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On some extension of Paley Wiener theorem

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Abstract: Paley Wiener theorem characterizes the class of functions which are Fourier transforms of $\mathbb{C}^\infty$ functions of compact support on $\mathbb{R}^n$ by relating decay properties of those functions or distributions at infinity with analyticity of their Fourier transform. The theorem is already proved in classical case : the real case with holomorphic Fourier transform on $L^2(\mathbb{R})$, the case of functions with compact support on $\mathbb{R}^n$ from Hörmander and the spherical transform on semi simple Lie groups with Gangolli theorem.

Let G be a locally compact unimodular group, K a compact subgroup of G , and δ an element of unitary dual $\hat{K}$ of K . In this work, we'll give an extension of Paley-Wiener theorem with respect to δ , a class of unitary irreducible representation of K , where G is either a semi-simple Lie group or a reductive Lie group with non-empty discrete series after introducing a notion of δ -orbital integral. If δ is trivial and one dimensional, we obtain the classical Paley-Wiener theorem.

Keywords: delta-orbital integral; reductive Lie group; spherical Fourier transform of type delta

MSC: 43, 22D, 22E, 46Em, 46H, 47L

1 Introduction

In this section, we shall study Paley-Wiener theorem in classical case.

Let $f \in \mathbb{C}^+$, then the following conditions are equivalent:

- i) $f \in \text{Hol}(\mathbb{C}^+)$ and $\sup_{0 \leq y \leq \infty} \int_{-\infty}^{+\infty} |f(x + iy)|^2 dx = C < \infty$
- ii) $f \in L^2(\mathbb{R})$ and $F \equiv 0$ on $]-\infty; 0[$ and $f(z) = \int_0^{+\infty} F(t)e^{-itz} dt, \forall z \in \Pi^+$.

The classical Paley–Wiener theorem uses the holomorphic Fourier transform on classes of square-integrable functions supported on $L^2(\mathbb{R})$.

Schwartz's Paley–Wiener theorem asserts that the Fourier transform of a distribution of compact support on $\mathbb{R}^n$ is an entire function on $\mathbb{C}^n$ and gives estimates on its growth at infinity. This theorem says that : An entire function F is the Fourier–Laplace transform of a distribution of compact support if and only if $\forall z \in \mathbb{C}^n$, there exists constants α and β such that

$$|F(z)| \leq \alpha(1 + |z|)^{-\beta} e^{R|Imz|}.$$

Let G be a connected semi-simple Lie group with finite center, $G = KAN$ an Iwasawa decomposition for G , where A is an abelian subgroup of G , N a nilpotent subgroup of G , K a compact subgroup of G , and $K^\sharp(G)$, the convolution algebra of continuous complex functions with compact support which are bi-invariant under K .

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Paley-Wiener Theorem exists for spherical functions, and is called Gangolli Theorem. this theorem says that for all $f \in K^\sharp(G)$ which vanishes outside the ball of radius R , the spherical Fourier-Laplace transform $\hat{f}$ of f defined by the rule

$$\hat{f}(v) = \int_G f(x) \phi_v(x) d_G(x), (\phi_v \in S(G \setminus K), \text{ the set of spherical functions on } G \text{ with respect to } K),$$

exists for all $v = \xi + i\eta \in \mathfrak{F}_c$ and is a W -invariant entire holomorphic function of v ($v \in \mathfrak{F}_c$), where W is the Weyl group, $\mathfrak{F}_c$ the complexification of $\mathfrak{F}$, and $\mathfrak{F}$ the real dual of $\mathcal{A}$. $\mathcal{A}$ is the lie algebra of A in the Iwasawa decomposition. Moreover, given any integer $\beta \geq 0$, there exists a constant $C_\beta > 0$ such that

$$|\hat{f}(\xi + i\eta)| \leq C_\beta (1 + \|\xi + i\eta\|)^{-\beta} e^{R\|\eta\|}, \xi, \eta \in \mathfrak{F}.$$

Conversely, if F is a W -invariant entire holomorphic function on $\mathfrak{F}_c$ with the property that there exists an $R > 0$ such that for any integer $\beta > 0$, there exists a constant $C_\beta > 0$ for which

$$|F(\xi + i\eta)| \leq C_\beta (1 + \|\xi + i\eta\|)^{-\beta} e^{R\|\eta\|}, \xi, \eta \in \mathfrak{F},$$

then there exists a unique function $f \in K(G)$, the algebra of continuous functions with compact support, such that $\hat{f} = F$; f vanishes outside the ball of radius R in A and is given by the formula

$$f(x) = [W]^{-1} \int_{\mathfrak{F}} F(v) \overline{\phi_v(x)} |c(v)|^{-2} d_{\mathfrak{F}}(v), x \in G,$$

where $c(v)$ is the Harish-Chandra's constant.

Let δ be a class of unitary irreducible representations of a compact subgroup K of G . Our goal is to establish and to prove the Paley-Wiener theorem with respect to the unitary dual of a compact group, in other words, to characterize functions which are Fourier transforms of type δ of some class of functions. As seen above, this result exists where δ is trivial and one dimensional. We'll start by defining generalized Abel according to δ .

2 Preliminaries

Let G be a locally compact unimodular group, K a compact subgroup of G , $\hat{K}$ the set of all equivalence classes of irreducible unitary representations of K .

For all class δ of $\hat{K}$, we denote by ξ_δ the character of δ , $d(\delta)$ the degree of δ and $\chi_\delta = d(\delta)\xi_\delta$.

If $\tilde{\delta}$ is the class of contragredient representations of δ in $\hat{K}$, we have $\chi_{\tilde{\delta}} = \bar{\chi}_\delta$ and thanks to the Schur Orthogonality relations, we can check that $\chi_{\tilde{\delta}} * \chi_\delta = \chi_{\tilde{\delta}}$. For all function $f \in K(G)$, the algebra of continuous functions with compact support, we set

$$\begin{aligned} \delta f(x) &= \bar{\chi}_\delta * f(x) = \int_K \chi_\delta(k) f(kx) dk \\ f_\delta(x) &= f * \bar{\chi}_\delta(x) = \int_K \chi_\delta(k^{-1}) f(xk) dk, \end{aligned}$$

(where dk is the normalized Haar measure on K), and

$$K_\delta(G) = \{f \in K(G), f = \delta f = f_\delta\}.$$

$K_\delta(G)$ is a subalgebra of $K(G)$ and the map $\bar{\chi}_\delta * f * \bar{\chi}_\delta$ is a projection of $K(G)$ onto $K_\delta(G)$.

Consider a Banach representation U of G on a Banach space E . Set $P(\delta) = U(\tilde{\chi}_\delta)$ et $E(\delta) = P(\delta)E$, $E(\delta)$ the closed subspace of E consisting of those vectors in E which transform under K according to δ . If $g = \tilde{\chi}_\delta * f * \tilde{\chi}_\delta$, we have $P(\delta)U(f)P(\delta) = U(g)$, $\forall f \in K(G)$; Thus, $E(\delta)$ is $U(f)$ -stable, $f \in K_\delta(G)$. Let $U_\delta(f)$ denote the restriction of $U(f)$ to $E(\delta)$; then we obtain a representation $f \mapsto U_\delta(f)$ of $K_\delta(G)$ onto $E(\delta)$. Let $\mathfrak{J}_c(G)$ the set of all functions f of $K(G)$ which are K -central (ie $f(kx) = f(xk) \forall k \in K$ and $x \in G$).

$\mathfrak{J}_c(G)$ is a subalgebra of $K(G)$ and the map $f \mapsto f_K$, with $f_K(x) = \int_K f(kxk^{-1})dk$, is a projection of $K(G)$ onto $\mathfrak{J}_c(G)$. For all $f, g \in K(G)$, we have the following properties

$$(f_K * g)_K = f_K * g_K = (f * g_K)_K \text{ and } (\tilde{\chi}_\delta * f)_K = \tilde{\chi}_\delta * f_K = (f * \tilde{\chi}_\delta)_K = f_K * \tilde{\chi}_\delta.$$

We set

$$K_\delta^\sharp(G) = K_\delta(G) \cap \mathfrak{J}_c(G).$$

If K is a compact subgroup of G and U a topological completely irreducible Banach representation of G on E , then the set of all operators $U_\delta(f)$, ($f \in K_\delta^\sharp(G)$) is the centralizer of representation $k \mapsto U_\delta(k)$ of K onto $E(\delta)$.

If we decompose the representation $k \mapsto U_\delta(k)$ of K onto $E(\delta)$ to obtain m irreducible equivalent representation, then the centralizer is isomorphic to the algebra $\mathbb{M}_m(\mathbb{C})$ of square matrices of order m . Thus, there exists an isomorphism $U_\delta(f) \mapsto u_\delta(f)$ of algebra $\{U_\delta(f), f \in K_\delta^\sharp(G)\}$ onto $\mathbb{M}_m(\mathbb{C})$ where $f \mapsto u_\delta(f)$ is an m -dimensional irreducible representation of $K_\delta^\sharp(G)$ and

$$\text{Tr}(U_\delta(f)) = d(\delta)\text{tr}(u_\delta(f)) \quad \forall f \in K_\delta^\sharp(G).$$

The algebra $K_\delta^\sharp(G)$ is isomorphic to algebra $U_{c,\delta}(G)$ of continuous functions ψ with compact support of G onto $F_\delta = \text{Hom}_{\mathbb{C}}(E_\delta, E_\delta)$ and which verify the relation

$$\psi(k_1 x k_2) = u_\delta(k_1)\psi(x)u_\delta(k_2).$$

A seminorm γ on G is a positive lower semicontinuous and bounded function on every compact of G such that

$$\gamma(xy) \leq \gamma(x)\gamma(y) \quad (x, y \in G).$$

A function f on G with values in a Banach space is called quasi-bounded function if there exists a seminorm γ on G such that

$$\sup_{x \in G} \frac{\|f(x)\|}{\gamma(x)} < \infty.$$

A spherical function ϕ (on G) of type δ is a quasi-bounded continuous function on G with values in $\text{End}_{\mathbb{C}}(E)$, (E is a finite dimensional vector space) such that

- i) $\phi(kxk^{-1}) = \phi(x)$
- ii) $\chi_\delta * \phi = \phi = \phi * \chi_\delta$
- iii) The map $u_\phi : f \mapsto \phi(f) = \int_G f(x)\phi(x)dx$ is an irreducible representation of algebra $K_\delta^\sharp(G)$.

If ϕ is a quasi-bounded continuous function on G with values in $\text{End}_{\mathbb{C}}(E)$ such that

$$\phi_K = \phi \text{ et } \chi_\delta * \phi = \phi.$$

Then the function ϕ is spherical function of type δ iff

$$\int_K \phi(kxk^{-1}y)dk = \phi(x)\phi(y), \forall x, y \in G.$$

As well, if U is an irreducible Banach representation of G over a space E such that δ occurs m times in the restriction of U to K , then there exists a function ϕ_δ^U defined on G which is spherical of type δ . The function ϕ_δ^U is said to be associated to the representation U .

Let $\mathcal{B}$ be a commutative Banach algebra with identity element e . Let $\mathcal{B}$ be an involutive normed complex algebra and $X_m(\mathcal{B})$ the set of all m -dimensional irreducible unitary representations of $\mathcal{B}$.

For all f in $\mathcal{B}$, a generalized Gelfand transform of f is a map denoted by $\mathcal{G}f$ of $X_m(\mathcal{B})$ onto the algebra $\mathbb{M}_m(\mathbb{C})$ of square matrices of order m which verifies

$$\begin{aligned} \mathcal{G}f: X_m(\mathcal{B}) &\longrightarrow \mathbb{M}_m(\mathbb{C}) \\ u &\longmapsto u(f) \end{aligned}$$

The homomorphism $f \mapsto \mathcal{G}f$ of $\mathcal{B}$ onto $\mathbb{M}_m(\mathbb{C})^{X_m(\mathcal{B})}$ is called the generalized Gelfand transformation associated to $\mathcal{B}$. If the algebra $\mathcal{B}$ is commutative, then irreducible unitary representations of $\mathcal{B}$ are one dimensional and thus we identify them with characters of $\mathcal{B}$. We get the usual definition of Gelfand transformation.

Let $\mathcal{G}_{m,\delta}(G)$ be the set of all spherical functions of type δ on G and height m ; thus, if ϕ is a function of $\mathcal{G}_{m,\delta}(G)$, there exists a representation $u_\delta^\phi \in X_m(K_\delta^\#(G))$ such that $u_\delta^\phi(f) = \int_G f(x)\phi(x)$ and conversely. This

result allows us to identify spaces $\mathcal{G}_{m,\delta}(G)$ with $X_m(K_\delta^\#(G))$ and then we can define spherical Fourier transform of type δ , $\mathcal{F}f$, accordingly the following diagram

Figure 1: Transformée de Fourier Sphérique de type δ

3 A Paley-Wiener theorem with respect to the unitary dual of a compact group

3.1 A Paley-Wiener on semi-simple Lie groups

Let G be a connected semi-simple Lie group with finite center, $G = KAN$ an Iwasawa decomposition for G , where A is an abelian subgroup of G , N a nilpotent subgroup of G , K a compact subgroup of G . Let's normalize the Haar measure

$$d_G(x) = h^{2\rho} d_K(k) d_A(h) d_N(n), x \in G, x = khn \in KAN,$$

where ρ is one-half the sum of the positive roots associated to Lie algebra of G .

Let $\delta \in \hat{K}$ and $\mu_\delta \in \delta$, an unitary irreducible representation of K onto the hilbert space E_δ . For every f dans $K_\delta^\sharp(G)$, let's consider the integral defined by

$$F_f^\delta(h) = h^\rho \int_K \int_N f(khn) \mu_\delta(k^{-1}) d_N(n) dk, h \in A.$$

We shall call the map $f \mapsto F_f^\delta$ the Abel transformation on G of type δ . We know that $K_\delta^\sharp(G)$ is isomorphic to $U_{c,\delta}(G)$ under the map $f \mapsto \psi_f^\delta$ defined by $\psi_f^\delta(x) = \int_K \mu_\delta(k^{-1}) f(kx) dk$. Then, for every f in $K_\delta^\sharp(G)$, we obtain that

$$F_f^\delta(h) = h^\rho \int_N \psi_f^\delta(hn) d_N(n), h \in A.$$

The Abel transformation is linear and one-to-one mapping of the algebra $K_\delta^\sharp(G)$ onto $K_\delta^\sharp(A)$. Let $N_K(A)$ and $Z_K(A)$, be respectively normalizer and centralizer of A in K . The Weyl group of the pair (G, A) can be identified with quotient $N_K(A)/Z_K(A)$. We say that a function f is W -invariante if, $\forall w \in W$, $w.f(x) = f(w^{-1}x) = f(x)$.

Theorem 3.1. *Let G be a connected semi-simple Lie group with finite center, and KAN its Iwasawa decomposition. For all $f \in K_\delta^\sharp(G, \text{End}_\mathbb{C}(E))$ which vanishes outside the ball of radius R in A , the spherical Fourier-Laplace transform of type δ , $\hat{f}_\delta$ of f defined by the rule*

$$\hat{f}_\delta(v) = \int_G f(x) \phi_{v,\delta}(x) d_G(x), \phi_{v,\delta} \in \mathcal{S}_{p,\delta}(G) \text{ with values in } \text{End}_\mathbb{C}(E),$$

exists for all $v = \xi + i\eta \in \mathfrak{F}_c$ and is a W -invariant entire holomorphic function of v ($v \in \mathfrak{F}_c$), where W is the Weyl group, $\mathfrak{F}_c$ the complexification of $\mathfrak{F}$, and $\mathfrak{F}$ the real dual of $\mathcal{A}$. Moreover, given any integer $\beta \geq 0$, there exists a constant $C_{\beta,\delta} > 0$ such that

$$\|\hat{f}_\delta(\xi + i\eta)\| \leq C_{\beta,\delta} (1 + \|\xi + i\eta\|)^{-\beta} e^{R\|\eta\|}, \xi, \eta \in \mathfrak{F}$$

Conversely, if F is a W -invariant entire holomorphic function on $\mathfrak{F}_c$ with the property that there exists an $R > 0$ such that for any integer $\beta > 0$, there exists a constant $C_{\beta,\delta} > 0$ for which

$$\|F(\xi + i\eta)\| \leq C_{\beta,\delta} (1 + \|\xi + i\eta\|)^{-\beta} e^{R\|\eta\|}, \xi, \eta \in \mathfrak{F}$$

then there exists a unique function $f \in K_\delta^\sharp(G, \text{End}_\mathbb{C}(E))$ such that $\hat{f}_\delta = F$; f vanishes outside the ball of radius R in A and is given by the formula

$$f(x) = [W]^{-1} \int_{\mathfrak{F}} F(v) \overline{\phi_{v,\delta}(x)} |c(v)|^{-2} d_{\mathfrak{F}}(v), x \in G$$

where $c(v)$ is the Harish-Chandra's constant.

Proof. (1) Suppose that f is an element of $K_\delta^\sharp(G, \text{End}_\mathbb{C}(E))$ which vanishes outside the ball of radius R in A . The spherical Fourier-Laplace transform of type δ , $\hat{f}_\delta$ of f defined by the rule

$$\hat{f}_\delta(v) = \int_G f(x) \phi_{v,\delta}(x) d_G(x), \phi_{v,\delta} \in \mathcal{S}_{p,\delta}(G),$$

exists for all $v = \xi + i\eta \in \mathfrak{F}_c$ and is a W -invariant entire holomorphic function of v ($v \in \mathfrak{F}_c$). Also, we see that the Abel transform F_f^δ of f has support in the ball of radius R in A ; Then, classical Paley-Wiener Theorem tells

us that the Euclidean Fourier transform $\hat{F}_f^\delta$ of F_f^δ is holomorphic on $\mathfrak{F}_c$ and has the property that for any given integer $\beta \geq 0$, there exists a constant $C_{\beta,\delta} > 0$ such that

$$|\hat{F}_f^\delta(\xi + i\eta)| \leq C_{\beta,\delta}(1 + \|\xi + i\eta\|)^\beta e^{R\|\eta\|}, \quad \xi, \eta \in \mathfrak{F}.$$

The result follows from the fact that $\hat{F}_f^\delta = \hat{f}_\delta$, indeed

$$\hat{f}_\delta(v) = \int_G f(x) \phi_{v,\delta}(x) d_G(x) = \int_K \int_A \int_N f(khn) \phi_{v,\delta}(khn) h^{2\rho} dk dh dn = \int_A \int_N f(hn) \phi_{v,\delta}(hn) h^{2\rho} dh dn$$

We have the following relation (1)

$$\int_G f(x) \phi_{v,\delta}(x) d_G(x) = \int_A h^{iv} F_f^\delta(h) d_A(h), \quad \phi_{v,\delta} \in \mathcal{G}_{p,\delta}(G), \quad (1)$$

The algebra $K_\delta^\sharp(G, \text{End}_{\mathbb{C}}(E))$ is isomorphic to algebra $U_{c,\delta}(G)$ of continuous functions with compact support of G in F_δ and which are u_δ -spherical on G . The isomorphism is defined by $f \mapsto \psi_f^\delta$, with $\psi_f^\delta(x) = \int_K u_\delta(k^{-1}) f(kx) dk$. Moreover, if δ is trivial and one dimensional, we have $\phi(x) = \int_K e^{(iv-\rho)H(xk)} dk$, and the relation (1)

$$\begin{aligned} \int_G \left(\int_K u_\delta(k^{-1}) f(kx) dk \right) \left(\int_K e^{(iv-\rho)H(xk)} dk \right) dx &= \int_A h^{iv} F_f^\delta(h) d_A(h) \\ \int_G \int_K u_\delta(k^{-1}) f(kx) e^{(iv-\rho)H(xk)} dk dx &= \int_A h^{iv} F_f^\delta(h) d_A(h) \\ \int_G f(x) \left(\int_K u_\delta(k^{-1}) e^{(iv-\rho)H(xk)} dk \right) dx &= \int_A h^{iv} F_f^\delta(h) d_A(h) \end{aligned}$$

Thus, with identification, $\phi_{v,\delta}(x) = \int_K u_\delta(k^{-1}) e^{(iv-\rho)H(xk)} dk$ so :

$$\begin{aligned} \hat{f}_\delta(v) &= \int_A \int_N \int_K f(khn) u_\delta(k^{-1}) e^{(iv-\rho)H(xk)} h^{2\rho} dk dh dn \\ &= \int_A \int_N \int_K f(khn) u_\delta(k^{-1}) e^{iv(h)} e^{-\rho(h)} h^{2\rho} dk dh dn \\ &= \int_A \int_N \int_K f(khn) u_\delta(k^{-1}) e^{iv(h)} h^{-\rho} h^{2\rho} dk dh dn \\ &= h^\rho \int_A \int_N \int_K e^{iv(h)} f(khn) u_\delta(k^{-1}) dk dh dn \end{aligned}$$

But $\psi_f^\delta(hn) = \int_K u_\delta(k^{-1}) f(khn) dk$. Thus, We have

$$\begin{aligned} \hat{f}_\delta(v) &= h^\rho \int_A \int_N e^{iv(h)} \psi_f^\delta(hn) dh dn \\ &= \int_A e^{iv(h)} F_f^\delta(h) dh \\ &= \hat{F}_f^\delta(h) \end{aligned}$$

We have the result.

(2) Suppose now that F is a W -invariant entire holomorphic function on $\mathfrak{F}_c$ with the property that there exists an $R > 0$, such that for any integer $\beta > 0$ there exists a constant $C_{\beta, \delta} > 0 > 0$ for which :

$$|F(\xi + i\eta)| \leq C_{\beta, \delta} (1 + \|\xi + i\eta\|)^\beta e^{R\|\eta\|}, \quad \xi, \eta \in \mathfrak{F}$$

Define a function $f \in K_\delta^\sharp(G, \text{End}_{\mathbb{C}}(E))$ by

$$f(x) = [W]^{-1} \int_{\mathfrak{F}} F(v) \overline{\phi_{v, \delta}(x)} |c(v)|^{-2} d_{\mathfrak{F}}(v), \quad x \in G$$

$f \in K_\delta^\sharp(G, \text{End}_{\mathbb{C}}(E))$ and f vanishes outside the ball of radius R in A . Indeed :

$$f(kxk^{-1}) = [W]^{-1} \int_{\mathfrak{F}} F(v) \overline{\phi_{v, \delta}(kxk^{-1})} |c(v)|^{-2} d_{\mathfrak{F}}(v) = f(x), \quad x \in G,$$

because $\phi_{v, \delta} \in \mathfrak{I}_c(G)$. Moreover, K is unimodular, we also have

$$\bar{\chi}_{\delta} * f = f.$$

And, by analytic continuation $\hat{f} = F$ on $\mathfrak{F}_c$.

It remains only to show that f is unique; Thus, suppose that g a function which verify the asserted properties above, then

$$\hat{f} = \hat{g} \Rightarrow \hat{F}_f^\delta = \hat{F}_g^\delta \Rightarrow F_f^\delta = F_g^\delta \Rightarrow f = g$$

Because the is one-to-one of $K_\delta^\sharp(G)$ onto $K_\delta^\sharp(A)$. □

3.2 A Paley-Wiener theorem on reductive Lie groups

Let G be a reductive Lie group with non-empty discrete series, $\mathfrak{G}$ the Lie algebra of G ; J a Cartan compact subgroup of G ; $\mathcal{U}$ an open subset of G which is completely invariant, γ a regular element of $\mathcal{U}$ ($\gamma \in \mathcal{U}_{reg}$).

One denotes by J^0 the identity component of J and $K = \gamma J^0 K$ is Cartan compact subgroup of G , $\mathcal{K}$ the Lie algebra of K , and the Weyl group $W(G; K)$ acts on $G/K \times K_{reg}$, dg a measure of G which is invariant on G/K .

As seen previously, the algebra $K_\delta^\sharp(\mathcal{U})$ is isomorphic to the algebra $U_{c, \delta}(\mathcal{U})$ of continuous functions with compact support ψ of $\mathcal{U}$ onto $F_\delta = \text{Hom}_{\mathbb{C}}(E_\delta, E_\delta)$ for any double Banach representation (u_δ, u_δ) of K .

Indeed, $\forall f \in K_\delta^\sharp(\mathcal{U})$, for $\psi_f^\delta(x) = \int_K u_\delta(k^{-1}) f(kx) dk$, we have $\psi_f^\delta(x) \in U_{c, \delta}(\mathcal{U})$. The isomorphism is defined

by the $f \mapsto \psi_f^\delta$.

The classical orbital integral on a reductive Lie group is defined by

$$J_G(f)(\gamma) = |\det(1 - \text{Ad}(\gamma^{-1}))_{\mathfrak{G}/\mathcal{K}}|^{1/2} \int_{G/K} f(g\gamma) dg, \quad f \in D(\mathcal{U}_{reg})$$

Thanks to the isomorphism of $K_\delta^\sharp(\mathcal{U})$ onto $U_{c, \delta}(\mathcal{U})$, we set

$$J_G^\delta(f)(\gamma) = |\det(1 - \text{Ad}(\gamma^{-1}))_{\mathfrak{G}/\mathcal{K}}|^{1/2} \int_{G/K} \psi_f^\delta(g\gamma) dg$$

The δ -orbital integral $J_G^\delta(f)(\gamma)$ is defined by

$$J_G^\delta(f)(\gamma) = |\det(1 - \text{Ad}(\gamma^{-1}))_{\mathfrak{G}/\mathcal{K}}|^{1/2} \int_K \int_{G/K} \mu_\delta(k) f(k.g\gamma) dg dk$$

If δ is trivial and one dimensional, we obtain the classical orbital integral.

We denote by $\mathcal{G}_{p,\delta}$ the set of spherical functions of type δ and height p . Let H be the factor of Lagrange decomposition of G which contains K , ($G = K.H$) and $\mathcal{H}$ the Lie algebra of H . For all $\phi \in \mathcal{G}_{p,\delta}$, $\exists u_\delta \in \hat{K}$ and $v \in \mathcal{H}_\mathbb{C}^*$ such that

$$\phi(kh) = \int_K u_\delta(k_1 k) (k_1 h)^{iv} dk_1$$

Thus, we can identify $\mathcal{G}_{p,\delta}$ with $\hat{K} \times \mathcal{H}_\mathbb{C}^*$. For any $\phi \in \mathcal{G}_{p,\delta}(G)$, $\exists (u_\delta, v) \in \hat{K} \times \mathcal{H}_\mathbb{C}^*$ and we set : $\|(u_\delta, v)\| = \|u_\delta\| + \|v\|$.

We denote by ρ_Δ one-half the sum of the positive roots associated to Lie algebra of G , κ the signature of the Weyl group W of (G, K) , and Δ a positive root system.

If m is a linear combination of roots of K , we denote by ξ_μ the corresponding character of K .

For all $r > 0$, we denote by $PW_\delta(K)_r$ the set of all functions $F: \mathcal{G}_{p,\delta} \rightarrow \text{End}(E)$ such that

1. $v \mapsto F(u_\delta, v)$ is holomorphic on $\mathbb{C}^*$
2. $\forall \beta \in \mathbb{N}$, $\exists C_\beta$ such that $|F(u_\delta, v)| \leq C_\beta (1 + \|u_\delta\| + \|v\|)^\beta e^{r\|Imv\|}$
3. $F(w.(u_\delta, v)\xi_{w.\rho_\Delta - \rho_\Delta}) = \kappa(w)F(u_\delta, v)$, $w \in W$.

Let put $\mathcal{PW}_\delta(K) = \bigcup_{r>0} \mathcal{PW}_\delta(K)_r$. We equip $\mathcal{PW}_\delta(K)$ with the topology defined by norms

$$S_k(F) = \sup_{\delta, v} (1 + \|u_\delta\| + \|v\|)^k |F(u_\delta, v)|$$

We denote by $I_{c,\delta}^\infty(G)$, the subspace of $K_\delta^\sharp(G)$ of infinitely differentiable functions.

Theorem 3.2.

$$\mathcal{F}(I_{c,\delta}^\infty(G)^\Delta) = \mathcal{PW}_\delta(K)^\Delta$$

where the map $f \mapsto \mathcal{F}f$ is the spherical Fourier transform of type δ .

$I_{c,\delta}^\infty(G)^\Delta$ is the set of all functions of $I_{c,\delta}^\infty(G)$ such that,

$$f(g^{-1}kg) = \kappa(g)\xi_{\rho_\Delta - g\rho_\Delta}f(k), \forall f \in I_{c,\delta}^\infty(G), \forall k \in K_{reg}$$

Proof. $\mathcal{PW}_\delta(K)$ is a close subspace of $\mathcal{PW}(K, \text{End}(E))$, then $\mathcal{PW}_\delta(K)$ is a Frechet space. Let put $Z_{\delta,p}(K)$ the set of all continuous functions of positive type ϕ of $\mathcal{G}_{p,\delta}(K)$. There exists $z \in (\text{End}(E))^*$ such that $\forall c_1, c_2, \dots, c_n \in \mathbb{C}$, $x_i \in G$ and $i \in \{1, 2, \dots, n\}$:

$$\langle \sum c_i \bar{c}_j \phi(x_i x_j^{-1}), z \rangle$$

Thanks to the Bochner theorem, there exist a measure $\hat{\mu}$ such that

$$f(kg) = \int_{Z_{\delta,p}(K)} F(u_\delta, v)(u_\delta, v)(kg) d\hat{\mu}(u_\delta, v),$$

$\forall F \in \mathcal{PW}_\delta(K)$, we have $\mathcal{F}f = F$.

$$\begin{aligned} f_K(x) &= \int_{Z_{\delta,p}(K)} F(u_\delta, v)(u_\delta, v)_K(x) d\hat{\mu}(u_\delta, v), \\ f * \chi_\delta(x) &= \int_{Z_{\delta,p}(K)} F(u_\delta, v)(u_\delta, v) * \chi_\delta(x) d\hat{\mu}(u_\delta, v), \end{aligned}$$

because $(u_\delta, v) \in \mathcal{G}_{p,\delta}(K)$ then $f \in I_{c,\delta}^\infty(G)$.

If $F \in \mathcal{PW}_\delta(K)^\Delta$, for $w \in W_C$, we have $F(g \cdot (u_\delta, v) \xi_{g \cdot \rho_\Delta - \rho_\Delta}) = \kappa(g)F(u_\delta, v)$, $g \in G$.

$$\begin{aligned} f(g \cdot h) &= \int_{Z_{\delta,p}(K)} F(u_\delta, v)(u_\delta, v)(g \cdot h) d\hat{\mu}(u_\delta, v) \\ &= \int_{Z_{\delta,p}(K)} F(g \cdot (u_\delta, v))(u_\delta, v)(h) d\hat{\mu}(u_\delta, v) \\ &= \int_{Z_{\delta,p}(K)} \kappa(g) \xi_{\rho_\Delta - g\rho_\Delta}(h) F(u_\delta, v)(u_\delta, v)(h) d\hat{\mu}(u_\delta, v) \\ &= \kappa(g) \xi_{\rho_\Delta - g\rho_\Delta}(h) f(h), \end{aligned}$$

Then $f \in I_{c,\delta}^\infty(G)^\Delta$. That prove that this map is surjective. $\square$

Let fix some conjugacy classes of Cartan subgroup of G , $K_1, K_2, \dots, K_k \in \text{Car}(G)$, and $\Delta_1, \Delta_2, \dots, \Delta_k$ the corresponding positive root system. Then $\forall f \in I_{c,\delta}^\infty(G)$, we have

$$\mathcal{F}(f)_{K_i} \in \mathcal{PW}_\delta(K)^{\Delta_i}.$$

Theorem 3.3. *The map F given by*

$$\begin{aligned} F : I_{c,\delta}^\infty(G) &\longrightarrow \bigoplus_{1 \leq i \leq k} \mathcal{PW}_\delta(K)^{\Delta_i}, \\ f &\longmapsto \sum_{i=1}^k \mathcal{F}f_{K_i}, \end{aligned}$$

is surjective.

Proof. Let put $F = \sum F_i \in \bigoplus_{1 \leq i \leq k} \mathcal{PW}_\delta(K)^{\Delta_i}$. Thanks to theorem 3.2, there exists $\phi_i \in I_{c,\delta}^\infty(K_i)^{\Delta_i}$ such that $\hat{\phi}_i = F_i$.

Let put $c_{G,K_i}(\gamma) = |K_i/Z(G, \gamma K_i^0)|$. The function $c_{G,K_i} \in I_{c,\delta}^\infty(K_i)^{\Delta_i}$ and there exists $f \in I_{c,\delta}^\infty(G)$ such that

$$b_{\Delta_i} J_G(f)_{K_i} = c_{G,K_i} \phi_i,$$

where b_Δ is the projection of $I_{c,\delta}(G)$ onto $I_{c,\delta}(G)^\Delta$ defined by $b_\Delta(\gamma) = \prod_{\alpha \in \Delta} \frac{1 - \xi_\alpha(\gamma^{-1})}{|1 - \xi_\alpha(\gamma^{-1})|}$, and we have

$$c_{G,K_i}^{-1}(b_{\Delta_i} J_G(f)_{K_i}) = \mathcal{F}f_{K_i}.$$

That prove that F is surjective. $\square$

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