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COMPUTING THE CHARACTERISTIC POLYNOMIAL OF A MULTIGRAPH

1. Introduction

Let $V = \{v_1, \dots, v_n\}$ be a set and E a family of elements of the Cartesian product $V\times V$. An element (v_i, v_j) of $V\times V$ can appear more than once in this family. The ordered pair $D = (V,E)$ is called directed multigraph (briefly: dimultigraph). Elements of V are vertices, elements of E are arcs of the dimultigraph. A multigraph G is the ordered pair $G = (V,U)$ where U is a family of subsets of V such that each element of U contains at most two elements of V. The elements of U are called the edges of G. The adjacency matrix $A(D) = [a_{ij}]$ of the dimultigraph $D = (V,E)$ with non-empty set V is the matrix with entry a_{ij} equals the number of arcs leading from the vertex v_i to the vertex v_j. By the characteristic polynomial of a dimultigraph D, written $F(D,x) = \sum_{i=0}^{n} a_i x^{n-i}$, we mean the characteristic polynomial of the adjacency matrix of D. If $D = (V,E) = K_0$ is empty dimultigraph (i.e. $V = \emptyset$), then it is both convenient and consistent to define the characteristic polynomial of K_0 as $F(K_0,x) = 1$. If two dimultigraphs have the same characteristic polynomial will be called cospectral. For graph-theoretical terms used without explicit definitions, see [1], [3].

The fundamental paper on the characteristic polynomials of simple graphs was published in 1957 by L.Collatz and U.Sinogowitz [2]. They obtained the relations between some of the coefficients a_i of $F(G,x)$ and certain graphical properties of G. A fundamental result in this area is due to H.Sachs [4]:

P r o p o s i t i o n 1. Let $F(D,x) = \sum_{i=0}^{n} a_i x^{n-i}$ be the characteristic polynomial of an arbitrary dimultigraph D. Then

$$a_i = \sum_{L_i \subset D} (-1)^{p(L_i)}, \quad i=1,\dots,n, \tag{1}$$

where the summation is taken over all linear directed subgraphs L_i (see [3] page 151) with exactly i vertices; $p(L_i)$ denotes the number of components of L_i.

For multigraphs, we have the following specialization of Proposition 1.

P r o p o s i t i o n 2. Let $F(G,x) = \sum_{i=0}^{n} a_i x^{n-i}$ be the characteristic polynomial of a multigraph G. Elementary figures are:

(a) a graph with two vertices (not necessary distinct) joined by an edge,

(b) an elementary cycle with p $(p \geqslant 3)$ vertices.

The basic figure U_i is every graph whose all components are elementary figures. Then

$$a_i = \sum_{U_i \subset G} (-1)^{p(U_i)} 2^{c(U_i)}, \quad i=1,\dots,n, \tag{2}$$

where $p(U_i)$ is the number of components of U_i, $c(U_i)$ is the number of components of U_i which are cycles of length $\geqslant 3$, and the summation is taken over all basic figures U_i with exactly i vertices which as partial subgraphs, are contained in G.

Several recurrence relations for computing the characteristic polynomial of simple graph are given by A.J.Schwenk [5] in 1974.

The present paper is concerned with the problem of computing the characteristic polynomial of dimultigraph (or multigraph) by describing the recurrence relations. Obviously, the theorems of Schwenk follow immediately from ours.

2. Recurrence relations

Theorem 1. Let v be a vertex of a dimultigraph $D = (V,E)$ and let $\overline{\varphi}(v)$ be the set of all elementary circuits of D containing v. Then

$$(3) \qquad F(D,x) = x \cdot F(D-v,x) - \sum_{c \in \overline{\varphi}(v)} F(D-V(c),x),$$

where $V(c)$ is the set of all vertices of circuit c and $D-v$, $D-V(c)$ are subdimultigraphs of D generated by $V \setminus \{v\}$ and $V \setminus V(c)$, respectively.

Proof. Let $F(D,x) = \sum_{i=0}^{n} a_i x^{n-i}$. Formula (1) expresses a_i in terms of the i-vertex linear directed subgraphs. Now, we establish a one-to-one correspondence between those linear directed subgraphs contributing to a_i on the left and those contributing to one of the terms on the right. Let L_i be a given linear directed subgraph of D and let $m_i = (-1)^{p(L_i)}$. We have two possibilities:

(i) If $v \notin V(L_i)$, let L_i' be the same linear directed subgraph of $D-v$. Since $L_i' = L_i$, we see that L_i' contributes m_i to the coefficient of x^{n-1-i} in $F(D-v,x)$, and thus supplies m_i toward the coefficient of x^{n-i} in $x \cdot F(D-v,x)$.

(ii) If $v \in V(c_k)$, $c_k \subset L_i \subset D$.
Let $L_{i-k}' = L_i - V(c_k) \subset D - V(c_k)$. Now, L_{i-k}' contributes $(-1)^{p(L_{i-k}')} = (-1)^{p(L_i - V(c_k))} = (-1)^{p(L_i)-1} = -m_i$ to $x^{n-k-(i-k)} = x^{n-i}$ in $F(D-V(c_k),x)$.
This completes the proof of the theorem.

A specialization of Theorem 1 for multigraphs is given below.

T h e o r e m 2. Let

1^o v be a vertex of a multigraph $G = (V,E)$,

2^o $\varphi(v)$ be the set of all elementary cycles of length $\geqslant 3$ containing v,

3^o $V(c)$ be the set of all vertices of the cycle c,

4^o $h(v,u)$ denotes the number of edges joining the vertices v and u of G,

5^o $l(v)$ be the number of loops incident to the vertex v.

Then

$$(4) \quad F(G,x) = x \cdot F(G-v,x) - l(v)F(G-v,x) -$$

$$- \sum_{u \text{ adj } v} (h(v,u))^2 \cdot F(G-v-u,x) - 2 \sum_{c \in \varphi(v)} F(G-V(c),x).$$

P r o o f . Let $\bar{G} = (V,E)$ be a dimultigraph obtained from $G = (V,E)$ by replacing any edge (not loops) of G by two oppositely directed arcs. Obviously, $F(G,x) = F(\bar{G},x)$ and $F(G-v,x) = F(\bar{G}-v,x)$. Note that

$$\sum_{c \in \bar{\varphi}(v)} F(\bar{G}-V(c),x) = \sum_{c_1 \in \bar{\varphi}(v)} F(\bar{G}-V(c_1),x) +$$

$$+ \sum_{c_2 \in \bar{\varphi}(v)} F(\bar{G}-V(c_2),x) +$$

$$+ \sum_{c_k \in \bar{\varphi}(v)} F(\bar{G}-V(c_k),x), \quad \text{for} \quad k \geqslant 3,$$

where $\bar{\varphi}(v)$ is the set of all elementary circuits containing v and c_i, $i=1,2,3,\ldots$ are circuits of length i. By simple considerations, we have

$$\sum_{c_1 \in \bar{\varphi}(v)} F(\bar{G}-V(c_1),x) = l(v)\cdot F(G-v,x),$$

$$\sum_{c_2 \in \bar{\varphi}(v)} F(\bar{G}-V(c_2),x) = \sum_{u \text{ adj } v} (h(v,u))^2\cdot F(G-v-u,x),$$

$$\sum_{c_k \in \bar{\varphi}(v)} F(\bar{G}-V(c_k),x) = 2\sum_{c \in \varphi(v)} F(G-V(c),x)$$

and from this our theorem follows.

If G is a simple graph, then by putting $l(v) = 0$ and $h(v,u) = 1$ in (4), we obtain, as the corollary, the theorem of Schwenk [5].

The next two theorems display the relations between the characteristic polynomial of dimultigraph D (or multigraph G) and the polynomials of D (or G) minus one arc (or edge) and vertices.

T h e o r e m 3. Let e be an arc of a dimultigraph $D = (V,E)$ and let $\bar{\varphi}(e)$ be the set of all elementary circuits of D containing e. Then

$$(5)\qquad F(D,x) = F(D-e,x) - \sum_{c \in \bar{\varphi}(e)} F(D-V(c),x).$$

P r o o f . Similarly as in the proof of Theorem 1 we consider two cases:

(i) $e \notin E(L_i)$, $L_i \subset D$. Let L_i' be the same linear directed subgraph of $D-e$. Obviously, $L_i' = L_i$ and L_i' contributes m_i to the coefficient of x^{n-i} in $F(D-e,x)$.

(ii) If $e \in E(c_k)$, $c_k \subset L_i \subset D$. Let $L_{i-k}' = L_i - V(c_k) \subset \subset D-V(c_k)$. In similar way as in the proof of Theorem 1 part (ii), we obtain the formula (5).

For multigraphs we have the following

T h e o r e m 4. Let

1° e be an edge joining two distinct vertices u and v of a multigraph $G = (V,E)$,

2^o $\varphi(e)$ be the set of all elementary cycles of length $\geqslant 3$ containing e,

3^o $V(c)$ be the set of all vertices of the cycle c and

4^o $h(u,v)$ denote the number of edges joining the vertices u and v of G.

Then

$$(6) \quad F(G,x) = F(G-e,x) - (2h(u,v)-1)\cdot F(G-u-v,x) - 2\sum_{c\in\varphi(e)} F(G-V(c),x).$$

P r o o f . Let a_1 and a_2 be two oppositely directed arcs (of $\bar{G}$) which are corresponding to edge e of G. By applying the formula (5) to $\bar{G}$ and a_1, and in the next step to $G-a_1$ and a_2, we have

$$(a) \quad F(G,x) = F(\bar{G},x) = F(\bar{G}-a_1,x) - \sum_{c\in\bar{\varphi}(a_1)} F(\bar{G}-V(c),x),$$

$$(aa) \quad F(\bar{G}-a_1,x) = F(\bar{G}-a_1-a_2,x) - \sum_{c\in\bar{\varphi}(a_2)} F(\bar{G}-a_1-V(c),x).$$

Note that

$$(*) \qquad \bar{G}-a_1-a_2 = \overline{G-e} \quad \text{and} \quad \bar{G}-a_1-V(c) = \bar{G}-V(c).$$

By the partition of $\bar{\varphi}(a_1)$ on two sets φ_1 and φ_2 containing the circuits of length two and length $\geqslant 3$, respectively, we see that

$$\sum_{c\in\varphi_1} F(\bar{G}-V(c),x) = h(u,v)\cdot F(\bar{G}-u-v,x) = h(u,v)\cdot F(G-u-v,x)$$

and

$$\sum_{c\in\varphi_2} F(\bar{G}-V(c),x) = \sum_{c_k\in\varphi(e)} F(G-V(c_k),x), \quad k \geqslant 3.$$

In similar way as above, let φ_3 and φ_4 be the sets of circuits of length two and of length $\geqslant 3$ respectively, of $\bar{G}-a_1$. Then, by (*),

$$\sum_{c\in\varphi_3} F(\bar{G}-V(c),x) = (h(u,v)-1)\cdot F(G-u-v,x)$$

and

$$\sum_{c\in\varphi_4} \bar{F}(G-V(c),x) = \sum_{c_k\in\varphi(e)} F(G-V(c_k),x), \quad k \geqslant 3.$$

By substituting (aa) to (a) and by above considerations, we obtain the relation (6).

If G is a simple graph, then by putting $h(u,v) = 1$ in (6) we obtain the theorem of Schwenk [5].

The coalescence of two rooted dimultigraphs D_1,r_1 and D_2,r_2, denoted $D_1\cdot D_2$, is the dimultigraph formed by identifying the two roots. The new root r is a cutvertex joining D_1 to D_2.

T h e o r e m 5. If D_1 and D_2 are two rooted dimultigraphs with roots r_1 and r_2, then

$$(7)\quad F(D_1\cdot D_2,x) = F(D_1,x)\cdot F(D_2-r_2,x) + F(D_1-r_1,x)\cdot F(D_2,x) - - x\, F(D_1-r_1,x)\cdot F(D_2-r_2,x).$$

P r o o f . By applying Theorem 1 to $D_1\cdot D_2$, we have

$$(a)\quad F(D_1\cdot D_2,x) = x\cdot F(D_1\cdot D_2-r,x) - \sum_{c\in\bar{\varphi}(r)} F(D_1\cdot D_2-V(c),x).$$

Since $D_1 \cdot D_r - r = (D_1 - r_1) \cup (D_2 - r_2)$, we obtain

$$\text{(b)} \qquad F(D_1 \cdot D_2 - r, x) = F(D_1 - r_1, x) \cdot F(D_2 - r_2, x).$$

The set of circuits $\bar{\varphi}(r)$ we can partition on two sets $\bar{\varphi}(r_1)$, $\bar{\varphi}(r_2)$ of circuits of D_1 and D_2 respectively. Thus

$$\text{(c)} \quad \sum_{c \in \bar{\varphi}(r)} F(D_1 \cdot D_2 - V(c), x) = \sum_{c \in \bar{\varphi}(r_1)} F(D_1 - V(c), x) \cdot F(D_2 - r_2, x) + \sum_{c \in \bar{\varphi}(r_2)} F(D_2 - V(c), x) \cdot F(D_1 - r_1, x).$$

From (3), we have

$$\text{(d)} \quad \sum_{c \in \bar{\varphi}(r_k)} F(D_k - V(c), x) = x \cdot F(D_k - r_k, x) - F(D_k, x), \quad k=1,2.$$

By the substitution of (b) to (a) and (d) to (c), and obtained new form of (c) to (a) we obtain, by elementary calculation, the formula (7).

Obviously, we can apply the formula (7) to multigraphs and simple graphs. For simple graphs the formula (7) was discovered by Schwenk in 1974 [5].

T h e o r e m 6. Let D_1, r_1 and D_2, r_2 be two disjoint rooted dimultigraphs. Let D be the dimultigraph formed by joining the roots with new arcs: $e_1, \ldots, e_{h_1}$ and $a_1, \ldots, a_{h_2}$ with initial endvertex r_1 and r_2, respectively. Then

$$\text{(8)} \quad F(D,x) = F(D_1,x) \cdot F(D_2,x) - h_1 \cdot h_2 \cdot F(D_1 - r_1, x) \cdot F(D_2 - r_2, x).$$

P r o o f . By (5), we have

$$F(D,x) = F(D - e_1, x) - \sum_{c \in \bar{\varphi}(e_1)} F(D - V(c), x).$$

Since the circuits c of $\bar{\varphi}(e_1)$ are formed by arc e_1 and $a_1,\ldots,a_{h_2}$, we have

$$\sum_{c\in\bar{\varphi}(e_1)} F(D-V(c),x) = h_2\cdot F(D-r_1-r_2,x) =$$

$$= h_2\cdot F(D_1-r_1,x)\cdot F(D_2-r_2,x).$$

Thus

$$F(D,x) = F(D-e_1,x) - h_2\cdot F(D_1-r_1,x)\cdot F(D_2-r_2,x).$$

By applying (5) do $D-e_1$ we have

$$F(D,x) = F(D-e_1-e_2,x) - 2h_2\cdot F(D_1-r_1,x)\cdot F(D_2-r_2,x).$$

Repeating the process for all arcs $e_3,\ldots,e_{h_1}$, we obtain

$$F(D,x) = F(D-e_1-\ldots-e_{h_1},x)-h_1h_2\cdot F(D_1-r_1,x)\cdot F(D_2-r_2,x).$$

Since $F(D-e_1-\ldots-e_{h_1},x) = F(D_1,x)\cdot F(D_2,x)$, from the above our theorem follows.

For multigraphs we have the following corollary.

C o r o l l a r y 1. Let G_1,r_1 and G_2,r_2 be two disjoint rooted multigraphs. The characteristic polynomial of the multigraph G formed by joining the roots with the new h edges is

$$F(G,x) = F(G_1,x)\cdot F(G_2,x) - h^2\cdot F(G_1-r_1,x)\cdot F(G_2-r_2,x).$$

If G_1,G_2 are simple graphs and $h = 1$, then we have the theorem of Schwenk [5].

A.J.Schwenk in [5] introduced the cospectrally rooted graphs. Analogously, two rooted dimultigraphs D_1,r_1 and D_2,r_2 are called cospectrally rooted if $F(D_1,x) = F(D_2,x)$ and $F(D_1-r_1,x) = F(D_2-r_2,x)$.

It is easy to see that Theorem 5 implies the following corollary.

C o r o l l a r y 2. If D_1 and D_2 cospectrally rooted and D is any rooted dimultigraph, then $F(D_1 \cdot D, x) = F(D_2 \cdot D, x)$.

Hence, the cospectrally rooted pair of dimultigraphs can be used to build larger cospectral pairs.

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