DEMONSTRATIO MATHEMATICA
Vol. XIV No 2 1981

Stanisław Tarnowski

ON A CERTAIN PROBLEM CONCERNING THE POLYHARMONIC POLYNOMIALS

1. In the paper ([1], p.86) B.Bondarenko proved that there exists exactly

$$h(n,s,p) = \binom{n+s}{s} - \binom{n+s-2p}{s} \tag{1}$$

linearly independent homogeneous p-harmonic polynomials of $s+1$ variables, of degree n. The proof of this assertion is rather complicated. In the present paper we give a simplified form of the proof.

2. First we shall introduce some notations.
Let $X = (x_0,\dots,x_s)$ and let

$$r^2(X) = \sum_{i=0}^{s} x_i^2.$$

Further, let

$$H_i^n(X), \quad i = 1,\dots,h(n,s,1) \tag{2}$$

denote the system of homogeneous linearly independent 1-harmonic polynomials of degree n, of $s+1$ independent variables.

By [1] the system (2) is unique.

By ([3], p.196) the following lemma is valid:

L e m m a 1. If $f(X)$ is a harmonic function, then the function $r^{2p-2} f(X)$ is a p-harmonic function.

Now we shall prove some generalization of the result received in ([2], p.119), that is, we shall prove

L e m m a 2. If $V_n(X)$ is a harmonic homogeneous function of degree n, $n \in N$, $m \in N$, then we have

$$\Delta\left(r^m V_n(X)\right) = m(n+s+m-1)r^{m-2} V_n(X). \tag{3}$$

We omit the simple proof.

3. Applying Lemmas 1,2 we shall prove the following theorem.

T h e o r e m 1. There exist exactly

$$h(n,s,p) = \binom{n+s}{s} - \binom{n+s-2p}{s}$$

linearly independent homogeneous p-harmonic polynomials of degree n, of $s+1$ variables.

P r o o f . Let us consider the system

$$W_i^{(n,1)}(X) = H_i^{(n)}(X), \quad i = 1,2,\dots,h(n,s,1), \tag{3_1}$$

where $H_i^{(n)}(X)$, $i = 1,2,\dots,h(n,s,1)$, are 1-harmonic homogeneous linearly independent polynomials of degree n, of $s+1$ variables. We also consider the system

$$W_i^{(n,2)}(X) = r^2 H_i^{(n-2)}(X), \quad i = 1,2,\dots,h(n-2,s,1), \tag{3_2}$$

where $H_i^{(n-2)}(X)$, $i = 1,2,\dots,h(n-2,s,1)$, are 1-harmonic homogeneous linearly independent polynomials of degree $n-2$ and $s+1$ variables,

....
....
....

and finally the system

$$(3_p) \quad W_i^{(n,p)}(X) = r^{2p-2}H_i^{(n-2p+2)}(X), \; i=1,\dots,h(n-2p+2,s,1),$$

where $H_i^{(n-2p+2)}(X)$, $i = 1,2,\dots,h(n-2p+2,s,1)$ are 1-harmonic homogeneous linearly independent polynomials of degree $n-2p+2$ and $s+1$ variables.

The number of all polynomials of the systems (3_i), $i = 1,2,\dots,p$ is given as follows

$$(4) \quad h(n,s,1) + h(n-2,s,1) + \dots + h(n-2p+2,s,1) = h(n,s,p).$$

Now it is sufficient to prove that the polynomials of the systems (3_i), $i = 1,2,\dots,p$ are linearly independent. In order to prove this assertion let us consider the polynomial

$$(5) \qquad W(X) = W_1(X) + W_2(X) + \dots + W_p(X),$$

where

$$W_1(X) = \sum_{i=1}^{h(n,s,1)} c_i^{(1)} W_i^{(n,1)}(X),$$

$$W_2(X) = \sum_{i=1}^{h(n-2,s,1)} c_i^{(2)} W_i^{(n,2)}(X),$$

$$\cdots\cdots \qquad \cdots\cdots$$

$$\cdots\cdots \qquad \cdots\cdots$$

$$\cdots\cdots \qquad \cdots\cdots$$

$$W_p(X) = \sum_{i=1}^{h(n-2p+2,s,1)} c_i^{(p)} W_i^{(n,p)}(X).$$

By Lemma 2 and (3_i) we obtain

$$(6) \qquad \Delta^p W(X) = 0$$

and

(7)
$$\Delta^{p-1} W(X) = \Delta^{p-1} W_p(X) =$$
$$= A(n,s,p) \sum_{i=1}^{h(n-2p+2,s,1)} C_i^{(p)} H_i^{(n-2p+2)}(X) = 0,$$

where $A(n,s,p)$ is the convenient positive constant. In addition, by the linear independence of polynomials $H_i^{(n-2p+2)}(X)$, $i = 1,2,\dots,h(n-2p+2,s,1)$, we receive the following conditions

(8)
$$C_i^{(p)} = 0, \quad i = 1,2,\dots,h(n-2p+2,s,1).$$

Similarly we have

(9)
$$\Delta^{p-2} W(X) = \Delta^{p-2} W_{p-1}(X) =$$
$$= A_1(n,s,p-1) \sum_{i=1}^{h(n-2p+4,s,1)} C_i^{(p-1)} H_i^{(n-2p+4)}(X) = 0$$

and consequently we obtain the conditions

(10)
$$C_i^{(p-1)} = 0, \quad i = 1,2,\dots,h(n-2p+4,s,1)$$

and also

(11)
$$C_i^{(1)} = 0, \quad i = 1,2,\dots,h(n,s,1)$$

which completes the proof of Theorem 1.

REFERENCES

[1] Б.А. Бондаренко: Полигармонические полиномы. Ташкент 1968.

[2] Е.В. Гобсон: Теория сферических и эллипсоидальных функций. Москва 1952.

[3] M. K r z y ż a ń s k i : Równania różniczkowe cząstkowe rzędu drugiego, cz.II. Warszawa 1962.

INSTITUTE OF APPLIED MECHANICS, TECHNICAL UNIVERSITY, KIELCE
Received December 21, 1979.