omega^n. \quad (27)$$

Corollary 1. $\Lambda = V_n \wedge V_{n-1}$ is a holomorphic Poisson bivector.

Proof: Λ is Poisson because the complex algebra $\mathfrak{g}^{1,0}$ is abelian. It is holomorphic because

$$\bar{\partial}(V_n \wedge V_{n-1}) = (\bar{\partial}V_n) \wedge V_{n-1} - V_n \wedge (\bar{\partial}V_{n-1}) = V_n \wedge V_n \wedge \left(\sum_{k=1}^{n-1} A_{n-1,k}^n \bar{\omega}^k \right) = 0.$$

□

Definition 1. Given an ascending basis $\{V_1, \dots, V_{n-1}, V_n\}$ for $\mathfrak{g}^{1,0}$ on a nilmanifold with abelian complex structure, we call $\Lambda = V_n \wedge V_{n-1}$ its canonical holomorphic Poisson structure, or simply a canonical Poisson structure of the abelian complex structure.

4 Cohomology of Canonical Poisson Structures

Since the canonical Poisson structure is invariant, given the quasi-isomorphism in Theorem 1, one could reduce the spectral sequence computation with E_1 -terms given in (13) to a computation of invariant objects. Namely, the d_1 map is reduced to

$$d_1^{p,q} = \text{ad}_\Lambda : H^q(\mathfrak{g}^{p,0}) \rightarrow H^q(\mathfrak{g}^{p+1,0}). \quad (28)$$

The goal of this section is to prove the following theorem, which generalizes a key result in [24].

Theorem 3. Let M be a nilmanifold with abelian complex structure. Let Λ be the canonical holomorphic Poisson structure associated to an ascending basis of the complex structure. The map $d_1^{p,q}$ is identically zero for all p, q if and only if there exists a vector X in $\mathfrak{g}^{1,0}$ such that $\text{ad}_\Lambda \bar{\omega}^n = \bar{\partial}X$.

We call such X , if exists, a potential vector for the canonical holomorphic Poisson structure. The proof below is similar to, but simplifies an argument in [24] due to the adoption of an ascending basis.

Proof: One direction of the computation is straightforward. Assume that the map $d_1^{p,q}$ is identically zero for all p, q . When $(p, q) = (0, 1)$, the space $H^1(\mathfrak{g}^{0,0})$ is precisely the space of invariant closed $(0, 1)$ -forms. Given

that the complex structure is abelian, it is apparent that every element in an ascending basis $\{\bar{\omega}^1, \dots, \bar{\omega}^n\}$ is $\bar{\partial}$ -closed. Furthermore, given that it is an ascending basis,

$$\iota_{V_n} d\bar{\omega}^j = 0 \text{ for all } 1 \leq j \leq n, \quad \text{and} \quad \iota_{V_{n-1}} d\bar{\omega}^k = 0 \text{ for all } 1 \leq k \leq n-1. \quad (29)$$

Therefore, the only possible non-trivial term in the computation of $d_1^{0,1}$ is contributed by

$$\text{ad}_\Lambda \bar{\omega}^n = \llbracket V_n \wedge V_{n-1}, \bar{\omega}^n \rrbracket = V_n \wedge \iota_{V_{n-1}} d\bar{\omega}^n.$$

It is equal to zero in $H^1(\mathfrak{g}^{1,0})$ as a cohomology class if and only if there exists an element X in $\mathfrak{g}^{1,0}$ such that

$$\text{ad}_\Lambda(\bar{\omega}^n) = \bar{\partial}X. \quad (30)$$

Next, we prove that the existence of a potential vector is a sufficient condition for the spectral sequence to degenerate. Suppose $\Gamma^{p,q}$ is an element in $\mathfrak{g}^{p,0} \otimes \mathfrak{g}^{*(0,q)}$ representing a class in $H^q(\mathfrak{g}^{p,0})$. Due to the direct sum decomposition $\mathfrak{g}^{*(0,1)} = \mathfrak{c}^{*(0,1)} \oplus \mathfrak{t}^{*(0,1)}$, we have

$$\mathfrak{g}^{p,0} \otimes \mathfrak{g}^{*(0,q)} = \mathfrak{g}^{p,0} \otimes \wedge^q (\mathfrak{c}^{*(0,1)} \oplus \mathfrak{t}^{*(0,1)}) = \mathfrak{g}^{p,0} \otimes (\mathfrak{c}^{*(0,1)} \otimes \mathfrak{t}^{*(0,q-1)} \oplus \mathfrak{t}^{*(0,q)}). \quad (31)$$

Therefore,

$$\Gamma^{p,q} = \bar{\omega}^n \wedge \sum_a (\Pi_a \wedge \bar{\Phi}_a) + \sum_b Y_b \wedge \bar{\Psi}_b, \quad (32)$$

where $\{\bar{\Phi}_a, a = 1, 2, \dots\}$ and $\{\bar{\Psi}_b, b = 1, 2, \dots\}$ are generated by the exterior products of $\{\bar{\omega}^1, \dots, \bar{\omega}^n\}$, and they form bases for $\mathfrak{t}^{*(0,q-1)}$ and $\mathfrak{t}^{*(0,q)}$ respectively. The terms Π_a and Y_b are elements in $\mathfrak{g}^{p,0}$.

Since $\llbracket V_n, \bar{\omega}^k \rrbracket = \iota_{V_n} d\bar{\omega}^k = 0$ for all k and $\llbracket V_{n-1}, \bar{\omega}^k \rrbracket = \iota_{V_{n-1}} d\bar{\omega}^k = 0$ for all $k \leq n-1$,

$$\begin{aligned} \text{ad}_\Lambda(\Gamma^{p,q}) &= \text{ad}_\Lambda(\bar{\omega}^n \wedge \sum_a (\Pi_a \wedge \bar{\Phi}_a)) = \llbracket V_n \wedge V_{n-1}, \bar{\omega}^n \wedge \sum_a (\Pi_a \wedge \bar{\Phi}_a) \rrbracket \\ &= V_n \wedge \llbracket V_{n-1}, \bar{\omega}^n \wedge \sum_a (\Pi_a \wedge \bar{\Phi}_a) \rrbracket = V_n \wedge \llbracket V_{n-1}, \bar{\omega}^n \rrbracket \wedge \sum_a (\Pi_a \wedge \bar{\Phi}_a) \\ &= \llbracket \Lambda, \bar{\omega}^n \rrbracket \wedge \sum_a (\Pi_a \wedge \bar{\Phi}_a) = (\bar{\partial}X) \wedge \sum_a (\Pi_a \wedge \bar{\Phi}_a). \end{aligned}$$

On the other hand, since $\Gamma^{p,q}$ is $\bar{\partial}$ -closed and all $(0, 1)$ -forms of an ascending basis are also $\bar{\partial}$ -closed,

$$\begin{aligned} \bar{\partial}\Gamma^{p,q} &= \bar{\partial}(\bar{\omega}^n \wedge \sum_a (\Pi_a \wedge \bar{\Phi}_a)) + \sum_b \bar{\partial}(Y_b \wedge \bar{\Psi}_b) \\ &= -\bar{\omega}^n \wedge \sum_a ((\bar{\partial}\Pi_a) \wedge \bar{\Phi}_a) + \sum_b (\bar{\partial}Y_b) \wedge \bar{\Psi}_b. \end{aligned}$$

In view of the nature of an ascending basis, every $(\bar{\partial}Y_b) \wedge \bar{\Psi}_b$ is an element in $\mathfrak{g}^{p,0} \otimes \mathfrak{t}^{*(0,q+1)}$ while $\bar{\omega}^n \wedge \sum_a ((\bar{\partial}\Pi_a) \wedge \bar{\Phi}_a)$ is in $\mathfrak{g}^{p,0} \otimes \mathfrak{c}^{*(0,1)} \otimes \mathfrak{t}^{*(0,q)}$. Therefore, $\bar{\partial}\Gamma^{p,q} = 0$ if and only if

$$\bar{\omega}^n \wedge \sum_a ((\bar{\partial}\Pi_a) \wedge \bar{\Phi}_a) = 0, \quad \text{and} \quad \sum_b (\bar{\partial}Y_b) \wedge \bar{\Psi}_b = 0.$$

In particular,

$$\bar{\partial} \sum_a (\Pi_a \wedge \bar{\Phi}_a) = \sum_a ((\bar{\partial}\Pi_a) \wedge \bar{\Phi}_a) = 0.$$

It follows from the previous paragraph that

$$\text{ad}_\Lambda(\Gamma^{p,q}) = (\bar{\partial}X) \wedge \sum_a (\Pi_a \wedge \bar{\Phi}_a) = \bar{\partial} \left(X \wedge \sum_a (\Pi_a \wedge \bar{\Phi}_a) \right). \quad (33)$$

In terms of spectral sequence, $d_1^{p,q}(\Gamma^{p,q}) = \text{ad}_\Lambda(\Gamma^{p,q}) \equiv 0$ as a cohomology class in $H^{q+1}(\mathfrak{g}^{p,0})$, concluding the proof of Theorem 3. $\square$

In subsequent computation, given any element $\Gamma^{p,q}$ in $\mathfrak{g}^{p,0} \otimes \mathfrak{g}^{*(0,q)}$, we represent its decomposition in (31) and (32) by

$$\Gamma^{p,q} = \bar{\omega}^n \wedge \alpha^{p,q-1} + \beta^{p,q}. \quad (34)$$

The proof above means that whenever $\bar{\partial}\Gamma^{p,q} = 0$, then $\bar{\partial}\alpha^{p,q-1} = 0$ and $\bar{\partial}\beta^{p,q} = 0$, and

$$\text{ad}_\Lambda(\Gamma^{p,q}) = \bar{\partial}(X \wedge \alpha^{p,q-1}) \quad (35)$$

when $\text{ad}_\Lambda \bar{\omega}^n = \bar{\partial}X$.

Theorem 4. When $\text{ad}_\Lambda(\bar{\omega}^n)$ is $\bar{\partial}$ -exact, there is a natural isomorphism

$$\phi : \oplus_{p+q=k} H^q(\mathfrak{g}^{p,0}) \rightarrow H_\Lambda^k.$$

Proof: Let

$$B^{p,q} = \mathfrak{g}^{p,0} \otimes \mathfrak{g}^{*(0,q)}. \quad (36)$$

As noted in [24], the bi-grading yields a natural filtration of the cohomology H_Λ^n :

$$H_\Lambda^k = F^0 H^k \supset F^1 H^k \supset \dots \supset F^\ell H^k \supset \dots \supset F^k H^k \supset F^{k+1} H^k = \{0\} \quad (37)$$

such that

$$H_\Lambda^k \cong \oplus_{m=0}^k \frac{F^m H^k}{F^{m+1} H^k}.$$

There is a natural injective map from

$$\psi : \frac{F^m H^k}{F^{m+1} H^k} \rightarrow H^{k-m}(\mathfrak{g}^{m,0})$$

defined as below. A non-trivial class in $\frac{F^m H^k}{F^{m+1} H^k}$ is presented by

$$\Gamma = \sum_{m \leq p \leq k, p+q=k} \Gamma^{p,q} = \Gamma^{k,0} + \Gamma^{k-1,1} + \dots + \Gamma^{m+1,k-m-1} + \Gamma^{m,k-m}$$

with $\Gamma^{m,k-m} \neq 0$ such that $\bar{\partial}_\Lambda \Gamma = 0$. Recall that $\bar{\partial}_\Lambda = \text{ad}_\Lambda + \bar{\partial}$, when we group $\bar{\partial}_\Lambda \Gamma$ by bi-degrees, we find

$$\bar{\partial}_\Lambda \Gamma = \text{ad}_\Lambda \Gamma^{k,0} + (\bar{\partial} \Gamma^{k,0} + \text{ad}_\Lambda \Gamma^{k-1,1}) + \dots + (\bar{\partial} \Gamma^{m+1,k-m-1} + \text{ad}_\Lambda \Gamma^{m,k-m}) + \bar{\partial} \Gamma^{m,k-m}.$$

Therefore, $\bar{\partial} \Gamma^{m,k-m} = 0$, and hence $\Gamma^{m,k-m}$ represents a class in $H^{k-m}(\mathfrak{g}^{m,0})$. One could verify that it represents a well defined image of $\psi(\Gamma)$.

To prove that the map ψ is surjective, recall the decomposition of $\Gamma^{p,q}$ as in (32), or equivalently (34) when $q \geq 1$.

$$\begin{aligned} \bar{\partial}_\Lambda(\Gamma^{p,q} - X \wedge \alpha^{p,q-1}) &= \text{ad}_\Lambda(\Gamma^{p,q} - X \wedge \alpha^{p,q-1}) + \bar{\partial}(\Gamma^{p,q} - X \wedge \alpha^{p,q-1}) \\ &= \text{ad}_\Lambda \Gamma^{p,q} - \bar{\partial}(X \wedge \alpha^{p,q-1}) - \text{ad}_\Lambda(X \wedge \alpha^{p,q-1}) + \bar{\partial} \Gamma^{p,q} \\ &= -\text{ad}_\Lambda(X \wedge \alpha^{p,q-1}). \end{aligned}$$

Note that $\alpha^{p,q-1}$ is in $\mathfrak{g}^{p,0} \otimes \mathfrak{t}^{*(0,q-1)}$ and the complex structure is abelian. The only non-trivial outcome in

$$\text{ad}_\Lambda(X \wedge \alpha^{p,q-1}) = \llbracket V_n \wedge V_{n-1}, X \wedge \alpha^{p,q-1} \rrbracket$$

is contributed by $\llbracket V_n, \bar{\omega}^k \rrbracket$ and $\llbracket V_{n-1}, \bar{\omega}^k \rrbracket$ for $1 \leq k \leq n-1$ in a fixed ascending basis. However, by nature of an ascending basis (20), these items are equal to zero. It follows that for all $q \geq 1$.

$$\bar{\partial}_\Lambda(\Gamma^{p,q} - X \wedge \alpha^{p,q-1}) = 0. \quad (38)$$

It shows that if

$$\Gamma^{p,q} = \bar{\omega}^n \wedge \alpha^{p,q-1} + \beta^{p,q},$$

represents a class in $H^q(\mathfrak{g}^{p,0})$, then

$$\hat{I} = -X \wedge \alpha^{p,q-1} + \Gamma^{p,q} \quad (39)$$

is $\bar{\partial}_\Lambda$ -closed. It represents a class in

$$\frac{F^{k-q} H^{p+q}}{F^{k-q+1} H^{p+q}} = \frac{F^p H^{p+q}}{F^{p+1} H^{p+q}}$$

such that $\psi(\hat{I}) = \Gamma^{p,q}$. Therefore the map ψ is an isomorphism. The map ϕ is its inverse. $\square$

Proposition 1. *A vector X is a potential vector of the canonical Poisson structure $V_n \wedge V_{n-1}$ associated to an ascending basis $\{\bar{\omega}^1, \dots, \bar{\omega}^n\}$ if and only if*

$$\llbracket X, \omega^k \rrbracket = \iota_X d\omega^k = 0 \text{ for all } 1 \leq k \leq n-1, \quad \text{and} \quad \iota_X d\omega^n = -\iota_{V_{n-1}} d\bar{\omega}^n. \quad (40)$$

Proof: Identity (26) states that

$$\bar{\partial}X = -\sum_{k=1}^n V_k \wedge (\iota_X d\omega^k).$$

Since

$$\text{ad}_\Lambda(\bar{\omega}^n) = V_n \wedge \llbracket V_{n-1}, \bar{\omega}^n \rrbracket = V_n \wedge \iota_{V_{n-1}} d\bar{\omega}^n \quad (41)$$

and $\{V_1, \dots, V_n\}$ is a basis for $\mathfrak{g}^{1,0}$, the conclusion of the proposition follows. $\square$

Remark The above observation corrects an error in the statement of Proposition 9.2 in [24] for omitting a requirement on the step of the nilmanifold.

5 Identification of Gerstenhaber Algebras

Let

$$\Gamma = (\Gamma^{p+q,0}, \Gamma^{p+q-1,1}, \dots, \Gamma^{p+1,q-1}, \Gamma^{p,q}, \dots, \Gamma^{1,p+q-1}, \Gamma^{0,p+q})$$

represents an element in $\oplus_{p+q=k} H^q(\mathfrak{g}^{p,0})$, the proof of Proposition 4 shows that it is lifted to

$$\begin{aligned} \phi(\Gamma^{p+q}) &= \phi(\Gamma^{p+q,0}, \Gamma^{p+q-1,1}, \dots, \Gamma^{p+1,q-1}, \Gamma^{p,q}, \Gamma^{p-1,q+1}, \dots, \Gamma^{1,p+q-1}, \Gamma^{0,p+q}) \\ &= \Gamma^{p+q,0} + (\Gamma^{p+q-1,1} - X \wedge \alpha^{p+q-1,0}) + \dots \\ &\quad + (\Gamma^{p,q} - X \wedge \alpha^{p,q-1}) \dots \\ &\quad + (\Gamma^{1,p+q-1} - X \wedge \alpha^{1,p+q-2}) + (\Gamma^{0,p+q} - X \wedge \alpha^{0,p+q-1}). \end{aligned} \quad (42)$$

Since $\Gamma^{p+q,0}$ is in $\mathfrak{g}^{p+q,0}$ and the complex structure is abelian, it is $\bar{\partial}_\Lambda$ -closed so long as it is $\bar{\partial}$ -closed. With Identity (38) above, we conclude that the map ϕ takes Γ^{p+q} into H_Λ^{p+q} .

If one arranges $\phi(\Gamma^{p+q})$ in terms of the bi-degree (p, q) in the decomposition of $\oplus_{p+q} B^{p,q}$, then

$$\begin{aligned} \phi(\Gamma^{p+q}) &= (\Gamma^{p+q,0} - X \wedge \alpha^{p+q-1,0}) + (\Gamma^{p+q-1,1} - X \wedge \alpha^{p+q-2,1}) \dots \\ &\quad + (\Gamma^{p,q} - X \wedge \alpha^{p-1,q}) + \dots \\ &\quad + (\Gamma^{1,p+q-1} - X \wedge \alpha^{0,p+q-1}) + \Gamma^{0,p+q}. \end{aligned} \quad (43)$$

Suppose that $\text{ad}_\Lambda(\bar{\omega}^n)$ is $\bar{\partial}$ -exact and X is a potential vector, so that ϕ is an isomorphism of vector spaces from $\oplus_{p+q=k} H^q(\mathfrak{g}^{p,0})$ to H_Λ^k . Recall that $\oplus_{p+q=k} H^q(\mathfrak{g}^{p,0})$ inherits the structure of a Gerstenhaber algebra. In fact, one may consider it to be the Gerstenhaber algebra of a trivial (zero) holomorphic Poisson structure; H_Λ^k has its own Gerstenhaber algebra structure. In the next theorem, we identify a situation when ϕ is an isomorphism of Gerstenhaber algebras.

Theorem 5. Suppose that Λ is the canonical holomorphic Poisson structure associated to an ascending basis of an abelian complex structure on a nilmanifold. If $\text{ad}_\Lambda(\bar{\omega}^n) = \bar{\partial}X$ for a vector X , then the map ϕ is an isomorphism of Gerstenhaber algebras if the action of $\text{ad}_X = \llbracket X, - \rrbracket$ on $\mathfrak{t}^{*(0,1)}$ is identically zero.

Proof: Recall the decomposition

$$B^{p,q} = \mathfrak{g}^{p,0} \otimes \{\bar{\omega}^n\} \otimes \mathfrak{t}^{*(0,q-1)} \oplus \mathfrak{g}^{p,0} \otimes \mathfrak{t}^{*(0,q)}. \quad (44)$$

When Γ is in $B^{p,q}$ and Y is in $B^{k,\ell}$, then they have decompositions

$$\Gamma = \bar{\omega}^n \wedge \alpha + \beta, \quad \text{and} \quad Y = \bar{\omega}^n \wedge \epsilon + \gamma, \quad (45)$$

where $\alpha \in \mathfrak{g}^{p,0} \otimes \mathfrak{t}^{*(0,q-1)}$, $\beta \in \mathfrak{g}^{p,0} \otimes \mathfrak{t}^{*(0,q)}$, $\epsilon \in \mathfrak{g}^{p,0} \otimes \mathfrak{t}^{*(0,\ell-1)}$ and $\gamma \in \mathfrak{g}^{p,0} \otimes \mathfrak{t}^{*(0,\ell)}$. Their exterior product has a corresponding decomposition.

$$\Gamma \wedge Y = \bar{\omega}^n \wedge (\alpha \wedge \gamma + (-1)^{p+q} \beta \wedge \epsilon) + \beta \wedge \gamma.$$

It follows (39) that

$$\phi(\Gamma \wedge Y) = \Gamma \wedge Y - X \wedge (\alpha \wedge \gamma + (-1)^{p+q} \beta \wedge \epsilon).$$

On the other hand, (39) and (45) together implies that

$$\phi(\Gamma) = \Gamma - X \wedge \alpha = \bar{\omega}^n \wedge \alpha + \beta - X \wedge \alpha \quad (46)$$

and

$$\phi(Y) = Y - X \wedge \epsilon = \bar{\omega}^n \wedge \epsilon + \gamma - X \wedge \epsilon. \quad (47)$$

It follows that

$$\begin{aligned} \phi(\Gamma) \wedge \phi(Y) &= \Gamma \wedge Y - (\bar{\omega}^n \wedge \alpha + \beta) \wedge X \wedge \epsilon - X \wedge \alpha \wedge (\bar{\omega}^n \wedge \epsilon + \gamma) \\ &= \Gamma \wedge Y - X \wedge (\alpha \wedge \gamma + (-1)^{p+q} \beta \wedge \epsilon) \\ &= \phi(\Gamma \wedge Y). \end{aligned}$$

Next we compare $\phi(\llbracket \Gamma, Y \rrbracket)$ with $\llbracket \phi(\Gamma), \phi(Y) \rrbracket$. Making use of (46) and (47) and after a tedious computation, we verify that

$$\begin{aligned} \llbracket \phi(\Gamma), \phi(Y) \rrbracket - \phi(\llbracket \Gamma, Y \rrbracket) &= -\bar{\omega}^n \wedge \llbracket \alpha, X \rrbracket \wedge \epsilon - \llbracket \beta, X \rrbracket \wedge \epsilon + X \wedge \llbracket \alpha, X \rrbracket \wedge \epsilon \\ &\quad - (-1)^{p+q+k+\ell} \alpha \wedge \llbracket X, \epsilon \rrbracket \wedge \bar{\omega}^n + (-1)^{p+q} \alpha \wedge \llbracket X, \gamma \rrbracket \\ &\quad + (-1)^{p+q+k+\ell} \alpha \wedge \llbracket X, \epsilon \rrbracket \wedge X. \end{aligned} \quad (48)$$

Since the complex structure is abelian, the action of $\text{ad}_X = \llbracket X, - \rrbracket$ on $\mathfrak{t}^{p,0}$ is identically zero for all p . Therefore, if the action of ad_X on $\mathfrak{t}^{*(0,1)}$ is also identically zero, then ad_X is identically zero on $\mathfrak{t}^{p,0} \otimes \mathfrak{t}^{*(0,q)}$ for all $p, q \geq 0$. It follows that

$$\llbracket \phi(\Gamma), \phi(Y) \rrbracket = \phi(\llbracket \Gamma, Y \rrbracket).$$

□

6 Applications

6.1 Two-steps case

As an application of various observations in the previous sections, we focus on two-steps nilmanifold with abelian complex structure. Suppose that the map

$$d\omega^n : \mathfrak{t}^{1,0} \rightarrow \mathfrak{t}^{*(0,1)}$$

is non-degenerate, so that $d\bar{\omega}^n$ is non-degenerate as well. It follows that $-\iota_{V_{n-1}}d\bar{\omega}^n$ is a non-zero vector. As $d\omega^n$ is non-degenerate, there exists X in $\mathfrak{t}^{1,0}$ such that

$$\iota_X d\omega^n = -\iota_{V_{n-1}} d\bar{\omega}^n.$$

By nature of an ascending basis and the assumption that the algebra is two-step, $d\omega^k = 0$ for $1 \leq k \leq n-1$. It follows that all conditions in Proposition 1 are satisfied. Therefore, X is a potential vector.

Furthermore, as $d\bar{\omega}^k = 0$ for all $1 \leq k \leq n-1$ as well, it follows that $\llbracket X, \bar{\omega}^k \rrbracket = \iota_X d\bar{\omega}^k = 0$ for all $\bar{\omega}^k$ with $1 \leq k \leq n-1$. As they span $\mathfrak{t}^{*(0,1)}$, then the potential vector satisfies the conditions in Theorem 5. As a consequence, we obtain the following observation.

Theorem 6. *On a two-step nilmanifold with abelian complex structure, if $d\omega^n$ is non-degenerate, the canonical Poisson cohomology with respect to an ascending basis is isomorphic as Gerstenhaber algebras to the one associated to the trivial (zero) Poisson structure. i.e.*

$$\left(H_{\Lambda}^k(M), \llbracket -, - \rrbracket, \wedge \right) \cong \left(\oplus_{p+q=k} H^q(M, \Theta^p), \llbracket -, - \rrbracket, \wedge \right).$$

The theorem above provides a theoretical explanation on a collection of examples in [24].

6.2 Four-dimension example

The only non-trivial real four-dimension example is the Kodaira surface, whose corresponding Lie algebra is a two-step nilpotent algebra [26]. In terms of an ascending basis $\{\omega^1, \omega^2\}$, its structure equations are

$$d\omega^1 = 0, \quad d\omega^2 = C\omega^1 \wedge \bar{\omega}^1$$

where C is a non-zero constant. When $\{V_1, V_2\}$ is the dual basis, it is apparent that

$$\iota_{V_1} d\bar{\omega}^2 = -\bar{C}\bar{\omega}^1, \quad \iota_{V_1} d\omega^2 = C\bar{\omega}^1.$$

Then $X = -\frac{\bar{C}}{C}V_2$ is a potential vector. In this case, the isomorphism is at the level of Gerstenhaber algebras as predicted in Theorem 6 above, which provides a theoretical explanation for a computation in [23].

6.3 A counterexample

Here we present an example to illustrate that the issue in this paper is non-trivial.

Consider a real vector space W_6 spanned by $\{X_1, X_2, X_3, X_4, Z_1, Z_2\}$. Define a Lie bracket by

$$\llbracket X_1, X_3 \rrbracket = -\frac{1}{2}Z_1, \quad \llbracket X_1, X_4 \rrbracket = -\frac{1}{2}Z_2, \quad \llbracket X_2, X_3 \rrbracket = -\frac{1}{2}Z_2, \quad \llbracket X_2, X_4 \rrbracket = \frac{1}{2}Z_1.$$

Consider $J \circ J = -\text{identity}$ and

$$JX_1 = X_2, \quad JX_4 = X_3, \quad JZ_2 = Z_1.$$

It is an abelian complex structure. A basis for $\mathfrak{g}^{1,0}$ is

$$V_1 = \frac{1}{2}(X_1 - iX_2), \quad V_2 = \frac{1}{2}(X_3 + iX_4), \quad V_3 = \frac{1}{2}(Z_1 + iZ_2). \quad (49)$$

The corresponding complex structure equations become

$$\llbracket \bar{V}_1, V_2 \rrbracket = -\frac{1}{2}V_3, \quad \llbracket \bar{V}_2, V_1 \rrbracket = \frac{1}{2}\bar{V}_3.$$

Let $\{\omega^1, \omega^2, \omega^3\}$ be the dual basis for $\mathfrak{g}^{*(1,0)}$, then

$$d\omega^1 = 0, \quad d\omega^2 = 0, \quad \text{and} \quad d\omega^3 = -\frac{1}{2}\omega^2 \wedge \bar{\omega}^1. \quad (50)$$

Since $d\bar{\omega}^3 = \frac{1}{2}\omega^1 \wedge \bar{\omega}^2$,

$$\iota_{V_1} d\bar{\omega}^3 = \frac{1}{2}\bar{\omega}^2.$$

The equation

$$\iota_X d\omega^3 = \iota_{V_1} d\bar{\omega}^3$$

does not have a solution. Therefore, the canonical holomorphic Poisson structure associated to nilmanifold with the given abelian complex structure fails to have a decomposition as noted in Theorem 3.

6.4 Six-dimensional, Type I Example

Real six-dimensional nilmanifolds with abelian complex structure could be divided into two types [6]. The structure equations of an ascending basis for type I abelian complex structures are given by

$$d\omega^1 = 0, \quad d\omega^2 = \omega^{1\bar{1}}, \quad d\omega^3 = B\omega^{1\bar{2}} + C\omega^{2\bar{1}}, \quad (51)$$

where $\omega^{k\bar{\ell}} = \omega^k \wedge \bar{\omega}^\ell$. It follows that

$$d\bar{\omega}^3 = -\bar{B}\omega^{2\bar{1}} - \bar{C}\omega^{1\bar{2}}.$$

Suppose that $X = c_1 V_1 + c_2 V_2$ is a potential vector field of the canonical Poisson structure, it satisfies the following set of constraints.

$$\iota_X d\omega^1 = 0, \quad \iota_X d\omega^2 = 0, \quad \iota_X d\omega^3 = -\iota_{V_2} d\bar{\omega}^3.$$

In terms of coordinates, these constraints become

$$c_1 \bar{\omega}^1 = 0, \quad c_1 B \bar{\omega}^2 + c_2 C \bar{\omega}^1 = \bar{B} \bar{\omega}^1. \quad (52)$$

Therefore, $c_1 = 0$ and $c_2 = \bar{B}/C$ if $BC \neq 0$. In other words, if the map $d\omega^3$ from $\mathfrak{t}^{1,0}$ to $\mathfrak{t}^{*(0,1)}$ is non-degenerate, then the canonical Poisson structure has a potential vector. Moreover, the Hodge-type decomposition is also an isomorphism of Gerstenhaber algebras.

There are two cases when $d\omega^3$ degenerates, but is non-trivial.

If $B = 0$, $\text{ad}_{V_2} \bar{\omega}^3 = 0$. It follows that

$$\bar{\partial}_A = \bar{\partial} + \text{ad}_A = \bar{\partial}$$

when it acts on $\oplus_{p,q} B^{p,q}$. Therefore, $\oplus_{p+q=k} H^q(\mathfrak{g}^{p,0})$ is naturally identical to H_A^k .

If $C = 0$ and $B \neq 0$, the system (52) does not have a solution.

This set of examples extends our scope beyond two-step nilmanifolds as seen in Section 6.3 above.

6.5 Six-dimensional, Type II Example

The structure equations of an ascending basis for Type II abelian complex structure in six-dimension are given by

$$d\omega^1 = 0, \quad d\omega^2 = 0, \quad d\omega^3 = A\omega^{1\bar{1}} + B\omega^{1\bar{2}} + C\omega^{2\bar{1}} + D\omega^{2\bar{2}}. \quad (53)$$

This is a two-step nilmanifold. Note that

$$d\bar{\omega}^3 = -\bar{A}\omega^{2\bar{1}} - \bar{B}\omega^{2\bar{2}} - \bar{C}\omega^{1\bar{2}} - \bar{D}\omega^{1\bar{1}}, \quad \iota_{V_2} d\bar{\omega}^3 = -\bar{B}\bar{\omega}^1 - \bar{D}\bar{\omega}^2.$$

If $X = c_1 V_1 + c_2 V_2$ is a potential vector field, then the sole constraint is $\iota_X d\omega^3 = -\iota_{V_2} d\bar{\omega}^3$, i.e.

$$c_1 A \bar{\omega}^1 + c_1 B \bar{\omega}^2 + c_2 C \bar{\omega}^1 + c_2 D \bar{\omega}^2 = -\bar{B} \bar{\omega}^1 - \bar{D} \bar{\omega}^2.$$

Equivalently,

$$(c_1, c_2) \begin{pmatrix} A & B \\ C & D \end{pmatrix} = (-\bar{B}, -\bar{D}). \quad (54)$$

Therefore, if $d\omega^3$ is non-degenerate, the canonical Poisson structure has a potential vector.

If $d\omega^3$ is a rank-1 map, there are various cases. Instead of investigating every case, we consider a particular situation. If $C = 0$ and $D = 0$, the constraints on a potential vector are reduced to

$$c_1 A = -\bar{B}, \quad c_2 B = 0.$$

- If $A \neq 0$ and $B \neq 0$, then there is a solution: $(c_1, c_2) = (-\bar{B}/A, 0)$.
- If $A = 0$ and $B \neq 0$, then there is no solution. This case corresponds exactly to the example found in Section 6.3.
- If $A \neq 0$ and $B = 0$, then there is a solution. $(c_1, c_2) = (0, 1)$.

The above observation points out the complication of the existence of potential vector when $d\omega^n$ degenerates. It will be a subject for future analysis.

7 Appendix

In this appendix, we provide a proof for an observation noted in the introduction.

Proposition 2. *Let Λ be a holomorphic Poisson structure. For each $t \neq 0$, there is an isomorphism of holomorphic Poisson cohomology.*

$$\phi_t : H_{t\Lambda}^k \cong H_\Lambda^k.$$

Recall that the cohomology spaces are respectively given by bi-complexes with operators

$$\bar{\partial}_{t\Lambda} = \text{ad}_{t\Lambda} + \bar{\partial} = t \text{ad}_\Lambda + \bar{\partial}, \quad \text{and} \quad \bar{\partial}_\Lambda = \text{ad}_\Lambda + \bar{\partial}.$$

We use the notation below.

$$\mathcal{C}^{p,q} = C^\infty(M, \wedge^p(T^{1,0}M) \otimes \wedge^q(T^{*(0,1)}M)).$$

Then

$$\bar{\partial} : \mathcal{C}^{p,q} \rightarrow \mathcal{C}^{p,q+1}, \quad \text{and} \quad \text{ad}_{t\Lambda} = t \text{ad}_\Lambda, \quad \text{ad}_\Lambda : \mathcal{C}^{p,q} \rightarrow \mathcal{C}^{p+1,q}.$$

Suppose that Ω^k is in $\oplus_{p+q=k} \mathcal{C}^{p,q}$, we represent it by its summands with corresponding bi-degrees:

$$\Omega^k = \Omega^{k,0} + \Omega^{k-1,1} + \dots + \Omega^{k-q,q} + \dots + \Omega^{1,k-1} + \Omega^{0,k}.$$

When we group $\bar{\partial}_{t\Lambda} \Omega^k$ with bi-degree, then

$$\begin{aligned} \bar{\partial}_{t\Lambda} \Omega^k &= t \text{ad}_\Lambda \Omega^{k,0} + (\bar{\partial} \Omega^{k,0} + t \text{ad}_\Lambda \Omega^{k-1,1}) + \dots + (\bar{\partial} \Omega^{k-q+1,q-1} + t \text{ad}_\Lambda \Omega^{k-q,q}) \\ &\quad + \dots + (\bar{\partial} \Omega^{1,k-1} + t \text{ad}_\Lambda \Omega^{0,k}) + \bar{\partial} \Omega^{0,k}. \end{aligned} \quad (55)$$

Therefore, Ω^k is in kernel of $\bar{\partial}_{t\Lambda}$ if and only if

$$\begin{aligned} \text{ad}_\Lambda \Omega^{k,0} &= 0, \quad \bar{\partial} \Omega^{k,0} + t \text{ad}_\Lambda \Omega^{k-1,1} = 0, \quad \dots, \quad \bar{\partial} \Omega^{k-q+1,q-1} + t \text{ad}_\Lambda \Omega^{k-q,q} = 0, \\ \dots, \quad \bar{\partial} \Omega^{1,k-1} + t \text{ad}_\Lambda \Omega^{0,k} &= 0, \quad \bar{\partial} \Omega^{0,k} = 0. \end{aligned} \quad (56)$$

When $t \neq 0$, this system of equations is equivalent to

$$\begin{aligned} \text{ad}_\Lambda \Omega^{k,0} &= 0, \quad \bar{\partial} \Omega^{k,0} + \text{ad}_\Lambda (t \Omega^{k-1,1}) = 0, \quad \dots, \quad \bar{\partial} (t^{q-1} \Omega^{k-q+1,q-1}) + \text{ad}_\Lambda (t^q \Omega^{k-q,q}) = 0, \\ \dots, \quad \bar{\partial} (t^{k-1} \Omega^{1,k-1}) + \text{ad}_\Lambda (t^k \Omega^{0,k}) &= 0, \quad \bar{\partial} (t^k \Omega^{0,k}) = 0. \end{aligned} \quad (57)$$

Therefore, for each $t \neq 0$, we consider the map ψ_t defined by

$$\begin{aligned}\psi_t(\Omega^k) &= \psi_t(\Omega^{k,0} + \Omega^{k-1,1} + \dots + \Omega^{k-q,q} + \dots + \Omega^{1,k-1} + \Omega^{0,k}) \\ &= \Omega^{k,0} + t\Omega^{k-1,1} + \dots + t^q\Omega^{k-q,q} + \dots + t^{k-1}\Omega^{1,k-1} + t^k\Omega^{0,k}.\end{aligned}\quad (58)$$

This is a one-to-one map from $\oplus_{p+q=k}\mathcal{C}^{p,q}$ onto $\oplus_{p+q=k}\mathcal{C}^{p,q}$. The computation in (56) and (57) shows that Ω^k is in the kernel of $\bar{\partial}_{t\Lambda}$ if and only if $\psi_t(\Omega^k)$ is in the kernel of $\bar{\partial}_\Lambda$.

Similarly, one could prove that if Ω^k is in $\oplus_{p+q=k}\mathcal{C}^{p,q}$, then there exists ω^{k-1} in $\oplus_{p+q=k-1}\mathcal{C}^{p,q}$ such that

$$\bar{\partial}_{t\Lambda}\omega^{k-1} = \Omega^k \quad \text{if and only if} \quad \bar{\partial}_\Lambda(t\psi_t(\omega^{k-1})) = \psi_t(\Omega^k). \quad (59)$$

Therefore, Ω^k is an image of $\bar{\partial}_{t\Lambda}$ if and only if $\psi(\Omega^k)$ is an image of $\bar{\partial}_\Lambda$. It follows that the map ψ_t induces an isomorphism from $H_{t\Lambda}^k$ to H_Λ^k .

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